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Abstract

Let $\pi(f)$ be a nearly ordinary automorphic representation of the multiplicative group of an indefinite quaternion algebra B over a totally real field F with associated Galois representation ρ_f. Let K be a totally complex quadratic extension of F embedding in B. Using families of CM points on towers of Shimura curves attached to B and K, we construct an Euler system for ρ_f. We prove that it extends to p-adic families of Galois representations coming from Hida theory and dihedral $\mathbb{Z}_p^d$-extensions. When this Euler system is non-trivial, we prove divisibilities of characteristic ideals for the main conjecture in dihedral and modular Iwasawa theory.

1. Introduction

1.1 Motivation and set-up

1.1.1 *Motivation.* Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. Let $K/\mathbb{Q}$ be a quadratic imaginary extension such that all primes dividing N split in K (the so-called Heegner hypothesis). Then the completed L-function $L(E/K, s)$ satisfies the symmetric functional equation

$$L(E/K, s) = \varepsilon_{K/F}(s)L(E/K, 2-s)$$

with $\varepsilon_{K/F}(1)$ equal to -1 by the Heegner hypothesis. In the celebrated work [GZ86], it is shown that the first derivative of $L(E/K, s)$ at $s = 1$ is equal up to normalization factors to the height of a certain Heegner point $z \in E(K)$. Comparing this result with the Birch–Swinnerton–Dyer conjecture yields the following.

Conjecture 1.1 [GZ86, Conjecture (2.2)]. If z has infinite order in $E(K)$, then it generates a subgroup of finite index and $[E(K) : z]^2 \overset{\bullet}{=} |Ш(E/K)|(\prod_{\ell|N} \mathrm{Tam}(E/\mathbb{Q}, \ell))^2$.

Here, the bullet indicates the presence of easy normalization factors and $\mathrm{Tam}(E/\mathbb{Q}, \ell)$ is the Tamagawa number of E at ℓ.

When E has good ordinary reduction at p, this conjecture was given a dihedral Iwasawa-theoretic transcription soon thereafter in [Per87, Théorème 4.1]: one crucial ingredient being that Heegner points give rise to universal norms for the dihedral $\mathbb{Z}_p$-extension of K, as already noted in [Maz84]. At least if the G_K-representation T_pE has large image, a substantial step towards Conjecture 1.1, namely that $|Ш(E/K)[p^\infty]| \leqslant [E(K) \otimes \mathbb{Z}_p : z]^2$, was then proved in [Kol90] using the fact that Heegner points form an Euler system. These results were subsequently generalized to incorporate dihedral Iwasawa theory in [Ber95, Theorem 3.2.1] and [How04a, Theorem B]. In another direction, it was shown in [Nek07, Theorem 3.2] and

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[Zha01, Theorem A] that the results of Gross–Zagier and Kolyvagin generalize to quaternionic Shimura curves over totally real fields; the dihedral Iwasawa-theoretic generalization being given in [How04b, Theorem B].

From the point of view of deformation theory, the introduction of the dihedral $\mathbb{Z}_p$-extension of K corresponds to the study of the variation of Conjecture 1.1 within the universal deformation ring of the central character of T_pE. However, the philosophy of the Equivariant Tamagawa Number Conjecture (as formulated for instance in [BF01, FP94, Kat93]) suggests that conjectures on special values of L-functions are best studied when they take into account the action of the largest algebra of endomorphisms acting on the object of interest, or in other words when they are expressed with coefficients in the universal deformation ring of the residual Galois representation. The aim of this article is to formulate a conjecture generalizing Conjecture 1.1 to this natural setting and to prove part of it using the method of Euler systems. The situation we have in mind is briefly the following: instead of an elliptic curve over $\mathbb{Q}$ and the Iwasawa algebra of the anticyclotomic $\mathbb{Z}_p$-extension, we consider a nearly ordinary automorphic representation $\pi(f)$ of the adelic points of the reductive group attached to a quaternion algebra B over a totally real field and the Hecke algebras of Hida theory which are often the universal deformation rings of the residual Galois representation of $\pi(f)$. The study of Conjecture 1.1 in Hida families of ordinary automorphic forms was first proposed in [How07, Conjecture 3.3.1] when $B^\times = \mathrm{GL}_2(\mathbb{Q})$. This text grew as an attempt to generalize [How07] to totally real fields and to give proofs of some results left as conjectures there.

1.1.2 *Set-up.* Let $F/\mathbb{Q}$ be a totally real field of degree d. Fix an Archimedean prime τ_1 and a finite place of finite places Σ_B such that $|\Sigma_B| \equiv d+1 \bmod 2$. Let B/F be the quaternion algebra ramified at $\Sigma_B \cup \{\tau_i \mid \infty, i>1\}$. Let $p>3$ be a rational prime and N be an ideal of $\mathcal{O}_F$ prime to p such that $v|Np$ implies that $v \notin S_B$. Let U be a compact open subgroup of the adelic points $\widehat{B}^\times$ of $B^\times$. In this introduction, we assume that $U = \prod_v U_v$, that U_v is maximal outside Np, that it is of type U_0 at $v|N$ and of type U_1^1 at $v|p$ (see § 2.2.1.2 for precise definitions). Let $\mathbf{T}$ be the Hecke algebra generated by Hecke operators outside $N \cup S_B$, by the nearly ordinary Hecke operators at $v|p$ and by the diamond operators (see § 3 for precise definitions). Let $\pi(f)$ be an automorphic representation of $\widehat{B}^\times$ arising from f, an eigencuspform for $\mathbf{T}$ of even weights $k \geqslant 2$ belonging to $S_k(U, \omega_f)$. We assume that $\pi(f)$ is nearly ordinary at p, in the sense for instance of [SW99, p. 21]. Let λ_f be the map sending elements of $\mathbf{T}$ to the corresponding eigenvalue of f (we will more generally use the notation λ_g for the same map attached to a nearly ordinary eigenform g) and let $L_\mathfrak{p}$ be a finite extension of $\mathbb{Q}_p$ containing the image of λ_f. Let $\mathcal{O}$ be the ring of integers of $L_\mathfrak{p}$ and $\mathfrak{m}_\mathcal{O}$ and $\mathbb{F}$ the maximal ideal and residue field of $\mathcal{O}$. The p-adic nearly ordinary Hecke algebra $\mathbf{T}_\infty^{\mathrm{ord}}$ of Hida theory (see [Hid86, Hid88, Hid89b, SW99, SW01], we use the convention of the latter two articles) acts on $S_k(U)$. The inclusion of the diamond operators makes $\mathbf{T}_\infty^{\mathrm{ord}}$ into a torsion-free Λ-module of finite type where Λ is a regular ring of dimension $2+d+\delta_{F,p}$, with $\delta_{F,p}$ the defect of Leopoldt's conjecture for F and p. The map λ_f from $\mathbf{T}_\infty^{\mathrm{ord}}$ to $\mathcal{O}$ defines a unique maximal ideal $\mathfrak{m}$ of $\mathbf{T}_\infty^{\mathrm{ord}}$. Because the Hecke algebra is locally étale at arithmetic points, λ_f also defines a unique minimal prime $\mathfrak{a} \subset \mathfrak{m}$. Let $(V(f), \rho_f, L_\mathfrak{p})$ (respectively $(T(f), \rho_f, \mathcal{O})$, respectively $(\overline{T}(f), \bar{\rho}_f, \mathbb{F})$) be the G_F-representation attached to f with coefficients in $L_\mathfrak{p}$ (respectively in $\mathcal{O}$ after a choice of lattice, respectively in $\mathbb{F}$ after reduction modulo $\mathfrak{m}_\mathcal{O}$). For $v|p$, the representation $\rho_f|G_{F_v}$ is reducible. Assume that it is G_{F_v}-distinguished for $v|p$, which is to say that the Jordan-Hölder factors of $\bar{\rho}_f^{ss}|G_{F_v}$ are distinct (Assumption 3.5). Assume that $\bar{\rho}_f$ is irreducible (Assumption 3.4) and let $R(\mathfrak{a})$ be $\mathbf{T}_\mathfrak{m}^{\mathrm{ord}}/\mathfrak{a}$. There then exists a G_F-representation $(\mathcal{T}(f), \rho_\mathfrak{m}, R(\mathfrak{a}))$ constructed by patching pseudorepresentations as in [Hid89b, Wil88] which is free of rank 2 over $R(\mathfrak{a})$. The G_F-representation $\mathcal{T}(\mathfrak{a})$ interpolates the G_F-representations attached to nearly ordinary modular forms in the Hida family of tame level N containing f in the following sense. If S is a complete local Noetherian domain which is an $\mathcal{O}$-algebra and if $\mathbf{Sp}$ is a morphism of $\mathcal{O}$-algebras between $R(\mathfrak{a})$ and S, let $T_{\mathbf{Sp}}$ be the S-specialization of $\mathcal{T}(f)$ attached to $\mathbf{Sp}$, this is to say the free S-module $\mathcal{T}(f)\otimes_{R(\mathfrak{a}),\mathbf{Sp}} S$ with G_F-action through $\rho_{\mathfrak{m}}$ on $\mathcal{T}(f)$ and trivial G_F-action on S. Then, if $\mathbf{Sp}$ is equal to λ_g for a nearly ordinary eigenform g, then T_{λ_g} is isomorphic as $S[G_F]$-module to $T(g)$. More generally, we call an S-module T an S-specialization of an $R(\mathfrak{a})[G]$-module T' whenever there is a map of $\mathcal{O}$-algebras $\mathbf{Sp}$ from R to S and $T = T'\otimes_{R(\mathfrak{a}),\mathbf{Sp}} S$ with the natural G-action on T' and trivial G-action on S.

Let $\{U(S)\}_S$ be a tower of compact open subgroups of U constant outside p (see § 2.2 for precise definitions) and $X(S)$ be the Shimura curve whose complex points are given by:

$$X(S)(\mathbb{C}) = B^\times\backslash(\mathbb{C}-\mathbb{R}\times \widehat{B}^\times/U(S)).$$

When $B=\mathcal{M}_2(\mathbb{Q})$, we take $X(S)$ to be the smooth compactification of the previous curve. The representation $\mathcal{T}(f)$ can often be realized geometrically as

$$M_{\mathfrak{m}}^{\mathrm{ord}} = \varprojlim_{S} H^1_{\mathrm{et}}(X(S)\times_F \bar{F},\mathcal{O})_{\mathfrak{m}}\otimes_{\mathbf{T}_{\mathfrak{m}}^{\mathrm{ord}}} R(\mathfrak{a}). \tag{1.1.1}$$

This is true in particular when F is equal to $\mathbb{Q}$ or under deformation conditions on $\bar{\rho}_f$ and is expected to hold very generally (see § 3.2.4 for details and precise statements).

Let K be a quadratic totally complex extension of F which embeds into B. Assume that $\bar{\rho}_f$ does not have residual complex multiplication by K (see Assumption 5.13 for details). The G_F-representation $T(f)$ is essentially self-dual. Assume that there exists a character χ of $\mathbb{A}_F^\times/F^\times$ such that $T(f)\otimes\chi$ is self-dual (see Assumption 3.10). There then exists a specific quotient R of $R(\mathfrak{a})$ of dimension $1+d$ and a self-dual G_F-representation $\mathcal{T}$ which is a quotient of $\mathcal{T}(f)$ such that the specialization T_{λ_f} of $\mathcal{T}$ attached to λ_f is equal to the self-dual twist of $T(f)$ (see Theorem 3.1 for details and precise statements).

Let D_∞ be the $\mathbb{Z}_p^d$-extension of K which is pro-dihedral over F and let Λ_a be the $(1+d)$-dimensional regular ring $\mathcal{O}[[\Gamma]]=\mathcal{O}[[\mathrm{Gal}(D_\infty/K)]]$. Let R_{Iw} be $R[[\Gamma]]$ and $\mathcal{T}_{\mathrm{Iw}}$ be $\mathcal{T}\otimes_R R_{\mathrm{Iw}}$ with action of G_K on both sides of the tensor product. For a system of well-chosen ideals and using coherent families of CM points in the tower of Shimura curves $\{U(S)\}$, we construct an Euler system $\{z(\mathfrak{c})\in H^1(K(\mathfrak{c}),\mathcal{T})\}_{\mathfrak{c}}$ where $K(\mathfrak{c})$ is the ring-class field of conductor $\mathfrak{c}$. This Euler system extends in the D_∞-direction in the sense that:

$$z(\mathfrak{c}p^\infty) = \varprojlim_n z(\mathfrak{c}p^n) \text{ belongs to } \varprojlim_n H^1(K(\mathfrak{c}p^n),\mathcal{T}) = H^1(K(\mathfrak{c}),\mathcal{T}_{\mathrm{Iw}}).$$

It was conjectured in [Maz84, Conjecture, p. 197] that the element $z_\infty = \mathrm{Cor}_{K(1)/K}\, z_\infty(1)$ is not a torsion element of $H^1(K,\mathcal{T}_{\mathrm{Iw}})$ (see also [CV04, § 1]).

A natural Iwasawa-theoretic generalization of Conjecture 1.1 would strive to express Galois cohomological invariants of $\mathcal{T}_{\mathrm{Iw}}$ in terms of the index of z_∞ inside $H^1(K,\mathcal{T}_{\mathrm{Iw}})$, just as Ш$(E/K)$ is described using the Heegner point z. However, several technical hurdles appear here. First of all, the first proposed generalization of Iwasawa theory to motives with coefficients in Hecke algebras given in [Gre89, Gre94] uses characteristic ideals, so requires the coefficient rings to be normal. The formulation given in [Kat93, Conjecture 3.2.2] covers general coefficient rings but takes as input a lisse sheaf over Spec $\mathbb{Z}[1/p]$. It is not known whether étale cohomology of towers of Shimura curves provides such an input (and this author suspects that they don't in

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general). Besides, when the L-function of the motive vanishes, which is our case of interest, [Kat93, Conjecture 3.2.2] requires some semi-simplicity properties of motives or almost equivalently the non-degeneracy of an R_{Iw}-valued height pairing. Such properties are as yet unknown. Hence, even the statements of the conjectures we discuss in this text are quite involved. Briefly speaking, we study the Selmer complex $\mathrm{R}\,\Gamma_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}})$ of $\mathcal{T}_{\mathrm{Iw}}$ as introduced in [Nek06]. Selmer complexes over Hecke algebras are not known to be perfect in general, but the rigidity of the automorphic type in Hida families allows us to construct a (non-canonical) modification of $\mathrm{R}\,\Gamma_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}})$ which is a perfect complex concentrated in degree $[0, 2]$. We study two different integral structures on the one-dimensional $\mathrm{Frac}(R_{\mathrm{Iw}})$-vector space $(\mathrm{Det}_{R_{\mathrm{Iw}}}\mathrm{R}\,\Gamma_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}}))\otimes_{R_{\mathrm{Iw}}}\mathrm{Frac}(R_{\mathrm{Iw}})$: one called the Euler structure coming from the existence of the Euler system for $\mathcal{T}_{\mathrm{Iw}}$ and one called the characteristic structure coming from the Fitting ideal of the second cohomology group of $\mathrm{R}\,\Gamma_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}})$. Equivalently, the Euler and characteristic structure can be viewed as choices of bases (b_{Eul}) and (b_{char}) of $(\mathrm{Det}_{R_{\mathrm{Iw}}}\mathrm{R}\,\Gamma_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}}))\otimes_{R_{\mathrm{Iw}}}\mathrm{Frac}(R_{\mathrm{Iw}})$. Our conjecture is that these integral structures coincide: we refer to § 6.2.1 and especially to Conjectures 6.12, 6.13 and Question 6.15 for details. When R_{Iw} is known to be regular and when z_∞ belongs to $\tilde{H}^1_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}})$ and is not torsion, our conjectures imply in particular the following equality:

$$\mathrm{char}_{R_{\mathrm{Iw}}}\tilde{H}^2_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}})_{\mathrm{tors}} = (\mathrm{char}_{R_{\mathrm{Iw}}}\tilde{H}^1_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}})/z_\infty)^2. \tag{1.1.2}$$

Here, $\tilde{H}^i_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}})$ is the ith cohomology group of the Selmer complex $\mathrm{R}\,\Gamma_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}})$. Under this hypothesis of regularity, our conjectures thus contain [How07, Conjecture 3.3.1]. For S a Cohen–Macaulay domain, let the support of a basis (b) of a free S-module of rank 1 inside $\mathrm{Frac}(S)$ be the set of height 1 primes such that $bS_{\mathcal{P}}$ is not equal to $S_{\mathcal{P}}$. A weaker form of our conjectures which does not require the regularity of R_{Iw} is that the support of (b_{Eul}) and (b_{char}) coincide.

1.2 Statement of results

This article establishes Conjecture 6.12 for the G_K-representation $T\otimes\Lambda_a$ and a slightly weaker statement for $\mathcal{T}_{\mathrm{Iw}}$. The reader should note that the theorems proved in the text are somewhat more precise and significantly more general than those stated in this section, where we have highlighted specific important instances. We refer to Theorems 6.1–6.3, Corollaries 6.19 and 6.20 for the strongest statements we obtain. We first state our results when $B^\times = \mathrm{GL}_2(\mathbb{Q})$, a situation which we refer to hereafter as the classical case.

THEOREM A (Classical case). *Assume $B^\times$ to be $\mathrm{GL}_2(\mathbb{Q})$. Let $s > 1$. Let $f \in S_k(\Gamma_0(N)\cap\Gamma_1(p^s), \omega^j)$ be a p-ordinary eigencuspform with central character a power of the Teichmüller character. Assume 3.4, 3.5, that all primes dividing N split in K and that $p\nmid 6\phi(N)$. Let λ_f (respectively $\lambda_{f,\infty}$) be the $\mathcal{O}$-algebra morphism from R_{Iw} to $\mathcal{O}$ (respectively Λ_a) equal to λ_f on R and the zero map on $\mathrm{Gal}(D_\infty/K)$ (respectively to λ_f on R and the identity on $\mathrm{Gal}(D_\infty/K)$). Let T and $T\otimes_{\mathcal{O}}\Lambda_a$ be the corresponding specializations of $\mathcal{T}_{\mathrm{Iw}}$. Let z_∞ be the non-torsion class in $\tilde{H}^1_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}})$ which is the first class constructed of the Euler system of Hida-theoretic Heegner points constructed in [How07].*

(i) *Let $z_f \in \tilde{H}^1_f(K_\Sigma/K, T)$ be the image of z_∞ under λ_f. Assume that z_f is not $\mathcal{O}$-torsion. Then Conjecture 6.14 is true. In particular, if T is non-exceptional, then:*

$$\ell_{\mathcal{O}}\tilde{H}^2_f(K_\Sigma, T)_{\mathrm{tors}} \leqslant 2\ell_{\mathcal{O}}(\tilde{H}^1_f(K_\Sigma/K, T)/\mathcal{O}z_f). \tag{1.2.1}$$

(ii) *Let* $z_{f,\infty} \in \tilde{H}^1_f(K_\Sigma/K, T \otimes_{\mathcal{O}} \Lambda_a)$ *be the image of* z_∞ *under* $\lambda_{f,\infty}$*. Then* $z_{f,\infty}$ *is not* Λ_a*-torsion and Conjecture 6.14 is true for* $T \otimes_{\mathcal{O}} \Lambda_a$*. In particular, if* T *is non-exceptional, then:*

$$\operatorname{char}_{\Lambda_a} \tilde{H}^2_f(K_\Sigma/K, T \otimes_{\mathcal{O}} \Lambda_a)_{\mathrm{tors}} | (\operatorname{char}_{\Lambda_a} \tilde{H}^1_f(K_\Sigma/K, T \otimes_{\mathcal{O}} \Lambda_a)/\Lambda_a z_{f,\infty})^2. \tag{1.2.2}$$

(iii) *The class* z_∞ *is not* R_{Iw}*-torsion and* $\tilde{H}^1_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}})$ *is of rank* 1*. The support of the characteristic structure is contained in the support of the Euler structure. Assume* R *to be a regular ring. Then:*

$$\operatorname{char}_{R_{\mathrm{Iw}}} \tilde{H}^2_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}})_{\mathrm{tors}} | \operatorname{char}_{R_{\mathrm{Iw}}} (\tilde{H}^1_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}})/R_{\mathrm{Iw}} z_\infty)^2. \tag{1.2.3}$$

Remark. Because Theorem A uses [How07, Theorem 3.3.1], its hypotheses are slightly different from those made on $\pi(f)$ in the bulk of the introduction. In terms of attribution, statement (1.2.1) goes back in essence to [Kol90]. Large parts of it can be found in [Nek92] (under slightly different hypotheses). Many cases of both statements (1.2.1) and (1.2.2) are proved or implicit in [Ber95, How04a, How07].[1]

THEOREM B (General case). *Assume that* $\pi(f)$ *is a nearly ordinary automorphic form which satisfies Assumptions 3.4, 3.5, 3.10, 5.13. Let* λ_f *(respectively* $\lambda_{f,\infty}$*) be the* $\mathcal{O}$*-algebra morphism from* R_{Iw} *to* $\mathcal{O}$ *(respectively* Λ_a*) equal to* λ_f *on* R *and the zero map on* $\operatorname{Gal}(D_\infty/K)$ *(respectively to* λ_f *on* R *and the identity on* $\operatorname{Gal}(D_\infty/K)$*). Let* T *and* $T \otimes_{\mathcal{O}} \Lambda_a$ *be the corresponding specializations of* $\mathcal{T}_{\mathrm{Iw}}$*. Let* z_∞ *be the first class of the Hida-theoretic Euler system of CM points constructed in § 4.*

(i) *Let* $z_f \in \tilde{H}^1_f(K_\Sigma/K, T)$ *be the image of* z_∞ *under* λ_f*. Assume that* z_f *is not* $\mathcal{O}$*-torsion and Assumption 5.10. Then Conjecture 6.14 is true for* T*. In particular, if* T *is not exceptional, then* $\tilde{H}^1_f(K_\Sigma/K, T)$ *is of rank* 1 *and:*

$$\ell_{\mathcal{O}} \tilde{H}^2_f(K, T)_{\mathrm{tors}} \leqslant 2\ell_{\mathcal{O}}(\tilde{H}^1_f(K, T)/\mathcal{O} z_f). \tag{1.2.4}$$

(ii) *Let* $z_{f,\infty} \in \tilde{H}^1_f(K_\Sigma/K, T \otimes_{\mathcal{O}} \Lambda_a)$ *be the image of* z_∞ *under* $\lambda_{f,\infty}$*. Assume* $z_{f,\infty}$ *is not* Λ_a*-torsion. Then Conjecture 6.14 is true for* $T \otimes_{\mathcal{O}} \Lambda_a$ *up to* p*. In particular, if* T *is not exceptional, then* $\tilde{H}^1_f(K_\Sigma/K, T \otimes_{\mathcal{O}} \Lambda_a)$ *is of rank* 1 *and there exists an* $\alpha \in \mathbb{N}$ *such that:*

$$\operatorname{char}_{\Lambda_a} \tilde{H}^2_f(K_\Sigma/K, T \otimes_{\mathcal{O}} \Lambda_a)_{\mathrm{tors}} | (\operatorname{char}_{\Lambda_a} \tilde{H}^1_f(K_\Sigma/K, T \otimes_{\mathcal{O}} \Lambda_a)/\Lambda_a p^\alpha z_{f,\infty})^2. \tag{1.2.5}$$

Under Assumption 5.10, α *can be chosen to be zero.*

(iii) *Assume that* z_∞ *is not* R_{Iw}*-torsion and Assumption 5.10. Then* $\tilde{H}^1_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}})$ *is of rank* 1 *and the support of the characteristic structure is contained in the support of the Euler structure. If in addition* R_{Iw} *is a regular ring*

$$\operatorname{char}_{R_{\mathrm{Iw}}} \tilde{H}^2_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}})_{\mathrm{tors}} | \operatorname{char}_{R_{\mathrm{Iw}}} (\tilde{H}^1_f(K_\Sigma/K, \mathcal{T}_{\mathrm{Iw}})/R_{\mathrm{Iw}} z_\infty)^2. \tag{1.2.6}$$

Remark. Statements (1.2.4) and (1.2.5) are proved or implicit in [How04b] under slightly different hypotheses.

We make a few comments on the frequency of the assumptions we require for Theorem B. When $\pi(f)$ is a non CM automorphic representation, Assumptions 3.4 and 5.13 are satisfied for all sufficiently large p by [Dim05]. The ring R is regular for sufficiently large p by the finiteness of $S_k(U)$ over $\mathbb{Z}$ and no example of a non-regular R is known under our working hypotheses to the best of our knowledge. In several special cases, it is known that z_∞ is not torsion. For instance,

[1] All errors remaining of course entirely ours.

if $\pi(f)$ has trivial central character, this follows from a comparison of the construction of z_∞ with [AN10, Theorem 4.3.1], [CV04, Theorem 4.10] and [CV05, Corollary 2.10]. In another direction, the class z_∞ can be shown to be non-torsion under some hypotheses on the non-vanishing of central L-functions using [YZZ08, Theorem 1.3.1]. In the absence of Assumption 5.10, the divisibility (1.2.6) is proved in the text up to an explicit modification of the Euler system studied which accounts for a possibly non-trivial algebraic μ-invariant.

The organization of this article is as follows. Sections 2 and 3 review the theory of nearly ordinary Galois representations arising from the étale cohomology of towers of Shimura curves and Hida theory for quaternionic automorphic forms. Though the results proved there are presumably well-known, see among others [Fuj99, Hid88, MW86, NP00, Oht95, SW99], their inclusion is necessary to ensure that all objects constructed are integral and of geometric nature. In § 4, we construct equivariant families of CM points indexed by conductors and levels in towers of Shimura curves. The images under the Abel–Jacobi map of these points are equivariant classes under the action of the Galois group of well-chosen abelian extensions of K and the action of the nearly ordinary Hecke algebra. In § 5, we show that these classes form an Euler/Kolyvagin system with coefficients in R_{Iw}, thus extending part of the results of [How07] from the classical case to F and B. The divisibilities of characteristic ideals which form the crux of Theorems A and B then follow from the method of Euler systems and techniques of descent reminiscent of [How04b, Och05] but applied at the level of complexes. The systematic use of Selmer complexes in the descent process, which is the main novelty of this text,[2] allows for the natural incorporation of forms with exceptional zeroes as well as the treatment of non-regular coefficient rings, and thus of Hecke algebras, see particularly Corollary 6.19 and Theorem 6.3. This is in contrast to most of the literature on the main conjecture for modular forms (for instance [EPW06, How04a, How04b, How07, Och05, Och06]) which relies in an essential way on the regularity of the coefficient ring and/or treats only non-exceptional forms. It also greatly simplifies and sharpens some of the arguments of [EPW06, Corollary 5.1.4], [How04b, § 3.3] and [Och05, § 3]. In particular, we show that our conjectures are stable by a fairly large class of base-change of rings of coefficients (see for instance Propositions 6.16 and 6.17) and prove that the main conjecture is true for all specializations in a Hida family if and only if it is true for a single one (see Corollary 6.20).

General notation. If A is a set, let $|A|$ be its cardinal. We fix an algebraic closure $\bar{F}$ of F. For each place v, we fix an algebraic closure $\bar{F}_v$ of F_v and an embedding of $\bar{F}$ into $\bar{F}_v$ extending $F \hookrightarrow F_v$. For p a rational prime, we fix an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and an embedding of $\bar{F} = \bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$. We fix an identification of $\mathbb{C}$ with $\bar{\mathbb{Q}}_p$ extending $\bar{F} \hookrightarrow \bar{\mathbb{Q}}_p$. Let $I_{F,\infty} = \{\tau_j\}_{1\leqslant j\leqslant d}$ be the set of real embeddings of F. If v is a finite place of F, let $\mathcal{O}_{F,v}$ be the ring of integers of F_v and ϖ_v a fixed uniformizing parameter of $\mathcal{O}_{F,v}$; if L is a finite extension of F, let L_v be $L \otimes_F F_v$ and $\mathcal{O}_{L,v}$ be $\mathcal{O}_L \otimes_{\mathcal{O}_F} \mathcal{O}_{F,v}$. Let $I_{F,v}$ be the set of field embeddings of F_v into $\bar{F}_v$. Let $I_{F,p}$ be the union of the $I_{F,v}$ for $v|p$. Our fixed identification of $\bar{\mathbb{Q}}_p$ with $\mathbb{C}$ identifies $I_{F,p}$ with $I_{F,\infty}$. For E a Galois extension of F or F_v, let G_E be the absolute Galois group of E. If w is a place of E above v, let E_w^{nr} be the maximal unramified extension of E_w and $I_w = \mathrm{Gal}(\bar{E}_v/E_v^{nr})$ be the inertia group of w. The geometric Frobenius morphism is written $\mathrm{Fr}(w)$ and the Artin reciprocity map is normalized by making ϖ_w correspond to $\mathrm{Fr}(w)$. If R is a complete local Noetherian ring, T is an R-module of finite type and i is an integer, let $H^i(E,T)$, $H^i(E/F,T)$, $H^i(E_w,T)$ and $H^1_{ur}(E_w,T)$ be respectively the continuous Galois cohomology

[2] The idea is implicit in [Kat99, § 9.6] and [Kat04, §§ 13.8 and 14.14].

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groups $H^i(G_E, T)$, $H^i(\mathrm{Gal}(E/F), T)$, $H^i(G_{E_w}, T)$ and $H^1(\mathrm{Fr}(w), T^{I_w})$. A G-representation (T, ρ, S) is an S-module T free of finite rank endowed with a continuous action of G via the group morphism ρ.

2. Towers of Shimura curves

2.1 Generalities on Shimura curves

2.1.1 *Notation.* In this subsection, we fix some notation relative to quaternionic Shimura curves, following [CV05, Del71, Mil05, Nek07]. Let S_B be a finite set of finite places such that:

$$|S_B| \equiv d-1 \mod 2.$$

Let $\mathrm{Ram}(B) = S_B \cup \{\tau_j\}_{j\neq 1}$ and B be the unique quaternion algebra over F whose set of ramification places is $\mathrm{Ram}(B)$. We refer to the case $B = \mathcal{M}_2(\mathbb{Q})$ as the classical case.

As in the introduction, let $p \geqslant 5$ be a prime such that $v \notin S_B$ if $v|p$, let $L_{\mathfrak{p}}$ be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}$ its ring of integers. We fix once and for all isomorphisms between B_v and $\mathcal{M}_2(F_v)$ for $v \notin S_B$. Let $\mathbf{G}$ be the reductive group whose set of points on a $\mathbb{Q}$-algebra A is $(B \otimes_{\mathbb{Q}} A)^\times$. Let $\mathbf{Z}$ be the center of $\mathbf{G}$ and $nr : \mathbf{G}(A) \longrightarrow (F \otimes_{\mathbb{Q}} A)^\times$ be the reduced norm. Let $\widehat{\mathbb{Z}}$ be the profinite completion of $\mathbb{Z}$ and for any abelian group A, let $\widehat{A}$ be $A \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. In particular, the group $\widehat{F}^\times$ is the group of finite idèles of F and $\widehat{B}^\times$ is the set of finite adelic points of $\mathbf{G}$. If g belongs to $\widehat{B}^\times$, the notation $g = g_v$ means that the local component of g at v is equal to g_v and that g is equal to the identity outside v. If A is a topological group, let A_+ be the connected component of the identity.

Let U_∞ be the centralizer of the image of

$$\begin{aligned} h_0 : \quad \mathbb{C}^\times &\longrightarrow \mathbf{G}(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{R}) \times \mathbb{H}^\times \times \cdots \times \mathbb{H}^\times \\ x + iy &\longmapsto \left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}, 1, \ldots, 1 \right) \end{aligned}$$

in $\mathbf{G}(\mathbb{R})$. Let U be a compact open subgroup of $\widehat{B}^\times$. In this article, we restrict our attention to subgroups U such that

$$U = \prod_v U_v$$

with U_v maximal for every v in S_B. To U is attached a quaternionic complex Shimura curve $X(U)^{an}$ over $\mathbb{C}$ given by the double quotient

$$X(U)^{an} = B^\times \backslash \mathbf{G}(\mathbb{A}_{\mathbb{Q}})/UU_\infty = B^\times \backslash (\mathbb{C} - \mathbb{R} \times \widehat{B}^\times / U)$$

or by its smooth compactification obtained by adjoining cusps in the classical case. The curve $X(U)^{an}$ is thus compact. We use the notation $[a, b]_U$, or $[a, b]$ when U is clear from the context, to denote an element of $X(U)^{an}$.

Definition 2.1 (Small subgroup). An open compact subgroup U is small if for any compact open normal subgroup $U' \subset U$, the right action of $(U \cap \mathbf{Z}(\mathbb{Q}) \cdot U')\backslash U$ on $X(U')^{an}$ is free.

According to [Fuj99, Proposition 4.9] (see also [Car86a, Corollaire 1.4.1.3] and [Mil05, Lemma 5.13]), a compact open group U admits a small normal subgroup of finite index. If U' is a normal compact open subgroup of a compact open U, the natural projection

$$X(U')^{an} \longrightarrow X(U)^{an}$$

is finite and flat. If moreover U is small, this covering is Galois.

2.1.2 *CM points and canonical model.*

2.1.2.1 *CM points.* Let L be a totally complex quadratic extension of F. Assume that places v in S_B either ramify or are inert in L. According to [Vig80, Théorème 3.8], there then exists an injective morphism of F-algebras $q : L \hookrightarrow B$ which we fix. It induces embeddings $q_v : L_v \hookrightarrow B_v$ for all places v and $\widehat{q} : \widehat{L}^\times \hookrightarrow \widehat{B}^\times$. The inclusions $L^\times \hookrightarrow B^\times \hookrightarrow B^\times \otimes_{F,\tau_1} \mathbb{R} \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{R})$ define an action of $q(L^\times)$ on $\mathbb{C} - \mathbb{R}$. Let z be the unique point of $\mathbb{C} - \mathbb{R}$ fixed by $q(L^\times)$ and with positive imaginary part. The set of CM points of $X(U)^{an}$ relative to L is the set:

$$CM(X(U), L) = \{x = [z, b] \in B^\times \backslash (\mathbb{C} - \mathbb{R} \times \widehat{B}^\times / U) \mid b \in \widehat{B}^\times\} \subset X(U)^{an}.$$

As every embedding of L in B is conjugate to q by an element of $B^\times$ by the Skolem–Noether theorem, this set does not depend on q. The set $CM(X(U), L)$ is dense for the complex topology, and hence for the Zariski topology. The same is of course also true for the union $CM(X(U))$ of CM points relative to L for all suitable L.

2.1.2.2 *Canonical model.* By [Shi70], the projective system of curves $\{X(U)^{an}\}_U$ indexed by compact open subgroups admits a canonical model $\{X(U)\}_U$ defined over $\operatorname{Spec} F$ and characterized by Shimura's reciprocity law at $CM(X(U), L)$:

$$\forall\, a \in \widehat{L}^\times, \forall\, [z, b] \in CM(X(U), L), \quad (\mathrm{rec}_L\, a) \cdot [z, b] = [z, \widehat{q}(a) b]. \tag{2.1.1}$$

Then $X(U)^{an} = (X(U) \times_{F,\tau_1} \mathbb{C})(\mathbb{C})$. The curve $X(U)$ is smooth over $\operatorname{Spec} F$.

Equation (2.1.1) implies that CM points relative to L are defined over L^{ab}. By definition of the canonical model and [Del71, Théorème 5.1], to show that a morphism of Shimura curves is defined over a finite extension of F, it is enough to show this on $CM(X(U))$. In particular, the right action of $\widehat{B}^\times$ on $\{X(U)\}_U$ is defined over F. By [KM85, Theorem, p. 508], the quotient $X(U)/\widehat{F}^\times$ is again a smooth curve over $\operatorname{Spec} F$.

2.1.2.3 *Connected components.* The strong approximation theorem and the norm theorem for quaternion algebras provide us with the following descriptions of the set of connected components of $X(U)^{an}$:

$$\pi_0(X(U)^{an}) \xrightarrow{\sim} \mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\widehat{\mathbb{Q}})/U \xrightarrow{\sim} \mathbf{Z}(\mathbb{Q})_+ \backslash \mathbf{Z}(\widehat{\mathbb{Q}})/nr(U). \tag{2.1.2}$$

The second isomorphism in (2.1.2) is the reduced norm. Shimura's reciprocity law for connected components (see for instance [Del71, § 3.4]) states that the action of G_F on $\pi_0(X(U) \times_F \bar{F})$ is through its abelian quotient G_F^{ab} and that it coincides with the action of $\widehat{B}^\times$ through the isomorphisms:

$$G_F^{ab} \underset{\mathrm{rec}_F}{\xleftarrow{\sim}} \mathbf{Z}(\mathbb{Q})_+ \backslash \mathbf{Z}(\widehat{\mathbb{Q}}) \xrightarrow{\sim} \mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\widehat{\mathbb{Q}}).$$

In particular, the action of G_F^{ab} on $\pi_0(X(U)^{an})$ is transitive, and $X(U)$ is a connected F-scheme. The structural morphism $X(U) \longrightarrow \operatorname{Spec} F$ admits a Stein factorization

$$X(U) \longrightarrow \mathcal{X}(U) \longrightarrow \operatorname{Spec} F$$

where the scheme $\mathcal{X}(U)$ is finite étale over $\operatorname{Spec} F$ and the morphism $X(U) \longrightarrow \mathcal{X}(U)$ is proper and smooth with geometrically connected fibers. Let $F[U]$ be the algebraic closure of F inside the function field of $X(U)$. The above implies that $\mathcal{X}(U)$ is equal to $\operatorname{Spec} F[U]$ where $F[U]$ is a finite abelian extension of F which can be considered as a subfield of $\mathbb{C}$ via $\tau_1 : F \hookrightarrow \mathbb{C}$.

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2.1.2.4 *Involution.* The group $\widehat{B}^\times$ admits a natural involutive morphism whose local expression is given by:

$$\begin{aligned} -alt_v : B_v^\times &\longrightarrow B_v^\times \\ b_v &\longmapsto \frac{1}{nr(b_v)} b_v. \end{aligned}$$

The morphism $-alt$ induces a morphism of Shimura curves defined as follows on complex points:

$$\begin{aligned} -alt : X(U)^{an} &\longrightarrow X(U^{-alt})^{an} \\ [x,b]_U &\longmapsto [x, b^{-alt}]_{U^{-alt}}. \end{aligned}$$

Lemma 2.2. *The morphism $-alt$ is defined over the subfield $F[U \cap U^{-alt}]$ of $\mathbb{C}$.*

Proof. According to the discussion of § 2.1.2.3, the field $F[U \cap U^{-alt}]$ is a well-defined subfield of $\mathbb{C}$. Let L/F be a CM extension embedding in B and let $x = [z,b] \in CM(X(U),L)$ be a CM point. Let $\sigma = \mathrm{rec}_L\, a$ be an element of G_L^{ab} fixing $F[U \cap U^{-alt}]$ and b_σ be $\widehat{q}(a)$. Because b_σ fixes $F[U \cap U^{-alt}]$, it is trivial in the abelian group $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\widehat{\mathbb{Q}})/(U \cap U^{-alt})$ and so can be written gu with $g \in \mathbf{G}(\mathbb{Q})$ and $u \in U \cap U^{-alt}$. Hence, $nr(u)^{-1} = u^{-1}u^{-alt} \in U^{-alt}$. Thus:

$$\begin{aligned} (-alt \circ \sigma) \cdot [z,b]_U &= \left[g^{-1}z, ub^{-alt}\frac{1}{nr(u)} \right]_{U^{-alt}} \\ &= [z, gub^{-alt}]_{U^{-alt}} = (\sigma \circ -alt) \cdot [z,b]_U. \end{aligned}$$

Thus, $-alt \circ \sigma$ is equal to $\sigma \circ -alt$ on $x \in CM(X(U),L)$. Hence, $-alt$ is a morphism of $F[U \cap U^{-alt}]L$-schemes. As this is true for all suitable L, it is a morphism of $F[U \cap U^{-alt}]$-scheme. □

2.1.3 *Hecke operators.* Let g be an element of $\widehat{B}^\times$. Right multiplication by g induces a finite flat F-morphism

$$[\cdot g] : \; X(U \cap gUg^{-1}) \longrightarrow X(U \cap g^{-1}Ug)$$

which defines the Hecke correspondence $T(g) = [UgU]$ on $X(U)$.

$$\begin{array}{ccc} X(U \cap gUg^{-1}) & \xrightarrow{[\cdot g]} & X(U \cap g^{-1}Ug) \\ \downarrow & & \downarrow \\ X(U) & \overset{[UgU]}{\dashrightarrow} & X(U) \end{array} \tag{2.1.3}$$

A finite idèle $a \in \widehat{F}^\times$ in particular induces a correspondence $\langle a \rangle$ called the diamond correspondence via the isomorphism:

$$\widehat{F}^\times \xrightarrow{\sim} \mathbf{Z}(\widehat{\mathbb{Q}}) \hookrightarrow \widehat{B}^\times.$$

The group of diamond correspondences acts on $X(U)$ via the finite group $G(U) = \widehat{F}^\times/F^\times(\widehat{F}^\times \cap U)$ which fits in the short exact sequence:

$$1 \longrightarrow \frac{F^\times \widehat{\mathcal{O}}_F^\times}{F^\times(\widehat{F}^\times \cap U)} \longrightarrow G(U) \longrightarrow \mathrm{Cl}(\mathcal{O}_F) \longrightarrow 1. \tag{2.1.4}$$

For $v \notin S_B$ a finite place of F and

$$g_v = \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} \tag{2.1.5}$$

we write:

$$T(v) = [U g_v U]. \tag{2.1.6}$$

The sub-algebra of the $\mathcal{O}$-algebra (see § 2.1.1) of all correspondences of $X(U)$ generated by the correspondences $\langle a \rangle$ and $T(v)$ is called the full classical Hecke algebra and written $\mathfrak{h}(U)$. As is apparent in (2.1.5), it depends in general on our choices of ϖ_v.

2.1.3.1 *Étale cohomology.* The first étale cohomology group of $X(U) \times \bar{F}$ with coefficients in $\mathcal{O}$

$$M(U) = H^1_{\text{et}}(X(U) \times_F \bar{F}, \mathcal{O})$$

is a finite $\mathcal{O}$-module which is free if U is small. It is endowed with a covariant and contravariant action of the full classical Hecke algebra $\mathfrak{h}(U)$. Poincaré duality induces a perfect alternating pairing:

$$\langle \cdot, \cdot \rangle : \; M(U) \times M(U) \longrightarrow H^2_{\text{et}}(X(U), \mathcal{O}) \longrightarrow \mathcal{O}(-1)$$

such that:

$$\langle \langle a \rangle_* x, y \rangle = \langle x, \langle a \rangle^* y \rangle, \quad \langle T(g)_* x, y \rangle = \langle x, T(g)^* y \rangle. \tag{2.1.7}$$

Choosing the covariant action makes $M(U)$ a $\mathfrak{h}(U)[G_F]$-module. Comparing the covariant and contravariant forms of (2.1.3) shows that $T(g)^*$ is equal to $T(g^{-1})_*$.

2.2 Change of levels

2.2.1 *Towers of levels.*

2.2.1.1 A level S is an almost everywhere zero family of non-negative integers (s_v) indexed by the finite places of F. The support $\operatorname{supp}(S)$ of a level S is the finite set of places such that s_v is not zero. We remark that an ideal $\mathcal{I}$ of $\mathcal{O}_F$ defines a level. Conversely, if S is a level, let S_p and S^p be the ideals:

$$S_p = \prod_{v|p} v^{s_v}, \quad S^p = \prod_{v\nmid p} v^{s_v}.$$

The set of all levels is partially ordered by the relation $S \leqslant S'$ if and only if $s_v \leqslant s'_v$ for all v. If $S \leqslant S'$, the level $S' - S$ is the family $(s'_v - s_v)$. A tower of levels $\{S\}$ is defined to be an infinite totally ordered family of distinct levels with fixed support. A tower of levels is said to go to infinity at an ideal $\mathcal{I}$ if and only if s_v goes to infinity for all v dividing $\mathcal{I}$. It is said to be ultimately constant outside a finite set of places if s_v stabilizes for v outside this finite set. All towers appearing henceforth are assumed to be of non-trivial level going to infinity at places dividing p, constant outside p and such that $\operatorname{supp}(S)$ is disjoint from S_B.

2.2.1.2 To a level S are attached four types of compact open subgroups called U_0, U_1, U^1 and U_1^1. For $v \notin S$, these groups are all maximal at v. For $v \in \operatorname{supp}(S)$, they are defined as follows.

(i) The group $U_{0,v}(S)$ is equal to:

$$\left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F,v}) \mid c \equiv 0 \bmod \varpi_v^{s_v} \right\}.$$

(ii) The group $U_v^1(S)$ is equal to:

$$\left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F,v}) \mid a-1, c \equiv 0 \bmod \varpi_v^{s_v} \right\}.$$

(iii) The group $U_{1,v}(S)$ is equal to:

$$\left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F,v}) \mid d-1, c \equiv 0 \bmod \varpi_v^{s_v} \right\}.$$

(iv) The group $U_{1,v}^1(S)$ is equal to:

$$\left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F,v}) \mid a-1, d-1, c \equiv 0 \bmod \varpi_v^{s_v} \right\}.$$

If $\{S\}$ is a tower of levels, a tower of groups $\{U(S)\}$ of type U_0, U_1, U^1, U_1^1 is a coherent choice of compact open groups of the given type for each level, that is to say a set of choices such that $U(S') \subset U(S)$ whenever $S \leqslant S'$. Moreover, we always assume that if S_0 is the smallest level, $U(S_0)$ is a small compact open group. Let $X_0(S)$, $X_1(S)$, $X^1(S)$ and $X_1^1(S)$ be the Shimura curves $X(U_0(S))$, $X(U_1(S))$, $X(U^1(S))$ and $X(U_1^1(S))$ respectively. Let $X^{tw}(S)$ be the curve $X_1^1(S)/\widehat{F}^\times$. For a given level S, let $\mathcal{O}_{F,1}^\times(S)$ be the subgroup of global units satisfying $x_v \equiv 1 \bmod \varpi_v^{s_v}$ for all v and $\mathcal{O}_{F,+,1}^\times(S)$ the subgroups of totally positive units of $\mathcal{O}_{F,1}^\times(S)$.

Example. When $\{S\} = \{Np^s\}_{s \geqslant 1}$ in the classical case, the tower of curves $\{X_1(S)\}$ is equal to the tower of modular curves $\{X_1(Np^s)\}$ of e.g. [MW86, § 8].

2.2.2 *Fields of constants and pairings.*

2.2.2.1 Let S be a level. Throughout § 2.2.2, let $X(S)$ be the Shimura curve $X_1(S)$ or $X^1(S)$ when there is no reason to distinguish between them. The geometrically connected components of $X_0(S)$ and $X(S)$ are defined over the narrow class field $F[0]$ of F because the reduced norm is onto $\mathcal{O}_{F,v}^\times$ for each v. However, the geometrically connected components of $X_1^1(S)$ are defined over an abelian extension $F[S]$ whose Galois group over $F[0]$ fits in the short exact sequence:

$$1 \longrightarrow \frac{\mathcal{O}_{F,+}^\times}{\mathcal{O}_{F,+,1}(S)} \longrightarrow \prod_{v \in S} (\mathcal{O}_{F,v}/\varpi_v^{s_v})^\times \longrightarrow \mathrm{Gal}(F[S]/F[0]) \longrightarrow 1. \tag{2.2.1}$$

As the tower of levels $\{S\}$ goes to infinity at p and is constant outside p, the inverse limit

$$G(\infty) = \mathrm{Gal}(F[\infty]/F) = \varprojlim_S \mathrm{Gal}(F[S]/F)$$

has a finite torsion part and $G(\infty)/G(\infty)_{\mathrm{tors}}$ is isomorphic to $\mathbb{Z}_p^{1+\delta_{F,p}}$. Beside, the curve $X_1^1(S)$ is endowed with an action of $\mathcal{O}_{F,v}^\times \times \mathcal{O}_{F,v}^\times$ for all $v|p$ via the diagonal subgroup of $B_v^\times$. The group $T_1 = \prod_{v \in S} (\mathcal{O}_{F,v}/\varpi_v^{s_v})^\times$ acts on $X_1^1(S)$ as:

$$\langle a \rangle_1 [z, b] = \left[z, b \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right],$$

and the group $T^1 = \prod_{v \in S} (\mathcal{O}_{F,v}/\varpi_v^{s_v})^\times$ acts on $X_1^1(S)$ as:

$$\langle a \rangle^1 [z, b] = \left[z, b \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \right].$$

The curve $X_1^1(S)/T_1$ (respectively $X_1^1(S)/T^1$) is equal to $X_1(S)$ (respectively to $X^1(S)$). For T equal to T_1 or T^1, let T_+ be the image of the quotient $\mathcal{O}_{F,+}^\times/\mathcal{O}_{F,+,1}^\times(S)$ in T. There is a diagram

$$\begin{array}{ccc} X_1^1(S) & \longrightarrow & X(S) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, F[S] & \longrightarrow & \mathrm{Spec}\, F[0] \end{array}$$

and hence a surjective morphism of $F[S]$-schemes $X_1^1(S) \longrightarrow X(S) \times_{F[0]} F[S]$. The identification of $\mathrm{Gal}(F[S]/F[0])$ with T/T_+ given by the reciprocity map shows that the coverings $X_1^1(S) \longrightarrow X(S)$ and $X(S) \times_{F[0]} F[S] \longrightarrow X(S)$ have the same Galois groups. Hence $X(S) \times_{F[0]} F[S]$ is isomorphic to $X_1^1(S)/T_+$.

2.2.2.2 According to Lemma 2.2, the morphism $-alt$ on $X_1^1(S)$ is an isomorphism of $F[S]$-schemes between the schemes $X_1^1(S)/T_{1,+}$ and $X_1^1(S)/T_+^1$, or equivalently between $X_1(S) \times_{F[0]} F[S]$ and $X^1(S) \times_{F[0]} F[S]$. Let $w_S \in \widehat{B}^\times$ be the element of $\widehat{B}^\times$ equal to the identity at $v \in S_B$ and whose local expression at $v \notin S_B$ is given by

$$w_{S,v} = \begin{pmatrix} 0 & -1 \\ \varpi^{s_v} & 0 \end{pmatrix} \tag{2.2.2}$$

using the fixed identification between $B_v^\times$ and $\mathrm{GL}_2(F_v)$.

LEMMA 2.3. *Right multiplication by w_S induces a morphism of F-schemes:*

$$[\cdot w_S]: \ X_1^1(S) \longrightarrow X_1^1(S).$$

Proof. As $U_1^1(S) = w_S^{-1} U_1^1(S) w_S$, this comes from the fact recalled in § 2.1.2.2 that right multiplication is defined over $\mathrm{Spec}\, F$. □

LEMMA 2.4. *The composition*

$$X_1^1(S) \xrightarrow{[\cdot w_S]} X_1^1(S) \xrightarrow{-alt} X_1^1(S)$$

defines an $F[S]$-morphism denoted by W_S. Consider an element a of $\widehat{F}^\times$, an element g of $\widehat{B}^\times$ such that the correspondence $T(g)$ is in the full classical Hecke algebra $\mathfrak{h}(U_1^1(S))$ and an element $\sigma \in G_F$ whose projection to G_F^{ab} is equal to $\mathrm{rec}_F\, a$. Then:

$$W_S \circ \langle a \rangle = \langle a^{-1} \rangle \circ W_S \tag{2.2.3}$$
$$W_S \circ T(g) = T(g^{-1}) \circ W_S \tag{2.2.4}$$
$$\sigma \circ W_S = W_S \circ \langle a^{-1} \rangle \circ \sigma. \tag{2.2.5}$$

The same properties are true for the morphisms defined by the compositions

$$X_1(S) \times_{F[0]} F[S] \xrightarrow{[\cdot w_S]} X^1(S) \times_{F[0]} F[S] \xrightarrow{-alt} X_1(S) \times_{F[0]} F[S]$$

and

$$X^1(S) \times_{F[0]} F[S] \xrightarrow{[\cdot w_S]} X_1(S) \times_{F[0]} F[S] \xrightarrow{-alt} X^1(S) \times_{F[0]} F[S]$$

except that $T(g)$ is then viewed as an element of $\mathfrak{h}(U_1(S))$ or $\mathfrak{h}(U^1(S))$.

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Proof. By Lemmas 2.2 and 2.3, the morphism W_S is a morphism of $F[S]$-schemes. Let L/F be a CM extension embedding in B and let $x=[z,b]\in CM(X_1^1(S),L)$ be a CM point. We first prove (2.2.4). For $v\notin S_B$, let g_v be the element of (2.1.6). Then $w_{S,v}^{-1}g_v w_{S,v}$ is equal to $(g^{-1})^{-alt}$ so:

$$(w_S^{-1}U_1^1(S)gU_1^1(S)w_S)^{-alt}=U_1^1(S)g^{-1}U_1^1(S).$$

This establishes (2.2.4). Applying this to the diagonal embedding of $a\in\widehat{F}^\times$ proves (2.2.3).

Finally, let $\alpha\in\widehat{L}^\times$ be such that $\mathrm{rec}_L\,\alpha$ is equal to the image of σ in G_L^{ab}. Then $N_{K/F}\alpha=a$ and:

$$\begin{aligned}\sigma\circ W_S\cdot x=[z,\widehat{q}(\alpha)b^{-alt}w_S^{-alt}]&=W_S\cdot\left[z,\widehat{q}(\alpha)\frac{1}{N_{L/F}\alpha}b\right]\\&=W_S\circ\langle a^{-1}\rangle\cdot x\end{aligned}$$

The same proof works verbatim for $X(S)\times_{F[0]}F[S]$. □

Remark. In the classical case, the morphism W_S coincides with the Fricke involution and the statements of Lemma 2.4 are well-known, see among others [MW84, p. 234].

2.2.3 *Action of $F^\times U_0/F^\times U$.*

2.2.3.1 Let $U(S)$ be one of the group $U_1(S)$, $U^1(S)$ or $U_1^1(S)$. The group $F^\times U_0(S)$ acts on the right on the curves $X(S)$, $X_1^1(S)$ and $X^{tw}(S)$. As S goes to infinity at p, this action gives rise to a group-algebra Λ.

Let $G_{U(S)}$ be the group $F^\times U_0(S)/F^\times U(S)$. The short sequence

$$1\longrightarrow\mathcal{O}_F^\times/\mathcal{O}_{F,1}^\times(S)\longrightarrow U_0(S)/U(S)\longrightarrow G_{U(S)}\longrightarrow 1 \tag{2.2.6}$$

is exact so $G_{U(S)}$ is isomorphic to $\mathcal{O}_F^\times U_0(S)/\mathcal{O}_F^\times U(S)$. Let $\bar{\mathcal{O}}_F^\times$ and $\bar{\mathcal{O}}_{F,+}^\times$ be respectively the p-adic closure of the image of $\mathcal{O}_F^\times$ and $\mathcal{O}_{F,+}^\times$ in $\mathcal{O}_{F,p}^\times$ through the diagonal embedding.

2.2.3.2 Assume first that $U(S)$ is equal to $U^1(S)$ or $U_1(S)$. By (2.2.6), we have:

$$G_{U(S)}\overset{\sim}{\longrightarrow}(\mathcal{O}_F^\times/\mathcal{O}_{F,1}^\times(S))\backslash\prod_{v\in\mathrm{supp}(S)}(\mathcal{O}_{F,v}/\varpi^{s_v})^\times. \tag{2.2.7}$$

Let G_∞ be the profinite inverse limit on S of the groups $G_{U(S)}$. There exists a finite group $G(S^p)$ such that:

$$G_\infty/G(S^p)\overset{\sim}{\longrightarrow}\bar{\mathcal{O}}_F^\times\backslash\left(\prod_{v|p}\mathcal{O}_{F,v}^\times\right). \tag{2.2.8}$$

Hence, the group G_∞ is of $\mathbb{Z}_p$-rank $1+\delta_{F,p}$. Let F_∞ be the maximal $\mathbb{Z}_p$-extension of F. Then $\mathrm{Gal}(F_\infty/F)$ is isomorphic to $\mathbb{Z}_p^{1+\delta_{F,p}}$ and the subgroup I of $\mathrm{Gal}(F_\infty/F)$ generated by the inertia groups I_v at $v|p$ is of finite index, and hence also isomorphic to $\mathbb{Z}_p^{1+\delta_{F,p}}$. We fix once and for all $1+\delta_{F,p}$ elements $(\gamma_i)_{1\leqslant i\leqslant\delta_{F,p}}$ in G_F whose images $(\bar{\gamma}_i)_i$ in $\mathrm{Gal}(F_\infty/F)$ generate I. The set of the inverse images of the $(\bar{\gamma}_i)_i$ through the global reciprocity map and the isomorphism (2.2.8) generate a direct summand Γ of $G_\infty/G(S^p)$ which is free of rank $1+\delta_{F,p}$. The group-algebra $\Lambda=\mathcal{O}[[\Gamma]]$ is non-canonically isomorphic to the power-series ring $\mathcal{O}[[X_1,\dots,X_{1+\delta_{F,p}}]]$ and thus regular of Krull dimension $2+\delta_{F,p}$.

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2.2.3.3 Now assume that $U(S)$ is equal to $U_1^1(S)$. By (2.2.6), we have:

$$G_{U(S)} \xrightarrow{\sim} (\mathcal{O}_F^\times/\mathcal{O}_{F,1}^\times(S))\backslash\Bigg(\prod_{v\in\mathrm{supp}(S)}(\mathcal{O}_{F,v}/\varpi^{s_v})^\times\Bigg)^2.$$

Let G_∞ be the profinite inverse limit on S of the groups $G_{U(S)}$. There exists a finite group $G(S^p)$ such that:

$$G_\infty/G(S^p) \xrightarrow{\sim} \bigg(\bar{\mathcal{O}}_F^\times\backslash\Big(\prod_{v|p}\mathcal{O}_{F,v}^\times\Big)^2\bigg) \xrightarrow{\sim} \bigg(\bar{\mathcal{O}}_F^\times\backslash\prod_{v|p}\mathcal{O}_{F,v}^\times\bigg)\times\prod_{v|p}\mathcal{O}_{F,v}^\times. \tag{2.2.9}$$

The last isomorphism of (2.2.9) being given by $(x,y)\mapsto(x,x^{-1}y)$. Hence, the group G_∞ is of $\mathbb{Z}_p$-rank $1+\delta_{F,p}+d$. As in the previous paragraph, we fix once and for all $1+\delta_{F,p}$ elements $(\gamma_i)_{1\leqslant i\leqslant\delta_{F,p}}$ in G_F whose images $(\bar{\gamma}_i)_i$ in $\mathrm{Gal}(F_\infty/F)$ generate I. We also fix once and for all d elements:

$$(y_{v_i,j})\in\prod_{v|p}\mathcal{O}_{F,v}^\times,\quad v_i|p, 1\leqslant j\leqslant[F_{v_i}:\mathbb{Q}_p].$$

The set of the inverse images of the $(\bar{\gamma}_i)_i$ through the global reciprocity map and the isomorphism (2.2.9) restricted on the first factor, together with the inverse images of the $y_{v_i,j}$ through the isomorphism (2.2.9) restricted on the second factor generate a direct summand Γ of $G_\infty/G(S^p)$ which is free of rank $1+\delta_{F,p}+d$. The group-algebra $\Lambda=\mathcal{O}[[\Gamma]]$ is non-canonically isomorphic to the power-series ring $\mathcal{O}[[X_1,\ldots,X_{1+\delta_{F,p}+d}]]$ and thus regular of Krull dimension $2+\delta_{F,p}+d$.

2.2.3.4 Finally, assume that $U(S)=U_1^1(S)$ and that we are considering the action of $G_{U(S)}$ and G_∞ on X^{tw}. Then the action of $\widehat{\mathcal{O}}_F^\times U_1^1(S)$ becomes trivial. Hence the action of $G_{U(S)}$ factors through its quotient $\mathcal{O}_F^\times U_0(S)/\widehat{\mathcal{O}}_F^\times U_1^1(S)$, which we denote by $G_{U(S)}^{tw}$. Writing $G_{U(S)}^{tw}$ as a quotient of $U_0(S)/U_1^1(S)$ as in (2.2.6) shows the following isomorphism:

$$G_{U(S)}^{tw} \xrightarrow{\sim} \prod_{v\in\mathrm{supp}(S)}(\mathcal{O}_{F,v}/\varpi^{s_v})^\times.$$

Let G_∞^{tw} be the inverse limit of $G_{U(S)}^{tw}$. There exists a finite group $G^{tw}(S^p)$ such that:

$$G_\infty^{tw}/G^{tw}(S^p) \xrightarrow{\sim} \prod_{v|p}\mathcal{O}_{F,v}^\times. \tag{2.2.10}$$

Keeping the notation of § 2.2.3.3, the inverse images of the $y_{v_i,j}$ through the isomorphism (2.2.10) generate a direct summand Γ^{tw} of $G_\infty^{tw}/G^{tw}(S^p)$ which is free of rank d. The group-algebra $\Lambda^{tw}=\mathcal{O}[[\Gamma^{tw}]]$ is non-canonically isomorphic to the power-series ring $\mathcal{O}[[X_1,\ldots,X_d]]$ and thus regular of Krull dimension $1+d$.

2.2.3.5 We now return to the general case and let U be equal to $U_1(S), U^1(S)$ or $U_1^1(S)$. The canonical surjection of G_F onto G_F^{ab} composed with the (inverse of the) global reciprocity map and the diagonal embedding

$$G_F \twoheadrightarrow G_F^{ab} \xrightarrow{\sim} \widehat{F}^\times/F_+^\times \to \widehat{B}^\times/B_+^\times \to \widehat{B}^\times/B^\times \tag{2.2.11}$$

maps G_F to $\mathfrak{h}(U)$. We write $\langle\mathrm{rec}_F^{-1}(\sigma_{|F^{ab}})\rangle$ for the image of $\sigma\in G_F$ through this map. We remark that $\langle\mathrm{rec}_F^{-1}(\sigma_{|F^{ab}})\rangle$ acts on $X(U)$ through its image in G_U. For a $\mathfrak{h}(U)[G_F]$-module M such that the action of $\sigma \in G_F$ on $x \in M$ is written σx, we denote by $M\langle n\rangle$ the $\mathfrak{h}(U)[G_F]$-module equal to M as a $\mathfrak{h}(U)$-module and with $\sigma \in G_F$ acting on $x \in M\langle n\rangle$ by $\sigma \cdot x = \langle \mathrm{rec}_F^{-1}(\sigma^n_{|F^{ab}})\rangle \sigma x$.

Lemma 2.5. *Let U be $U_1(S), U^1(S)$ or $U_1^1(S)$, $X(S)$ be $X(U(S))$ and $M(S)$ be $H^1_{\mathrm{et}}(X(S)\times_F \bar{F}, \mathcal{O})$. Poincaré duality and W_S give rise to a twisted pairing on étale cohomology defined by:*

$$\begin{aligned}(\cdot,\cdot):\ M(S)\times M(S) &\longrightarrow \mathcal{O}(-1)\\ (x,y) &\longmapsto \langle x, W_S y\rangle.\end{aligned} \tag{2.2.12}$$

This pairing induces an isomorphism of $\mathfrak{h}(U)[G_F]$-modules:

$$\alpha:\ M(S) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(M(S),\mathcal{O})(-1)\langle -1\rangle.$$

Proof. Poincaré duality shows that α is an isomorphism of $\mathcal{O}$-modules. The assertions (2.1.7) and the remark following them together with the assertions (2.2.3) and (2.2.4) of Lemma 2.4 show that it is an isomorphism of $\mathfrak{h}(U)$-modules. Let σ be in G_F and a be $\mathrm{rec}_F^{-1}(\sigma_{|F^{ab}})$. Let x be in $M(S)$. The computation

$$\begin{aligned}\chi_{cyc}^{-1}(\sigma)(\langle a^{-1}\rangle\sigma)\alpha(x) &= \chi_{cyc}^{-1}(\sigma)\langle x, W_S\langle a\rangle\sigma^{-1}\,\cdot\rangle\\ &= \chi_{cyc}^{-1}(\sigma)\langle x, \sigma^{-1}W_S\,\cdot\rangle\\ &= \alpha(\sigma x)\end{aligned}$$

in which the second equality, which comes from the assertion (2.2.5), shows that the image of α is the $\mathfrak{h}(U)[G_F]$-module $\mathrm{Hom}_{\mathcal{O}}(M(S),\mathcal{O})(-1)\langle -1\rangle$. □

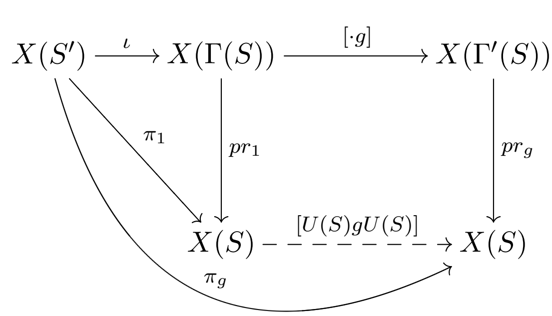
2.2.4 *Change of levels.*

2.2.4.1 Throughout § 2.2.4, and not in accord with the conventions of § 2.2.2, the compact open subgroups $U_1(S), U^1(S)$ or $U_1^1(S)$ are denoted by $U(S)$, the curve $X(U(S))$ is denoted by $X(S)$ and $M(S)$ stands for $H^1_{\mathrm{et}}(X(S)\times_F \bar{F},\mathcal{O})$. We establish that the cohomology groups $M(S)$ together with the pairing $(\cdot,\cdot)$ of Lemma 2.5 satisfy compatibilities in towers of levels $\{S\}$ after projection to the ordinary part. Let $\{X(S)\}_S$ be a tower of Shimura curves endowed with compatible choices of Hecke operators and of elements w_S. Let $S \leqslant S'$ be two distinct levels, necessarily equal outside p.

Define $t \in \widehat{B}^\times$ and $g \in \widehat{B}^\times$ by:

$$\forall\ v|p, t_v = \begin{pmatrix}1 & 0\\ 0 & \varpi_v\end{pmatrix}, \quad g_v = t_v^{s'_v - s_v}. \tag{2.2.13}$$

Let $\Gamma(S)$ and $\Gamma'(S)$ be the groups $U(S)\cap gU(S)g^{-1}$ and $g^{-1}U(S)g \cap U(S)$. Then $U(S')$ is included in $\Gamma(S)$. The following diagram and its covariant and contravariant incarnations define the fundamental maps between the various Shimura curves of interest and their cohomological realizations.



(2.2.14)

Let $eM(S)$ be the greatest direct summand of $M(S)$ on which $T(g^{-1})_*$ is invertible. Let $eM(\Gamma'(S))$ be the greatest direct summand of $X(\Gamma'(S))$ on which $[\Gamma'(S)g^{-1}\Gamma'(S)]_*$ is invertible. Let ξ denote the cohomological map $pr_{1*}[g]^*$. Comparing diagram (2.1.3) with (2.2.14) shows that $T(g^{-1})_* = T(g)^*$ is equal to $pr_{1*}[\cdot g]^* pr_g^*$. The map

$$pr_g^* : \ eM(S) \longrightarrow eM(\Gamma'(S))$$

therefore has inverse $T(g^{-1})_*^{-1} \circ \xi$. Consequently

$$\pi_{g*} = pr_{g*}pr_g^*(pr_g^*)^{-1}[g]_*\iota_* = C(pr_g^*)^{-1}[g]_*\iota_*$$

where C is the cardinal of the finite quotient $U(S)/\Gamma'(S)$, thus the product of the cardinals of residue fields at $v|p$. The group $eM(S')$ has no $\mathbb{Z}_p$-torsion so the map

$$C^{-1}\pi_{g*} = (pr_g^*)^{-1}[g]_*\iota_*$$

is well-defined and satisfies:

$$T(g^{-1})_*C^{-1}\pi_{g*} = \xi[g]_*\iota_* = \pi_{1*}. \tag{2.2.15}$$

Finally:

$$\iota^*\iota_* = \iota^*[g]^*pr_g^*(pr_g^*)^{-1}[g]_*\iota_* = \pi_g^*C^{-1}\pi_{g*}. \tag{2.2.16}$$

We now compare $(\cdot,\cdot)_S$ and $(\cdot,\cdot)_{S'}$. We recall that if G is a finite group and if X is a left $\mathcal{O}[G]$-module then the map

$$\begin{aligned} \Phi : \mathrm{Hom}_{\mathcal{O}}(X,\mathcal{O}) &\longrightarrow \mathrm{Hom}_{\mathcal{O}[G]}(X,\mathcal{O}[G]) \\ f \quad &\longmapsto \sum_{\gamma\in G} f(\gamma(\cdot))[\gamma^{-1}] \end{aligned}$$

is an isomorphism of $\mathcal{O}[G]$-modules. Applying this result to $eM(S)$and $G_{U(S)}$, we can and will view $(\cdot,\cdot)_S$ as being $\mathcal{O}[G_{U(S)}]$-valued.

Lemma 2.6. *Let ϕ be the map from $G_{U(S')}$ to $G_{U(S)}$ induced by the inclusion $U_0(S')\subset U_0(S)$. Then:*

$$\phi((x,y)_{S'}) = (\pi_{1*}x, C^{-1}\pi_{g*}y)_S.$$

Proof. For $\gamma\in G_S$, let $G_{S'}(\gamma)$ be the set of elements of $G_{S'}$ sent to γ by ϕ.

$$\begin{aligned} \phi((x,y)_{S'}) &= \phi(\Phi((x,\cdot)_{S'})(y)) = \phi\Bigg(\sum_{\gamma\in G_{S'}}(x,\gamma_*y)_{S'}[\gamma^{-1}]\Bigg) \\ &= \sum_{\gamma\in G_S}\Bigg(\Bigg(x, \sum_{\gamma'\in G_{S'}(\gamma)}\gamma'_*y\Bigg)_S\Bigg)[\gamma^{-1}] = \sum_{\gamma\in G_S}\Bigg(\Bigg(x, \sum_{\gamma'\in G_{S'}(1)}\gamma'_*\gamma_*y\Bigg)_S\Bigg)[\gamma^{-1}]. \end{aligned}$$

The kernel of ϕ is equal to $(F^\times U_0(S')\cap F^\times U(S))/F^\times U(S') = F^\times\Gamma(S)/F^\times U(S')$. By smallness of $U(S)$, this last group is also the Galois group of the covering $X(S')\longrightarrow X(\Gamma(S))$. Hence:

$$\begin{aligned} \phi((x,y)_{S'}) &= \sum_{\gamma\in G_S}((x,\iota^*\iota_*\gamma_*y)_S)[\gamma^{-1}] = \sum_{\gamma\in G_S}((x,\pi_g^*C^{-1}\pi_{g*}\gamma_*y)_S)[\gamma^{-1}] \\ &= \langle x, W_{S'}\pi_g^*C^{-1}\pi_{g*}y\rangle_{S'}. \end{aligned}$$

The equality $gw_S = w_{S'}$ implies that $\pi_1^*W_S = W_{S'}\pi_g^*$. Hence:

$$\begin{aligned} \phi((x,y)_{S'}) &= \langle x, \pi_1^*W_SC^{-1}\pi_{g*}y\rangle_{S'} \\ &= (\pi_{1*}x, C^{-1}\pi_{g*}y)_S. \end{aligned}$$

□

Definition 2.7 (Hecke operators). Let $S \leqslant S'$ be two levels. Let $T(S)$, $T(-S)$, $T(S'-S)$ and $T(S-1)$ denote the Hecke operators:

$$T(S) = \prod_{v|p} T(t_v^{-s_v})_*, \quad T(-S) = \prod_{v|p} T(t_v^{s_v})_*,$$
$$T(S'-S) = \prod_{v|p} T(t_v^{s_v - s'_v})_*, \quad T(S-1) = \prod_{v|p} T(t_v^{1-s_v})_*.$$

Proposition 2.8. *Let M_∞^{ord} be the inverse limit on the level:*

$$M_\infty^{\mathrm{ord}} = \varprojlim_{\pi_{1*}} eH^1_{\mathrm{et}}(X(S) \times_F \bar{F}, \mathcal{O}).$$

The $\Lambda[\mathrm{Gal}(\bar{F}/F)]$-module M_∞^{ord} is endowed with a perfect alternating Hecke-equivariant pairing $(\cdot,\cdot)_\Lambda$ defined by

$$\begin{aligned} (\cdot,\cdot)_\Lambda :\ M_\infty^{\mathrm{ord}} \times M_\infty^{\mathrm{ord}} &\longrightarrow \Lambda(-1) \\ (x, y) &\longmapsto (x_S, T(S-1)y_S)_S. \end{aligned} \tag{2.2.17}$$

Proof. As the Galois and Hecke actions are not affected by the inverse limit, if the formula (2.2.17) is well-defined, then it defines a Hecke-equivariant pairing by Lemma 2.5. If $\pi_{1*}y_{S'} = y_S$, then by (2.2.15):

$$C^{-1}\pi_{g*}T(S'-1)y_{S'} = T(S-S')\pi_{1*}T(S'-1)y_{S'}.$$

The group U is small so Hecke operators commute with projections. Hence:

$$C^{-1}\pi_{g*}T(S'-1)y_{S'} = T(S-1)y_S.$$

Lemma 2.6 shows that the pairings $(\cdot,\cdot)_S$ are compatible with the transition map from $G_{U(S')}$ to $G_{U(S)}$. □

Remark. Though the projection π_{g*} was important in the previous proofs, it does not appear in the statement of Proposition 2.8. Accordingly, the projection between curves $X(S')$ and $X(S)$ in the tower of curves $\{X(S)\}_S$ is henceforth taken with respect to the natural π_1 projection and is denoted by $\pi_{S'/S}$.

3. Hida theory

Let $\{U(S)\}_S$ be a tower of groups of type U_1, U^1 or U_1^1, and U be a group in this tower. Let $\mathcal{O}'$ be the ring of integers of a finite extension of $\mathbb{Q}_p$.

3.1 Ordinary Hecke algebra

3.1.1 *Automorphic representations and Hecke algebras.*

3.1.1.1 *General definitions.* A weight k is an element of $\mathbb{Z}[I_{F,\infty}]$; an arithmetic weight is a weight such that $k_\tau \geqslant 2$ for all $\tau \in I_{F,\infty}$ and such that the k_τ have constant parity; a parallel weight is an integral multiple of the weight t such that $t_\tau = 1$ for all $\tau \in I_{F,\infty}$; a parallel defect is a weight $\nu \in \mathbb{Z}[I_{F,\infty}]$ such that ν_τ is non-negative for all τ and such that one ν_τ at least is zero. To an arithmetic weight k are attached $(m, \nu, \mu) \in \mathbb{Z}[I_{F,\infty}]^2 \times \mathbb{Z}$ as follows:

$$m = k - 2t. \tag{3.1.1}$$

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The weight ν is the parallel defect such that $\mu \cdot t = m + 2\nu$ is a parallel weight. The integer μ is called the parallel type of k. For $k \in \mathbb{Z}[I_{F,\infty}]$ and $x \in \mathbb{C}^{I_{F,\infty}}$, let $x^k \in \mathbb{C}$ be $\prod_{\tau|\infty} x_v^{k_\tau}$. If x belongs to $\mathbb{C}$, we view it as being in $\mathbb{C}^{I_{F,\infty}}$ through the diagonal embedding. If x belongs to $\bar{\mathbb{Q}}_p$, we view it as being in $\mathbb{C}$ via our fixed identification of these two fields.

We refer to [Nek06, § 12.3], [Shi78, § 1], [SW99, § 3.1] or [Hid06, § 2.3] for definitions and basic properties of holomorphic cuspforms for $\mathbf{G}(\mathbb{A}_{\mathbb{Q}})$. For k an arithmetic weight, let $S_k(U)$ and S_k be respectively the $\mathbb{C}$-vector space of holomorphic cuspforms of weight k and level U and the space of all cuspforms on B of weight k. Then S_k is an admissible representation of $\mathbf{G}(\widehat{\mathbb{Q}})/\widehat{F}^\times$ under the right action of $\mathbf{G}(\widehat{\mathbb{Q}})$ on $f \in S_k$. This representation decomposes into a direct sum of irreducible admissible representations V_π. Then $S_k(U)$ is the direct sum of the V_π^U such that $V_\pi^U \neq 0$. To $f \in S_k(U)$ is thus attached an automorphic representation $\pi(f)$ of $\mathbf{G}(\mathbb{A}_{\mathbb{Q}})$. The conductor of f is the conductor ideal of $\pi(f)$ defined as in [Cas73, Theorems 1 and 4]. At the finite places v, the automorphic representations $\pi(f)_v$ are either principal series, twisted Steinberg or supercuspidal representations (see for instance [BH06, 9.11]). We call the class containing $\pi(f)_v$ the automorphic type of $\pi(f)_v$. Thanks to our fixed identification of $\bar{F}_v$ with $\mathbb{C}$, the full classical Hecke algebra $\mathfrak{h}(U)$ acts on $S_k(U)$ by:

$$[UgU] \cdot f(x) = \sum_{g_i} f(xg_i^{-1}), \quad [UgU] = \coprod_{g_i} Ug_i.$$

We recall that ν is the parallel defect of k. For $v|p$, let $T_0(v)$ be $\varpi_v^{-\nu} T(v)$, where we view ϖ_v as belonging to $\mathbb{C}$ as explained above.

The Hecke algebra $\mathbf{T}_k(U, \mathbb{Z}_p)$ is the sub-algebra of $\mathrm{End}_{\mathbb{C}}(S_k(U))$ generated by the image of G_∞, by the images of the $T_0(v)$ for $v|p$ and by the images of the $T(v)$ for $v \notin S_B \cup \mathrm{supp}(S)$, $v \nmid p$. It does not depend on our choices of ϖ_v for $v \nmid p$. As $S_k(U) = \bigoplus V_\pi^U$, the Hecke algebra $\mathbf{T}_k(U, \mathbb{Z}_p)$ acts on each V_π^U. Let $\mathbf{T}_k(U, \mathcal{O}')$ be $\mathbf{T}_k(U, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}'$. When k is equal to $2t$, $\mathbf{T}_k(U, \mathcal{O}')$ coincides with the sub-algebra of $\mathrm{End}(H^1_{\mathrm{et}}(X(U) \times_F \bar{F}, \mathcal{O}'))$ generated by G_∞, the operators $T(v)$ for $v|p$ and $T(v)$ for $v \notin S_B \cup \mathrm{supp}(S)$.

Under the central action of $\mathbb{A}_F^\times$, the space $S_k(U)$ decomposes in the direct sum

$$\bigoplus_\phi S_k(U, \phi)$$

where ϕ runs through all the characters of $F^\times \backslash \mathbb{A}_F^\times / (F \otimes \mathbb{R})_+^\times U$ satisfying $\phi_\tau(-1) = (-1)^{k_\tau}$ for all $\tau \in I_{F,\infty}$. When $U = U_1^1$, let $S_k(U^{tw})$ be the direct sum

$$\bigoplus_{\phi_0} S_k(U, \phi_0)$$

where ϕ_0 runs through all the characters of $F^\times \backslash \mathbb{A}_F^\times / (F \otimes \mathbb{R})_+^\times U$ as above satisfying in addition that ϕ restricted to $\widehat{F}^\times$ is trivial. The Hecke algebra $\mathbf{T}_k(U^{tw}, \mathbb{Z}_p)$ is the sub-algebra of $\mathrm{End}_{\mathbb{C}}(S_k(U^{tw}))$ generated by the image of G_∞, by the images of the $T_0(v)$ for $v|p$ and by the images of the $T(v)$ for $v \notin S_B \cup \mathrm{supp}(S)$, $v \nmid p$.

3.1.1.2 *Nearly ordinary representations.* A form $f \in S_k(U)$ is called an eigenform if it is an eigenvector under the action of all but finitely many Hecke operators. Two eigenforms f and g in $S_k(U)$ are said to be equivalent in the sense of Atkin–Lehner if their eigenvalues coincide for all but finitely many Hecke operators. An eigenform $f \in S_k(U)$ is a newform if for all eigenforms $g \in S_k(U')$ equivalent to f in the sense of Atkin–Lehner, the conductor $\mathfrak{c}(\pi(g))$ of g is divisible by the conductor $\mathfrak{c}(\pi(f))$ of f. A form $f \in S_k(U^{tw})$ is called an eigenform (respectively a newform) if $f \in S_k(U)$ is an eigenform (respectively a newform).

Let $f \in S_k(U)$ be a newform and let $\pi(f)$ be the attached automorphic representation. Then $\pi(f)$ is said to be nearly ordinary at $v|p$ if there exists a v-good line on $V_{\pi(f)_v}^{U_v}$, this is to say a line on which $T_0(v)$ acts via a unit in the ring of integers of $\bar{\mathbb{Q}}_p$. In that case, $\pi(f)_v$ is either a principal series representation $\pi(\eta_v|\cdot|_v^{-1/2}, \xi_v|\cdot|_v^{-1/2})$ or a twisted Steinberg representation $\pi(\xi_v|\cdot|^{-1/2}, \xi_v|\cdot|^{1/2})$ with $\varpi_v^{-\nu}\xi_v(\varpi_v)$ a unit in the ring of integers of $\bar{\mathbb{Q}}_p$ (here we recall that ν is the parallel defect of k defined in § 3.1.1.1). In both cases, the v-good line is then unique (see for instance [Hid89a, Corollary 2.2]). The representation $\pi(f)$ is said to be ordinary at $v|p$ if the character ξ_v is moreover unramified. It is said to be (nearly) ordinary if it is (nearly) ordinary at all $v|p$. The form f is said to be (nearly) ordinary (at $v|p$) if $\pi(f)$ is (nearly) ordinary (at $v|p$). In the classical case, f is nearly ordinary if and only if it is a twist of an ordinary modular form by a character of $1+p\mathbb{Z}_p$ identified via the cyclotomic character with the Galois group of the unique $\mathbb{Z}_p$-extension of $\mathbb{Q}$. For a general totally real field F, this does not hold anymore. Note also that the property of being nearly ordinary depends on our choice of an embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$.

3.1.1.3 *Nearly ordinary Hecke algebra.* For each nearly ordinary automorphic representation π, let $w(\pi) = \otimes w(\pi, v) \in \bigotimes_v V_{\pi,v}^{U_v}$ be a vector such that for all $v|p$, the vector $w(\pi, v)$ spans a v-good line. The nearly ordinary subspace $S_k^{\mathrm{ord}}(U)$ of $S_k(U)$ is the span in $\bigoplus_\pi V_\pi^U$ of the $w(\pi)$. The nearly ordinary subspace $S_k^{\mathrm{ord}}(U^{tw})$ of $S_k(U^{tw})$ is the intersection of $S_k^{\mathrm{ord}}(U)$ with $S_k(U^{tw})$. For $U = U_1, U^1, U_1^1$ or U^{tw}, the nearly ordinary Hecke algebra $\mathbf{T}_k^{\mathrm{ord}}(U, \mathbb{Z}_p)$ is the sub-algebra of $\mathrm{End}_{\mathbb{C}}(S_k^{\mathrm{ord}}(U))$ generated over $\mathbb{Z}_p$ by the same operators as the Hecke algebra. If v is such that U_v is $\mathrm{GL}_2(\mathcal{O}_{F,v})$, the space $V_{\pi,v}^{U_v}$ is of dimension 1 by [Cas73, Theorem 1]. Hence, the algebra $\mathbf{T}_k^{\mathrm{ord}}(U, \mathbb{Z}_p)$ does not depend on our choices of $w(\pi, v)$.

Let $\mathbf{T}_k^{\mathrm{ord}}(U, \mathcal{O}')$ be $\mathbf{T}_k^{\mathrm{ord}}(U, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}'$. When k is equal to $2t$, $\mathbf{T}_k^{\mathrm{ord}}(U, \mathcal{O}')$ coincides with the sub-algebra of $\mathrm{End}(eH^1_{\mathrm{et}}(X(U) \times_F \bar{F}, \mathcal{O}'))$ generated by the same elements as $\mathbf{T}_k(U, \mathcal{O}')$ (where $X(U)$ is understood to be $X^{tw}(U)$ if U is equal to U^{tw}). If M is a $\mathbf{T}_{2t}(U, \mathcal{O}')$-module, let $e^{\mathrm{ord}}M$ be the greatest direct summand on which the operators $T(v)$ act in an invertible way for all $v|p$.

The semi-local ring $\mathbf{T}_k^{\mathrm{ord}}(U, \mathcal{O}')$ is a finite, flat and reduced $\mathcal{O}'$-algebra. If $S' \geqslant S$, there is a natural map from $\mathbf{T}_k^{\mathrm{ord}}(U(S'), \mathcal{O}')$ to $\mathbf{T}_k^{\mathrm{ord}}(U(S), \mathcal{O}')$. Let $\mathbf{T}_{\infty,k}^{\mathrm{ord}}(U, \mathcal{O}')$ be:

$$\mathbf{T}_\infty^{\mathrm{ord},k}(U, \mathcal{O}') = \varprojlim_S \mathbf{T}_k^{\mathrm{ord}}(U(S), \mathcal{O}').$$

If k and k' are arithmetic weights, the algebras $\mathbf{T}_{\infty,k}^{\mathrm{ord}}(\mathcal{O}')$ and $\mathbf{T}_{\infty,k'}^{\mathrm{ord}}(\mathcal{O}')$ are isomorphic by [Hid89b, Theorem 2.3]. When $U = U_1, U^1$ or U_1^1, we denote them by the common symbol $\mathbf{T}_\infty^{\mathrm{ord}}(\mathcal{O}')$ or sometimes $\mathbf{T}_\infty^{\mathrm{ord}}$ when U and $\mathcal{O}'$ are unimportant or clear from the context. When $U = U^{tw}$, we denote them by $\mathbf{T}_\infty^{tw}(\mathcal{O})$. The inclusion of G_∞ (respectively of G_∞^{tw} when U is equal to U^{tw}) inside $\mathbf{T}_\infty^{\mathrm{ord}}(\mathcal{O}')$ (respectively $\mathbf{T}_\infty^{tw}(\mathcal{O})$) endows it with a structure of Λ-algebra (respectively Λ^{tw}-algebra).

3.2 Arithmetic specializations and Galois representations

3.2.1 *Arithmetic specializations of* Λ. The map from G_F to G_∞ of equation (2.2.11) followed by the projection to $G_\infty/G(S^p)$, then the projection to the direct summand Γ and finally by the

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inclusion of Γ inside $\Lambda^\times$ defines a character:

$$\chi_\Gamma:\ G_F \longrightarrow \Lambda^\times.$$

As p is odd, the group Γ is uniquely 2-divisible so χ_Γ^{-1} admits a canonical square-root $\chi_\Gamma^{-1/2}$ which we fix.

Let k be an arithmetic weight of parallel type μ. If the tower $\{U(S)\}_S$ is of type U_1^1, let $\nu\in\mathbb{Z}[I_{F,\infty}]$ be a weight such that $k-2t+2\nu$ is a parallel weight; otherwise fix ν to be the parallel weight 0. By our fixed identification of $I_{F,\infty}$ with $I_{F,p}$, each $\tau\in I_{F,\infty}$ is attached to a unique $\sigma_\tau\in I_{F,p}$. Let ϕ_ν be the character:

$$\begin{aligned}\phi_\nu:\prod_{v|p}\mathcal{O}_{F,v}^\times &\longrightarrow \bar{\mathbb{Q}}_p^\times\\ (a_v)_v &\longmapsto \prod_{v|p}\ \prod_{\sigma_\tau\in I_{F,v}}\sigma_\tau(a_v)^{\nu_\tau}.\end{aligned}$$

In the above, we are identifying $\bar{F}_v$ with $\bar{\mathbb{Q}}_p$. If the tower $\{U(S)\}_S$ is of type U_1 or U^1, then the character ϕ_ν is trivial. In the U_1^1 case, the composition of the isomorphism (2.2.9) with the projection on the second factor allows us to view ϕ_ν as a character of G_∞. In both cases, we can thus view ϕ_ν as a character of G_∞. Let ε be a finite character of G_∞. An arithmetic specialization λ of weight k, character ε and defect ν is an $\mathcal{O}$-algebra morphism from Λ to $\bar{\mathbb{Q}}_p$ which induces the character $\varepsilon\chi_{cyc}^{-\mu}\phi_\nu$ on Γ (the minus sign coming from our choice of normalization of the reciprocity map). The difference between ν and the parallel defect ν_0 of k is called the cyclotomic twist of λ. The set $\operatorname{Spec}^{\text{arith}}\Lambda$ of arithmetic primes of Λ is the subset of $\operatorname{Spec}\Lambda$ of the kernels of arithmetic specializations. If $\mathfrak{p}\in\operatorname{Spec}^{\text{arith}}\Lambda$ is the kernel of λ, let $\mathcal{O}_\lambda$ be $\Lambda/\mathfrak{p}$.

Arithmetic specializations of Λ which factor through Λ^{tw} are called arithmetic specializations of Λ^{tw}; the subset of their kernels is denoted by $\operatorname{Spec}^{\text{arith}}\Lambda^{tw}$.

3.2.2 *Control theorems.*

Lemma 3.1. *Let M^{ord} be the Λ-module:*

$$M^{\text{ord}}=\varprojlim_S e^{\text{ord}}H^1_{\text{et}}(X(S)\times_F\bar{F},\mathcal{O}).$$

Then M^{ord} is free of finite rank. Let λ be an arithmetic specialization of Λ of weight k, character ε and parallel defect ν with values in $\mathcal{O}_\lambda$. Then there exists a level S such that the following isomorphism of $\mathcal{O}_\lambda[G_{U(S)}]$-modules holds:

$$M^{\text{ord}}\otimes_{\Lambda,\lambda}\mathcal{O}_\lambda \xrightarrow{\sim} e^{\text{ord}}H^1_{\text{et}}(X(S)\times_F\bar{F},\mathcal{F}_k(S))\otimes_{\Lambda,\lambda}\mathcal{O}_\lambda. \tag{3.2.1}$$

Here, $\mathcal{F}_k(S)$ is the usual sheaf of $\mathcal{O}$-modules of weight k (see for instance [Car86b, § 2.1]). The ring $\mathbf{T}^{\text{ord}}_\infty(\mathcal{O})$ is finite and torsion-free as a Λ-module, hence semi-local. Let $\mathcal{P}\in\operatorname{Spec}\mathbf{T}^{\text{ord}}_\infty(\mathcal{O})$ be a prime above the kernel $\mathfrak{p}$ of λ. Then $M^{\text{ord}}_{\mathcal{P}}$ is free of rank 2 over $\mathbf{T}^{\text{ord}}_\infty(\mathcal{O})_{\mathcal{P}}$ and $\mathbf{T}^{\text{ord}}_\infty(\mathcal{O})_{\mathcal{P}}/\Lambda_{\mathfrak{p}}$ is an unramified extension of regular rings.

Proof. Statement (3.2.1) is for instance [Hid07, Theorem 1.2(2)]. The fact that M^{ord} is a free Λ-module and that $\mathbf{T}^{\text{ord}}_\infty(\mathcal{O})$ is a torsion-free Λ-module then follows as in [Hid07, Theorem 1.2(2) and (4)] or [SW99, Corollary 3.4].

By [Car86b, § 2.2.24], $e^{\text{ord}}H^1_{\text{et}}(X(S)\times_F\bar{F},\mathcal{F}_k(S))\otimes_{\mathbf{T}^{\text{ord}}_\infty(\mathcal{O})}\mathbf{T}^{\text{ord}}_\infty(\mathcal{O})_{\mathcal{P}}/\mathcal{P}$ is free of rank 2 over $\mathbf{T}^{\text{ord}}_\infty(\mathcal{O})_{\mathcal{P}}/\mathcal{P}$. Equation (3.2.1) then implies that the same is true for $M^{\text{ord}}_{\mathcal{P}}/\mathcal{P}$. By the lemma of Nakayama–Azumaya–Krull,[3] there is thus a surjection of $\mathbf{T}^{\mathrm{ord}}_\infty(\mathcal{O})_{\mathcal{P}}$-modules from $\mathbf{T}^{\mathrm{ord}}_\infty(\mathcal{O})^2_{\mathcal{P}}$ onto $M^{\mathrm{ord}}_{\mathcal{P}}$. Let X be its kernel. The short sequence

$$0 \longrightarrow X \otimes_{\Lambda_{\mathfrak{p}}} \Lambda_{\mathfrak{p}}/\mathfrak{p} \longrightarrow \mathbf{T}^{\mathrm{ord}}_\infty(\mathcal{O})^2_{\mathcal{P}} \otimes_{\Lambda_{\mathfrak{p}}} \Lambda_{\mathfrak{p}}/\mathfrak{p} \longrightarrow M^{\mathrm{ord}}_{\mathcal{P}} \otimes_{\Lambda_{\mathfrak{p}}} \Lambda_{\mathfrak{p}}/\mathfrak{p} \longrightarrow 0$$

is left-exact because M^{ord} is Λ-free. Comparing dimensions shows that $X \otimes_{\Lambda_{\mathfrak{p}}} \Lambda_{\mathfrak{p}}/\mathfrak{p}$ is zero and thus that X vanishes. Hence $M^{\mathrm{ord}}_{\mathcal{P}}$ is a free $\mathbf{T}^{\mathrm{ord}}_\infty(\mathcal{O})_{\mathcal{P}}$-module of rank 2. The equation (3.2.1) then implies that $\mathbf{T}^{\mathrm{ord}}_\infty(\mathcal{O})^2_{\mathcal{P}} \otimes_{\Lambda_{\mathfrak{p}}} \Lambda_{\mathfrak{p}}/\mathfrak{p}$ is isomorphic to two copies of $\mathbf{T}^{\mathrm{ord}}(\mathcal{O})_{\mathcal{P}}/\mathcal{P}$, and thus that $\mathbf{T}^{\mathrm{ord}}_\infty(\mathcal{O})_{\mathcal{P}}/\mathfrak{p}$ is a field. Hence $\mathbf{T}^{\mathrm{ord}}_\infty(\mathcal{O})_{\mathcal{P}}$ is an unramified extension of the regular ring $\Lambda_{\mathfrak{p}}$. □

A comparable statement for the tower of curves $\{X^{tw}(S)\}$ can be extracted in the same way from the axiomatic method of [Hid07]. Once the previous lemma is granted, it can also be deduced as follows.

Corollary 3.2. *The Λ^{tw}-module M^{tw}*

$$M^{tw} = \varprojlim_{S} e^{\mathrm{ord}} H^1_{\mathrm{et}}(X^{tw}(S) \times_F \bar{F}, \mathcal{O})$$

is isomorphic to $M^{\mathrm{ord}} \otimes_\Lambda \Lambda^{tw}$ and thus free of finite rank. Let λ be an arithmetic specialization of Λ^{tw} of weight k, character ϵ and parallel defect ν with values in $\mathcal{O}_\lambda$. Then there exists a level S such that the following isomorphism of $\mathcal{O}_\lambda[G^{tw}_{U(S)}]$-modules holds:

$$M^{tw} \otimes_{\Lambda^{tw},\lambda} \mathcal{O}_\lambda \overset{\sim}{\longrightarrow} e^{\mathrm{ord}} H^1_{\mathrm{et}}(X^{tw}(S) \times_F \bar{F}, \pi_*\mathcal{F}_k(S)) \otimes_{\Lambda^{tw},\lambda} \mathcal{O}_\lambda. \tag{3.2.2}$$

Here, $\pi_*\mathcal{F}_k(S)$ is the push-forward of the sheaf $\mathcal{F}_k(S)$ on X^{tw}. The ring $\mathbf{T}^{tw}_\infty(\mathcal{O})$ is isomorphic to $\mathbf{T}^{\mathrm{ord}}_\infty(\mathcal{O}) \otimes_\Lambda \Lambda^{tw}$, hence finite and torsion-free as a Λ^{tw}-module, hence semi-local. Let $\mathcal{P} \in \operatorname{Spec} \mathbf{T}^{tw}_\infty(\mathcal{O})$ be a prime above the kernel $\mathfrak{p}$ of λ. Then $M^{tw}_{\mathcal{P}}$ is free of rank 2 over $\mathbf{T}^{tw}_\infty(\mathcal{O})_{\mathcal{P}}$ and $\mathbf{T}^{tw}_\infty(\mathcal{O})_{\mathcal{P}}/\Lambda^{tw}_{\mathfrak{p}}$ is an unramified extension of regular rings.

Proof. In view of the proof of Lemma 3.1, it is enough to prove the first assertion and (3.2.2). Both results amount to proving:

$$e^{\mathrm{ord}} H^1_{\mathrm{et}}(X^{tw}(S) \times_F \bar{F}, \pi_*\mathcal{F}_k(S)) \overset{\sim}{\longrightarrow} e^{\mathrm{ord}} H^1_{\mathrm{et}}(X(S) \times_F \bar{F}, \mathcal{F}_k(S)) \otimes_\Lambda \Lambda^{tw}. \tag{3.2.3}$$

By smallness of U, the action of $\mathbf{Z}(\mathbb{Q})$ on $X(S)$ is free. Hence the projection π from $X(S) \times_F \bar{F}$ to $X^{tw} \times_F \bar{F}$ is an étale morphism which is a torsor under the action of $\mathbf{Z}(\mathbb{Q})$. By [SGA4, Exposé XVII Théorème 4.3.1], there is thus an isomorphism:

$$\mathrm{R}\,\Gamma_{\mathrm{et}}(X(S) \times_F \bar{F}, \mathcal{F}_k(S)) \overset{\mathrm{L}}{\otimes}_\Lambda \Lambda^{tw} \overset{\sim}{\longrightarrow} \mathrm{R}\,\Gamma_{\mathrm{et}}(X^{tw}(S) \times_F \bar{F}, \pi_*\mathcal{F}_k(S)). \tag{3.2.4}$$

Let Y be a quotient curve of $X(S)$ covering $X^{tw}(S)$. On the étale cohomology of degree 2, $T(v)^*$ acts by multiplication by the degree of the Hecke correspondence, so by multiplication by a power of p for $v|p$. Hence, $T(v)^*$ is topologically nilpotent. The group $e^{\mathrm{ord}} H^2(Y \times_F \bar{F}, \pi_{X(S),Y*}\mathcal{F}_k(S))$ is thus zero. Combining this with the spectral sequence of (3.2.4) establishes the isomorphism (3.2.3). □

3.2.3 *Galois representation.* Let k be an arithmetic weight of parallel type μ and cyclotomic twist ν. Let $f \in S_k(U)$ be a nearly ordinary eigenform for all Hecke operators and let $\mathcal{O}$ be a discrete valuation ring containing all the Hecke eigenvalues of f as well as the images of its central character. Let $\pi(f)$ be the attached automorphic representation. Let λ_f be the $\mathcal{O}$-algebras morphism from $\mathbf{T}^{\mathrm{ord}}_\infty(\mathcal{O})$ to $\mathcal{O}$ sending a Hecke operator to the corresponding eigenvalue of f.

[3] We follow the citation practice of H. Matsumura and abbreviate this lemma as NAK.

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By the last statement of Lemma 3.1, there is a unique maximal ideal $\mathfrak{m}$ of $\mathbf{T}^{\mathrm{ord}}_{\infty}(\mathcal{O})$ and a unique minimal prime ideal $\mathfrak{a}\subset\mathfrak{m}$ such that λ_f factors through $R(\mathfrak{a})=(\mathbf{T}^{\mathrm{ord}}_{\infty})_{\mathfrak{m}}/\mathfrak{a}$.

$$\begin{array}{ccc} \mathbf{T}^{\mathrm{ord}}_{\infty} & \xrightarrow{\lambda_f} & \mathcal{O} \\ {\scriptstyle\lambda_{\mathfrak{m}}}\downarrow & \nearrow & \\ R(\mathfrak{a}) & & \end{array}$$

The ring $R(\mathfrak{a})$ is a complete local Noetherian domain of dimension $2+\delta_{F,p}$ or $2+d+\delta_{F,p}$ depending on whether U is equal to U_1, U^1 or U_1^1. Let $\mathbb{F}$ be the residual field of $R(\mathfrak{a})$. We recall the following theorem of [Car86b, Hid89a, Oht82, Wil88].

Proposition 3.3. *To $\pi(f)$ is attached a two-dimensional continuous, irreducible G_F-representation $(V(f),\rho_f,L_{\mathfrak{p}})$ satisfying the following properties.*

(i) *If τ is a complex conjugation, $\det\rho_f(\tau)$ is equal to -1.*

(ii) *Let v be a finite place not in $S_B\cup\mathrm{supp}(S)$. The G_{F_v}-representation $V(f)$ is unramified:*

$$\det(1-\mathrm{Fr}(v)X|V(f))=1-\lambda_f(T(v))X+\lambda_f(\langle\varpi_v\rangle)N_{F/\mathbb{Q}}vX^2.$$

(iii) *Let $v\nmid p$ be a place at which $V(f)$ is ramified. Then I_v acts through an infinite quotient if and only if $\pi(f)_v$ is a Steinberg representation. Otherwise, the group G_{F_v} acts irreducibly if and only if $\pi(f)_v$ is a supercuspidal representation and reducibly if and only if $\pi(f)_v$ is in the principal series.*

(iv) *Let $v\nmid p$ be a finite place. To $V(f)_v$ is attached a representation of the Weil–Deligne group of F_v which is pure of weight $\mu+2\nu+1$ (see [Bla06, § 1.10]). In particular, for all lifts σ of $\mathrm{Fr}(v)$ to G_{F_v} and all embeddings ι of $\bar{\mathbb{Q}}$ into $\mathbb{C}$, the two eigenvalues α_1 and α_2 of σ acting on $V(f)_v$ are algebraic integers such that $|\iota(\alpha_i)|=(Nv)^{(\mu+2\nu+1)/2}$ if $\pi(f)_v$ is principal series or supercuspidal and such that $|\iota(\alpha_1)|=(Nv)^{(\mu+2\nu+2)/2}$ and $|\iota(\alpha_2)|=(Nv)^{(\mu+2\nu)/2}$ if $\pi(f)_v$ is Steinberg.*

(v) *Let v be a finite place dividing p. The G_{F_v}-representation $V(f)$ fits in an exact sequence of non-trivial G_{F_v}-representations:*

$$0\longrightarrow V(f)_v^+\longrightarrow V(f)\longrightarrow V(f)_v^-\longrightarrow 0. \qquad (3.2.5)$$

If f is ordinary, then $V(f)_v^+$ can be chosen to be unramified.

There exists a semi-simple residual representation $(\overline{T}(f),\bar{\rho}_{\mathfrak{m}},\mathbb{F})$ satisfying the properties (i), (ii) and (v) with λ_f replaced by $\lambda_f \bmod \mathfrak{m}$. If $\bar{\rho}_{\mathfrak{m}}$ is irreducible, there exist irreducible representations $(T(f),\rho_f,\mathcal{O})$ and $(\mathcal{T}(f),\rho_{\mathfrak{m}},R(\mathfrak{a}))$ satisfying the properties (i), (ii) and (v) with λ_f replaced by $\lambda_{\mathfrak{m}}$ in the latter case.

Proof. See [Car86b, Théorème A] and [SW99, § 3.3]. □

Suppose that λ_f factors through $\mathbf{T}^{tw}_{\infty}$, or equivalently that f belongs to $S_k^{\mathrm{ord}}(U^{tw})$. By the last statement of Corollary 3.2, there is a unique maximal ideal $\mathfrak{m}$ of $\mathbf{T}^{tw}_{\infty}(\mathcal{O})$ and a unique minimal prime ideal $\mathfrak{a}\subset\mathfrak{m}$ such that λ_f factors through $R^{tw}(\mathfrak{a})=(\mathbf{T}^{tw}_{\infty})_{\mathfrak{m}}/\mathfrak{a}$.

$$\begin{array}{ccc} \mathbf{T}^{tw}_{\infty} & \xrightarrow{\lambda_f} & \mathcal{O} \\ {\scriptstyle\lambda_{\mathfrak{m}}}\downarrow & \nearrow & \\ R^{tw}(\mathfrak{a}) & & \end{array}$$

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The ring $R^{tw}(\mathfrak{a})$ is a complete local Noetherian domain of dimension $1+d$. If $\bar{\rho}_{\mathfrak{m}}$ is irreducible, the G_F-representation $\mathcal{T}(f)\otimes_{\Lambda}\Lambda^{tw}$ is an irreducible representation $(\mathcal{T}^{tw}(f), \rho_{\mathfrak{m}}, R^{tw}(\mathfrak{a}))$ satisfying the properties (i), (ii) and (v) with λ_f replaced by $\lambda_{\mathfrak{m}}$.

Henceforth, we make the following essential assumption.

Assumption 3.4. The G_F-representation $\bar{\rho}_{\mathfrak{m}}$ is irreducible.

Statement (i) of Proposition 3.3 then implies that $\bar{\rho}_{\mathfrak{m}}$ is absolutely irreducible.

According to statement (v) of Proposition 3.3, there exist characters $\bar{\chi}_1$ and $\bar{\chi}_2$ of G_{F_v} such that the G_{F_v}-representation $\bar{\rho}_{\mathfrak{m}}|G_{F_v}$ fits in the exact sequence:

$$0\longrightarrow\bar{\chi}_1\longrightarrow\bar{\rho}_{\mathfrak{m}}\longrightarrow\bar{\chi}_2\longrightarrow 0. \tag{3.2.6}$$

The representation $\bar{\rho}_{\mathfrak{m}}|G_{F_v}$ is said to be distinguished if its image is not scalar. Henceforth, we make the following assumption.

Assumption 3.5. For all $v|p$, the representation $\bar{\rho}_f|G_{F_v}$ is distinguished.

Under Assumptions 3.4 and 3.5, property (v) holds for $(\mathcal{T}(f), \rho_{\mathfrak{m}}, R(\mathfrak{a}))$ with $\mathcal{T}(f)_v^+$ and $\mathcal{T}(f)_v^-$ free of rank 1 over $R(\mathfrak{a})$.

3.2.4 *Commutative algebra properties of $R(\mathfrak{a})$ and geometric realization of $\mathcal{T}(f)$.* Let $M_{\mathfrak{m}}^{\mathrm{ord}}$ be defined by:

$$M_{\mathfrak{m}}^{\mathrm{ord}}=\varprojlim_{S} e_{\mathfrak{m}}^{\mathrm{ord}}H^1_{\mathrm{et}}(X(S)\times_F\bar{F},\mathcal{O})\otimes_{\mathbf{T}_\infty^{\mathrm{ord}}}R(\mathfrak{a}). \tag{3.2.7}$$

Lemma 3.6. *Let A be a complete local regular ring and B a local domain which is a finite A-module. Suppose that there exists a finite B-module M satisfying the following properties.*

(i) *The A-module M is free and $M\otimes_B\mathrm{Frac}(B)$ is of dimension* 2.

(ii) *There exists a G_F-representation $(M\otimes_B\mathrm{Frac}(B),\rho,\mathrm{Frac}(B))$ such that $\mathrm{Tr}(\rho)$ has values in B.*

(iii) *There exists an absolutely irreducible G_F-representation $(V,\bar{\rho},B/\mathfrak{m}B)$ such that $\mathrm{Tr}(\bar{\rho})$ is equal to $\mathrm{Tr}(\rho)\bmod\mathfrak{m}_B$*

Then there exists a unique G_F-representation (T,ϱ,B) of rank 2 such that $\varrho\bmod\mathfrak{m}_B=\bar{\rho}$ and such that $\mathrm{Tr}(\varrho)=\mathrm{Tr}(\rho)$. Moreover B is a Cohen–Macaulay ring and a free A-module.

Proof. Under assumptions (ii) and (iii), the existence and unicity of the G_F-representation (T,ϱ,B) of rank 2 such that $\varrho\bmod\mathfrak{m}_B=\bar{\rho}$ and such that $\mathrm{Tr}(\varrho)=\mathrm{Tr}(\rho)$ is by [Nys96, Théorème 1]. Then M is isomorphic to a G_F-stable submodule of T. According to [Car94, Théorème 4], this implies that there exists an ideal I of B such that M is isomorphic as $B[G_F]$-module to I^2 seen as a submodule of T identified with B^2. Assumption (i) implies that M, and so I, contains an A-regular sequence of length equal to the dimension of A, which is also the dimension of B. The depth of B, seen either as ring or as A-module, is thus equal to its dimension so B is a Cohen–Macaulay ring and a free A-module. □

Lemma 3.6 applied to $A=\Lambda$ or Λ^{tw}, $B=R(\mathfrak{a})$ or $R^{tw}(\mathfrak{a})$ and M equal to $M_{\mathfrak{m}}^{\mathrm{ord}}$ shows that $R(\mathfrak{a})$ and $R^{tw}(\mathfrak{a})$ are Cohen–Macaulay rings. When the ideal I of the previous lemma is principal, that is to say when there is an isomorphism of $R(\mathfrak{a})[G_F]$-modules

$$\mathcal{T}(f)\overset{\sim}{\longrightarrow}M_{\mathfrak{m}}^{\mathrm{ord}}=\varprojlim_{S} e_{\mathfrak{m}}^{\mathrm{ord}}H^1_{\mathrm{et}}(X(S)\times_F\bar{F},\mathcal{O})\otimes_{\mathbf{T}_\infty^{\mathrm{ord}}}R(\mathfrak{a}) \tag{3.2.8}$$

and when $R(\mathfrak{a})$ is a Gorenstein ring, we say that $\mathcal{T}(f)$ arises geometrically. In fact, under our hypotheses, when (3.2.8) holds, the proof of [Maz77, Lemma 15.1] shows that $R(\mathfrak{a})$ is a Gorenstein ring and conversely, when $R(\mathfrak{a})$ is Gorenstein, the isomorphism (3.2.8) holds by [Til97, Appendix by B. Mazur]. Consequently, $\mathcal{T}(f)$ arises geometrically whenever one of the two conditions is satisfied.

There are several ways to establish (3.2.8) but none known to this author is necessarily completely satisfying in general. The first is simply to *assume* this result, which has the merit of being straightforward but has rather obvious shortcomings as well. The second is to proceed as in [Car94] and to assume that $\bar{\rho}_{\mathfrak{m}}$ has residual multiplicity 1, which means that there is an isomorphism:

$$M_{\mathfrak{m}}^{\mathrm{ord}}/\mathfrak{m}M_{\mathfrak{m}}^{\mathrm{ord}} \xrightarrow{\sim} \overline{T}(f).$$

This implies that there exists a level S such that $M_{\mathfrak{m}}^{\mathrm{ord}}(S) = e_{\mathfrak{m}}^{\mathrm{ord}} H^1_{\mathrm{et}}(X(S) \times_F \bar{F}, \mathcal{O})$ is a free $\mathbf{T}^{\mathrm{ord}}(U(S), \mathcal{O})$-module of rank 2 and that $\mathbf{T}^{\mathrm{ord}}(U(S), \mathcal{O})$ is a Gorenstein ring. By [MW86, Lemma, p. 249] applied to the Cohen–Macaulay Λ-module $M_{\mathfrak{m}}^{\mathrm{ord}}$, this implies in turn that (3.2.8) holds. Under our running hypotheses, the residual multiplicity 1 hypothesis is known to hold in particular in the classical case when $\bar{\rho}_{\mathfrak{m}}$ is ramified at p or unramified but with non-scalar image of $\mathrm{Fr}(p)$, hence under hypotheses 3.4 and 3.5 thanks to [MT90, Théorème 7] (but not under the hypothesis 3.4 alone as is apparently claimed in that same theorem, see [KW08, Appendix B]). Finally, (3.2.8) can be established using the deformation theoretic results of [Fuj99]. We summarize this in the next proposition.

PROPOSITION 3.7. *Assume one of the following.*

(i) *The ring $R(\mathfrak{a})$ is a Gorenstein ring and $M_{\mathfrak{m}}^{\mathrm{ord}}$ is free of rank 2 over $R(\mathfrak{a})$.*

(ii) *The algebra B is equal to $\mathcal{M}_2(\mathbb{Q})$.*

(iii) *There exists a level S such that:*

$$\dim_{\mathbb{F}} e_{\mathfrak{m}}^{\mathrm{ord}} H^1_{\mathrm{et}}(X(S) \times_F \bar{F}, \mathcal{O}/\varpi)[\mathfrak{m}] = 2.$$

In other words, the representation $\bar{\rho}_{\mathfrak{m}}$ is of residual multiplicity 1.

(iv) *The prime p is strictly greater than 5 or $p = 5$ and $[F(\zeta_p) : F] > 2$. The $G_{F(\zeta_p)}$-representation $\bar{\rho}_{\mathfrak{m}}$ is absolutely irreducible. Either the representation $\mathcal{T}(f)$ is minimally ramified in the sense of [Fuj99, Definition 3.32] or Ihara's lemma is known to hold in parallel weight $(2, \ldots, 2)$ (in particular, this last condition is satisfied if $F = \mathbb{Q}$).*

Then $R(\mathfrak{a})$ is a Gorenstein ring and $M_{\mathfrak{m}}^{\mathrm{ord}}$ is free of rank 2 over $R(\mathfrak{a})$. If in addition λ_f restricted to Λ factors through Λ^{tw}, then $R^{tw}(\mathfrak{a})$ is a Gorenstein ring and $M_{\mathfrak{m}}^{tw} = M_{\mathfrak{m}}^{\mathrm{ord}} \otimes_{R(\mathfrak{a})} R^{tw}(\mathfrak{a})$ is free of rank 2 over $R^{tw}(\mathfrak{a})$.

Proof. Under (ii), this is [MT90, Théorème 7]. Under (iii), this is [Car94]. Under (iv), it is shown in [Fuj99, Theorem 0.2] that the universal deformation ring of $\bar{\rho}_{\mathfrak{m}}$ in the sense of [Fuj99, Definitions 3.3, 3.13 and 3.32] is a complete intersection ring, so in particular a Gorenstein ring. Because Ihara's lemma (Hypothesis 5.9 of [Fuj99]) is true by [DT94, Theorem 2] when $F = \mathbb{Q}$, the last requirement of (iv) is satisfied when $F = \mathbb{Q}$. Then Lemma 3.1 and [Til97, Appendix by B. Mazur] imply as above that condition (iii) is satisfied. The results of [Car94, Fuj99, MT90] which we quote typically establish that

$$M = \varprojlim_{S} e_{\mathfrak{m}}^{\mathrm{ord}} H^1_{\mathrm{et}}(X(S) \times_F \bar{F}, \mathcal{O})$$

is free of rank 2 over $\mathbf{T}^{\mathrm{ord}}_{\mathfrak{m}}=(\mathbf{T}^{\mathrm{ord}}_{\infty})_{\mathfrak{m}}$ and that $\mathbf{T}^{\mathrm{ord}}_{\mathfrak{m}}$ is a Gorenstein ring. Combining this with Proposition 2.8 and forgetting the Galois action yields:

$$M \xrightarrow{\sim} \mathrm{Hom}_{\Lambda}(M,\Lambda) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{T}^{\mathrm{ord}}_{\mathfrak{m}}}(M, \mathrm{Hom}_{\Lambda}(\mathbf{T}^{\mathrm{ord}}_{\mathfrak{m}},\Lambda)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{T}^{\mathrm{ord}}_{\mathfrak{m}}}(M,\mathbf{T}^{\mathrm{ord}}_{\mathfrak{m}}).$$

Reducing modulo $\mathfrak{a}$ shows that $M^{\mathrm{ord}}_{\mathfrak{m}}$ is free of rank 2 over $R(\mathfrak{a})$ and that there is a perfect pairing on $M^{\mathrm{ord}}_{\mathfrak{m}}$ inducing an isomorphism between $M^{\mathrm{ord}}_{\mathfrak{m}}$ and $\mathrm{Hom}_{R(\mathfrak{a})}(M^{\mathrm{ord}}_{\mathfrak{m}}, R(\mathfrak{a}))$. Hence $R(\mathfrak{a})$ is Gorenstein by [Maz77, Lemma 15.1].

If in addition λ_f restricted to Λ factors through Λ^{tw}, then $M^{tw}_{\mathfrak{m}} = M^{\mathrm{ord}}_{\mathfrak{m}} \otimes_{R(\mathfrak{a})} R^{tw}(\mathfrak{a})$ is free of rank 2 over $R^{tw}(\mathfrak{a})$. As above, this implies that $R^{tw}(\mathfrak{a})$ is a Gorenstein ring. □

Remark. The last requirement of condition (iv) can very probably be eliminated in view of announcements of C. Cheng, giving a proof of Ihara's lemma in low weight for quaternionic Shimura curves. We are grateful to B. Howard for pointing out to us that the Gorenstein property for $\mathbf{T}^{\mathrm{ord}}_{\mathfrak{m}}$ implied the Gorenstein property for $R(\mathfrak{a})$.

3.2.5 *Specializations.* Let A be a complete local Noetherian Λ-algebra. An $\mathcal{O}$-algebra morphism λ from A to a complete local Noetherian domain S with finite residue field of characteristic p is called an S-specialization of A. If (M,ρ,A) is a G_F-representation, the G_F-representation $(T_\lambda,\rho_\lambda,S)$ whose underlying S-module is $M\otimes_{A,\lambda} S$ endowed with the trivial G_F-action on S and the ρ-action on M is called an S-specialization of M. For instance, the map λ_f is an $\mathcal{O}$-specialization of $R(\mathfrak{a})$ and $T(f)$ is the corresponding $\mathcal{O}$-specialization of $\mathcal{T}(f)$. An arithmetic specialization λ of $R(\mathfrak{a})$ is a specialization of $R(\mathfrak{a})$ which coincides with an arithmetic specialization after restriction to Λ. When A is a finite Λ-algebra, in particular when A is equal to $\mathbf{T}^{\mathrm{ord}}_{\infty}$ or $R(\mathfrak{a})$, an arithmetic prime $\mathcal{P}\in\mathrm{Spec}^{\mathrm{arith}} A$ of A is a prime ideal $\mathcal{P}$ whose restriction to Λ is in $\mathrm{Spec}^{\mathrm{arith}}\,\Lambda$. If $\mathcal{P}\in\mathrm{Spec}^{\mathrm{arith}} R(\mathfrak{a})$ is over $\mathfrak{p}\in\Lambda$, then the last assertion of Lemma 3.1 implies that $R(\mathfrak{a})_{\mathcal{P}}/\Lambda_{\mathfrak{p}}$ is an unramified extension of regular rings.

The sets of arithmetic primes (respectively of arithmetic primes containing $\mathfrak{a}$) and of arithmetic primes of fixed weight (respectively of fixed weight containing $\mathfrak{a}$) are Zariski-dense in $\mathrm{Spec}\,\mathbf{T}^{\mathrm{ord}}_{\infty}$ (respectively in $\mathrm{Spec}\,R(\mathfrak{a})$), see for instance [SW99, Lemma 3.8]. This implies the following lemma.

Lemma 3.8. *If M is an $R(\mathfrak{a})$-module of finite type such that $M\otimes_{R(\mathfrak{a})} R(\mathfrak{a})_{\mathcal{P}}/\mathcal{P}$ is zero for all arithmetic primes $\mathcal{P}$ of fixed weight containing $\mathfrak{a}$, then M is $R(\mathfrak{a})$-torsion.*

Arithmetic specializations of $R(\mathfrak{a})$ correspond by [Hid88, Corollary 3.5] to automorphic eigenforms. For $\mathcal{P}\in\mathrm{Spec}^{\mathrm{arith}} R(\mathfrak{a})$ an arithmetic prime, we write $f_{\mathcal{P}}$ and $\pi(f_{\mathcal{P}})$ for the corresponding eigenform and automorphic representation and $V(f_{\mathcal{P}})$ for its attached G_F-representation with coefficients in $\bar{\mathbb{Q}}_p$. We use the same notation for an arithmetic specialization λ.

Lemma 3.9. *Let λ and λ' be two arithmetic specializations of $R(\mathfrak{a})$ with values in S and S' corresponding to two automorphic representations $\pi(\lambda)$ and $\pi(\lambda')$. Let $v\nmid p$ be a finite place. Then $\pi(\lambda)_v$ and $\pi(\lambda')_v$ have the same automorphic type. The rank of the $R(\mathfrak{a})$-module $\mathcal{T}(f)^{I_v}$ is equal to the rank of the S-module $T_\lambda^{I_v}$. The set of specializations μ such that the rank of $\mathcal{T}(f)^{I_v}$ is not equal to the rank of $T_\mu^{I_v}$ is of codimension at least 1.*

Proof. Let $\mathcal{V}(f)$ be the G_{F_v}-representation $\mathcal{T}(f)\otimes_{R(\mathfrak{a})}\mathrm{Frac}(R(\mathfrak{a}))$. The representation $\mathcal{V}(f)$ is continuous and the residue field of $R(\mathfrak{a})$ is finite of characteristic $p\nmid N_{F/\mathbb{Q}}v$. By Grothendieck's monodromy theorem [ST68, p. 515], there thus exists an open normal subgroup U of the tame inertia group I_v^t such that $\rho_{\mathfrak{m}}|U$ is unipotent.

Assume first that the action of U on $\mathcal{V}(f)$ is trivial. In that case the action of U on V_λ and $V_{\lambda'}$ is also trivial. Because $\mathrm{Tr}(\rho_{\mathfrak{m}})(I_v)$ is a finite subset of $\bar{\mathbb{Q}}_p$ and because arithmetic primes do not contain p, the two I_v-pseudorepresentations $\mathrm{Tr}(\rho_\lambda)$ and $\mathrm{Tr}(\rho_{\lambda'})$ are both equal to $\mathrm{Tr}(\rho_{\mathfrak{m}})$ and so isomorphic. The $\bar{\mathbb{Q}}_p[I_v]$-modules V_λ and $V_{\lambda'}$ are thus isomorphic (see for instance [Ser71, § 2.3, Corollaire 4] for this classical result of Frobenius). In particular, they are both reducible or both irreducible. In the latter case, $V_\lambda^{I_v}$ is trivial so the same is true for $\mathcal{V}(f)$ and both $\pi(\lambda)_v$ and $\pi(\lambda')_v$ are supercuspidal. In the former case, by [Cli37, Theorem 1], the I_v-representation $\mathcal{V}(f)$ is the sum of two one-dimensional I_v-stable subspaces which are either permuted under the action of G_{F_v}, in which case $\mathcal{V}(f)$ is an irreducible G_{F_v}-representation, or fixed under the action of G_{F_v}, in which case $\mathcal{V}(f)$ is a reducible G_{F_v}-representation. In both cases, the action of I_v on an I_v-stable subspace of $\mathcal{V}(f)$ is given by a character of finite order. This action stays the same after choosing the lattice $\mathcal{T}(f)$, localizing at an arithmetic prime and taking reduction modulo the maximal ideal. Hence, the dimension of $V(\lambda)^{I_v}$ is equal to the dimension of $\mathcal{V}(f)^{I_v}$. The action of G_{F_v} on the set of I_v-stable subspaces of $\mathcal{V}(f)$ is given by a matrix with integral entries so it also stays the same after choosing the lattice $\mathcal{T}(f)$, localizing at an arithmetic prime and taking reduction modulo the maximal ideal. The G_{F_v}-representations V_λ and $V_{\lambda'}$ are thus irreducible if and only if the G_{F_v}-representation $\mathcal{V}(f)$ is. If they are both reducible, the automorphic representations $\pi(\lambda)_v$ and $\pi(\lambda')_v$ are both principal series. In the other eventuality, they are both supercuspidal.

Assume now that $\mathcal{V}(f)|U$ is a non-trivial unipotent representation. Then $\mathcal{V}(f)^U$ is of dimension 1. If $\mathcal{P}$ is an arithmetic prime, $\mathcal{T}(f)^U_{\mathcal{P}}$ is then a saturated torsion-free submodule of rank 1 of a free module of rank 2 over a regular (hence factorial) ring, and thus is free of rank 1. Moreover, there exists an arithmetic specialization λ such that V_λ^U is of dimension 1 and thus such that V_λ has non-trivial monodromy. For such a λ, the representation $\pi(\lambda)_v$ is Steinberg so we wish to show that $\pi(\lambda')_v$ is also Steinberg. Let N be the monodromy operator of $\mathcal{V}(f)|U$, which is non-trivial by assumption. Let $w|v$ be a finite place of $\bar{F}^U$. The G_{F_w}-representation V_λ is pure so the eigenvalue of $\mathrm{Fr}(w)$ acting on V_λ/V_λ^U is non-zero. Thus the eigenvalue β of $\mathrm{Fr}(w)$ acting on $\mathcal{V}(f)/\mathcal{V}(f)^U$ is also non-zero and the ratio of the eigenvalue α of $\mathrm{Fr}(w)$ acting on $\mathcal{V}(f)^U$ with β is well-defined. By Grothendieck's monodromy theorem, this ratio is equal to $\chi_\Gamma(\mathrm{Fr}(w))$. Hence, the eigenvalues of $\mathrm{Fr}(w)$ under $V_{\lambda'}$ have distinct weights and so $V_{\lambda'}$ has non-trivial monodromy. This implies that $\pi(\lambda')_v$ is a Steinberg representation.

Let a and b be respectively rank $\mathcal{T}(f)^{I_v}$ and rank $T_\lambda^{I_v}$. The inequality $0 \leqslant a \leqslant b \leqslant 2$ proves that $a=b$ if $a=2$ or $b=0$. Combined with the above, this shows that it remains to compare a and b when $\pi(\lambda)_v$ is unramified Steinberg. Hence, we can assume that $b=1$ and that the action of I_v on $\mathcal{T}(f)^U$ is through a finite-order character ψ which becomes trivial after specialization by λ. Because $p \notin \ker \lambda$, the character ψ is then trivial. In that case, we have $a=1=b$.

Finally, assume that the set C of specializations μ such that $a \neq b$ has codimension zero. The previous paragraphs show that all arithmetic specializations of $\mathcal{T}(f)$ are then Steinberg at v and that $\mathcal{T}(f)^{I_v}$ is of rank 1. Choose an element $e \in \mathcal{T}(f)$ which is not fixed by I_v. Then e is not fixed by I_v in $\mathcal{V}(f)$ either and this contradicts our assumption on the codimension of C. □

3.2.6 *Galois representation with coefficients in* $R^{tw}(\mathfrak{a})$. Let ω be the central character of f. The G_F-representation $V(f)$ has determinant $\omega\chi_{cyc}^{-(\mu+2\nu+1)}$. Let ω_f be the finite-order part of $\omega\chi_{cyc}^{-(\mu+2\nu+1)}$. Consider the following assumption.

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Assumption 3.10. There exists a finite-order character χ of $\mathbb{A}_F^\times/F^\times$ such that χ^2 is equal to the finite-order part ω_f of $\omega\chi_{cyc}^{-(\mu+2\nu+1)}$.

The obstruction for Assumption 3.10 to be true is the 2-torsion in G_∞. By (2.2.7), Assumption 3.10 is thus true for instance when ω_f is trivial on the 2-torsion of the right-hand side of (2.2.7). This is the case in particular if f is of level U_0 outside p and if there is only one prime $\mathfrak{p}$ above p in F; indeed, ω_f is then trivial on $\mathcal{O}_{F,v}^\times$ for $v\nmid p$ and on $\{\pm 1\}$, which is the 2-torsion of $\mathcal{O}_{F,\mathfrak{p}}^\times$.

Henceforth, we assume consistently Assumption 3.10 and fix a χ as in the assumption. Then the central character of $\pi(f)\otimes\chi^{-1}$ factors though a group without 2-torsion so has a canonical square-root ψ. Let f^{tw} be the form $f\otimes\chi^{-1}\psi^{-1}$ and $\pi(f^{tw})$ be the automorphic representation $\pi(f)\otimes\chi^{-1}\psi^{-1}$. Then $\pi(f^{tw})$ has trivial central character. Let λ_f^{tw} be the morphism from $\mathbf{T}_\infty^{\mathrm{ord}}(\mathcal{O})$ to $\mathcal{O}$ attached to f^{tw}. If f belongs to $S_k(U_1^1)$, then f^{tw} belongs to $S_k(U^{tw})$ so λ_f^{tw} factors through a complete local domain $R^{tw}(\mathfrak{a})$ of dimension $1+d$. If f belongs to $S_k(U_1)$ of $S_k(U^1)$, then f^{tw} also belongs to $S_k(U^{tw})$. However, the parallel defect of the weight of f^{tw} is a parallel weight, or equivalently, the parallel defect of f^{tw} is entirely accounted for by a cyclotomic twist. Hence, λ_f^{tw} restricted to Λ further factors through a power-series ring of dimension 2. In a slight abuse of notation, we also denote this ring by Λ^{tw}. The same arguments as in Corollary 3.2 show that there then exists a complete local Noetherian domain $R^{tw}(\mathfrak{a})$ of dimension 2 through which λ_f^{tw} factors. Likewise, Proposition 3.3 and the results of § 3.2.4 extend under the same hypotheses.

In either case, we denote by R the ring $R^{tw}(\mathfrak{a})$, which is thus of dimension 2 if $f\in S_k(U_1)$ or $f\in S_k(U^1)$ and of dimension $1+d$ if $f\in S_k(U_1^1)$.

We summarize the results of the preceding subsections in a theorem.

Theorem 3.1. *Let $R(\mathfrak{a})$ be $\mathbf{T}_{\mathfrak{m}}^{\mathrm{ord}}/\mathfrak{a}$. We assume 3.4 and 3.5. Then there is a free $R(\mathfrak{a})$-module $\mathcal{T}(f)$ of rank 2 unramified as G_F-module outside $S_B\cup\mathrm{supp}(S)$. Let v be a finite place outside $S_B\cup\mathrm{supp}(S)$. Then:*

$$\det(1-X\,\mathrm{Fr}(v)|\mathcal{T}(f))=1-\lambda_{\mathfrak{m}}(T(v))X+\lambda_{\mathfrak{m}}(\langle\varpi_v\rangle)N_{F/\mathbb{Q}}\varpi_vX^2. \tag{3.2.9}$$

At $v|p$, the $R(\mathfrak{a})[G_{F_v}]$-module $\mathcal{T}(f)$ fits in a short exact sequence

$$0\longrightarrow\mathcal{T}(f)_v^+\longrightarrow\mathcal{T}(f)\longrightarrow\mathcal{T}(f)_v^-\longrightarrow 0$$

whose terms are all free of positive ranks.

Assume in addition that Assumption 3.10 holds. Let R be $R^{tw}(\mathfrak{a})$. Then the R-module $\mathcal{T}=\mathcal{T}(f^{tw})(1)$ is self-dual:

$$\mathcal{T}\xrightarrow{\sim}\mathrm{Hom}_R(\mathcal{T},R)(1). \tag{3.2.10}$$

If moreover one of the conditions of Proposition 3.7 holds then:

$$\mathcal{T}(f)\xrightarrow{\sim}\varprojlim_S e_{\mathfrak{m}}^{\mathrm{ord}}H^1_{\mathrm{et}}(X(S)\times_F\bar{F},\mathcal{O})\otimes_{\mathbf{T}_\infty^{\mathrm{ord}}}R(\mathfrak{a}).$$

The pairing of Proposition 2.8 induces an isomorphism

$$\mathcal{T}(f)(1)\xrightarrow{\sim}\mathrm{Hom}_{R(\mathfrak{a})}(\mathcal{T}(f),R(\mathfrak{a}))\otimes\omega\chi_\Gamma \tag{3.2.11}$$

which recovers (3.2.10) when $f=f^{tw}$.

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4. Equivariant classes in towers of Shimura curves

4.1 CM points on $X(S)$

4.1.1 *Galois action on CM points.* We fix a totally complex quadratic extension K of F as in § 2.1.2.1. For v a finite place of F, we write K_v for $K \otimes_F F_v$ and $\mathcal{O}_{K,v}$ for $\mathcal{O}_K \otimes \mathcal{O}_{F,v}$. For a non-zero ideal c of $\mathcal{O}_F$, let $\mathcal{O}_c$ be the $\mathcal{O}_F$-order of conductor c in $\mathcal{O}_K$:

$$\mathcal{O}_c = \mathcal{O}_F + c\mathcal{O}_K.$$

Let $x = [z, b]$ be a CM point. Let $Z(x)$ and $Z_0(x)$ be the compact open subgroups of $\widehat{K}^\times$ equal to $\widehat{q}^{-1}(bUb^{-1})$ and $\widehat{q}^{-1}(bU\widehat{\mathcal{O}}_F^\times b^{-1})$. By definition, the group $Z_0(x)$ contains $\mathcal{O}_{F,v}^\times$ for all finite places v. The statement (2.1.1) implies that:

$$\mathrm{Gal}(K(x)/K) \xrightarrow{\sim} \widehat{K}^\times / K^\times Z(x).$$

Let $K_0(x)$ be the abelian extension corresponding to $Z_0(x)$. The compact open group U_v is maximal at almost every finite place, so b_v is in U_v at almost every place, so $Z(x)$ is equal to $\mathcal{O}_{K,v}^\times$ at almost every place. Let $c(x)$ be the smallest ideal of $\mathcal{O}_F$ such that $\widehat{\mathcal{O}}_{c(x)}^\times \subset Z_0(x)$ and let $K_0(c(x))$ be the corresponding extension by class field theory:

$$\widehat{K}^\times / K^\times \widehat{F}^\times \widehat{\mathcal{O}}_{c(x)}^\times \xrightarrow{\sim} \mathrm{Gal}(K_0(c(x))/K).$$

Let Z be a compact open subgroup of $\widehat{K}^\times$ and K_Z the corresponding abelian extension. If $Z' \subset Z$ is an inclusion of such subgroups, then the Galois group $\mathrm{Gal}(K_{Z'}/K_Z)$ fits in the short exact sequence:

$$1 \longrightarrow \frac{K^\times \cap Z}{K^\times \cap Z'} \longrightarrow \frac{Z}{Z'} \xrightarrow{\mathrm{rec}_K} \mathrm{Gal}(K_{Z'}/K_Z) \longrightarrow 1. \tag{4.1.1}$$

Definition 4.1 (Ideal of bad places). Let $\mathcal{I}_0$ be the intersection of the ideals generated by $u - 1$ with $u \in (\mathcal{O}_K)_{\mathrm{tors}}^\times$ and $u \neq 1$.

4.2 Variation of CM points

4.2.1 *Coherent families in the tower $X(S)$.*

Definition 4.2 (Coherent conductors). Let $\mathscr{C}(p)$ be the set of ideals $\mathfrak{c}_p$ such that places dividing $\mathfrak{c}_p$ are above p. For x a CM point, let $\mathscr{L}_1^{(p)}(x)$ be the set of finite places l of $\mathcal{O}_F$ such that l is inert in K, is not in $S_B \cup \mathrm{supp}(S)$, does not divide $c(x)$ and $l\mathcal{O}_K$ does not divide $\mathcal{I}_0$. For $n \geqslant 0$, let $\mathscr{L}_n(x)$ be the set of ideals $\mathfrak{c}$ of $\mathcal{O}_F$ equal to the product of n distinct elements of $\mathscr{L}_1^{(p)}(x)$ with an ideal $\mathfrak{c}_p$ of $\mathscr{C}(p)$. Let $\mathscr{L}(x)$ be the union of $\mathscr{L}_n(x)$. For $l \in \mathscr{L}_1^{(p)}(x)$, let λ be the unique place of K above l. The residue fields of l and λ are written $k(l)$ and $k(\lambda)$ respectively.

Remark. The set $\mathscr{L}(x)$ contains the ideal $\mathcal{O}_F$.

Definition 4.3 (Coherent families of CM points). Let $\{S\}_{S \geqslant S_0}$ be a tower of levels. Let $x = [z, b(x)]$ be a CM point on $X(S_0)$ with $b(x)$ trivial at places above p and such that $Z(x)$ is included in $\widehat{\mathcal{O}}_{\mathfrak{c}_0}^\times$ with $\mathfrak{c}_0 \nmid \mathcal{I}_0$. Let $\mathfrak{c}$ be an ideal of $\mathscr{L}(x)$ and S a level. Let $b(\mathfrak{c}, S)$ be the element of $\widehat{B}^\times$ defined by the following conditions.

(i) If $v \nmid \mathfrak{c}$ and $v \nmid p$, then $b(\mathfrak{c}, S)_v$ is equal to the identity.

(ii) If $v | \mathfrak{c}$ and $v \nmid p$, then:

$$b(\mathfrak{c}, S)_v = \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.2.1}$$

(iii) If $v|p$, let t_v be $\operatorname{ord}_v \mathfrak{c}$. Then:

$$b(\mathfrak{c}, S)_v = \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v^{s_{0,v}-s_v-t_v} \end{pmatrix}. \tag{4.2.2}$$

Define:

$$x(\mathfrak{c}, S) = [z, b(x)b(\mathfrak{c}, S)] \in CM(X(S), K).$$

Remark. The point $x(\mathcal{O}_F, S_0)$ is equal to x. The definition (4.2.2) is the correct one to treat all possible changes of levels in the tower of levels $\{S\}_{S \geqslant S_0}$ (see especially Propositions 4.5 and 4.9 below). If we were to restrict ourselves to changes of levels such that $s'_v - s_v$ is equal to $\operatorname{ord}_v p$, which is enough to establish the main results of this article, then we could follow the choice of [How04b, § 1.2] and consider coherent families of CM points such that (4.2.2) is replaced by:

$$b(\mathfrak{c}, S)_v = \begin{pmatrix} \varpi_v^{s_v-s_{0,v}+t_v} & 0 \\ 0 & 1 \end{pmatrix}.$$

Henceforth, all levels S are in the tower $\{S\}_{S \geqslant S_0}$. Let $Z_0(\mathfrak{c}, S)$, $Z(\mathfrak{c}, S)$, $K_0(\mathfrak{c}, S)$ and $K(\mathfrak{c}, S)$ be respectively $Z_0(x(\mathfrak{c}, S))$, $Z(x(\mathfrak{c}, S))$, $K_0(x(\mathfrak{c}, S))$ and $K(x(\mathfrak{c}, S))$.

Lemma 4.4. *Let $\mathfrak{c}$ be an ideal of $\mathscr{L}(x)$ and S a level. Then $K^\times \cap Z_0(\mathfrak{c}, S) = \mathcal{O}_F^\times$ and $K^\times \cap Z(\mathfrak{c}, S) = \mathcal{O}_{F,1}(S)^\times$.*

Proof. This is an adaptation of [Nek07, Proposition 2.10] to our situation. □

Remark. Lemma 4.4 justifies the condition $Z(x) \subset \widehat{\mathcal{O}}_{\mathfrak{c}_0}^\times$ of Definition 4.3. Starting with an arbitrary CM point x and enlarging its conductor if necessary, this condition is easily achieved. It is however only imposed for convenience of notation: in its absence, all the results of this article remain valid after multiplying the equivariant classes $z(\mathfrak{c}, S)$ of Definition 4.11 below by the error term $[K^\times \cap Z_0(\mathfrak{c}, S) : \mathcal{O}_F^\times]^{-1}$.

4.2.2 *Galois and Hecke actions on the points $x(\mathfrak{c}, S)$.*

Proposition 4.5. *Let $\mathfrak{c}l$ be in $\mathscr{L}(x)$ and $S \leqslant S'$ be two levels. Let $G_{\mathfrak{c}l/\mathfrak{c}}$ and $G_{S'/S}$ be respectively the relative Galois groups $\operatorname{Gal}(K(\mathfrak{c}l, S)/K(\mathfrak{c}, S))$ and $\operatorname{Gal}(K_0(\mathfrak{c}, S')/K_0(\mathfrak{c}, S))$. Let $T(l)$ be the Hecke correspondence at l and $T(S)$ be the Hecke correspondence defined in (2.7). Then:*

$$G_{\mathfrak{c}l/\mathfrak{c}} \xrightarrow{\sim} \operatorname{Gal}(K_0(\mathfrak{c}l, S)/K_0(\mathfrak{c}, S)) \xrightarrow{\sim} k(\lambda)^\times / k(l)^\times \tag{4.2.3}$$

$$T(l) \cdot x(\mathfrak{c}, S) = \sum_{\sigma \in G_{\mathfrak{c}l/\mathfrak{c}}} \sigma \cdot x(\mathfrak{c}l, S) \tag{4.2.4}$$

and

$$G_{S'/S} \xrightarrow{\sim} \prod_{v \in S} \mathcal{O}_{F,v} / \varpi_v^{s'_v - s_v} \tag{4.2.5}$$

$$T(S' - S) \cdot x(\mathfrak{c}, S) = \pi_{S'/S}\Bigg(\sum_{\sigma \in G_{S'/S}} \sigma \cdot x(\mathfrak{c}, S') \Bigg). \tag{4.2.6}$$

Proof. By Lemma 4.4, $K^\times \cap Z_0(\mathfrak{c}, S)$ and $K^\times \cap Z_0(\mathfrak{c}l, S)$ are both equal to $\mathcal{O}_F^\times$. As l belongs to $\mathscr{L}_1(x)$, it divides neither $c(x)\mathcal{I}_0$ nor p so that:

$$\frac{Z(\mathfrak{c}, S)}{Z(\mathfrak{c}l, S)} \xrightarrow{\sim} \frac{Z_0(\mathfrak{c}, S)}{Z_0(\mathfrak{c}l, S)} \xrightarrow{\sim} \mathrm{Gal}(K_0(\mathfrak{c}l, S)/K_0(\mathfrak{c}, S)).$$

The quotient $Z(\mathfrak{c}, S)/Z(\mathfrak{c}l, S)$ is trivial outside l so we localize at l. We compute more generally the quotient $q_l^{-1}(gU_lg^{-1})/q_l^{-1}(gb(\mathfrak{c}, S)_lU_lb(\mathfrak{c}, S)_l^{-1}g^{-1})$ for $g \in B_l$. The extension K_λ/F_l is unramified so by [Vig80, Théorème 3.1], there exists an optimal embedding t of $\mathcal{O}_{K,\lambda}$ in U_l. This embedding is conjugated to q_l by the theorem of Skolem–Noether. As $q_l^{-1}(gU_lg^{-1}) = t^{-1}(g_1Ug_1^{-1})$ for some g_1, for the computation of the quotient $q_l^{-1}(gU_lg^{-1})/q_l^{-1}(gb(\mathfrak{c}, S)_lU_lb(\mathfrak{c}, S)_l^{-1}g^{-1})$, we can and do assume that q_l is optimal. Then $q_l^{-1}(gU_lg^{-1})$ is equal to $q_l^{-1}(gU_lg^{-1} \cap U_l)$. Let r be $\mathrm{ord}_{\varpi_l}(\det g)$. Let $\mathcal{O}_{K,\lambda,r}^\times$ denote the kernel of the reduction modulo ϖ_l^r from $\mathcal{O}_{K,\lambda}^\times$ to $(\mathcal{O}_{K,\lambda}/\varpi_l^r)^\times$. Then $q_l^{-1}(gU_lg^{-1})$ is equal to $\mathcal{O}_{F,l}^\times\mathcal{O}_{K,\lambda,r}^\times$. As $\mathrm{ord}_{\varpi_l}(\det gb(c, S)_l)$ is equal to $r+1$, the quotient $q_l^{-1}(gU_lg^{-1})/q_l^{-1}(gb(\mathfrak{c}, S)_lU_lb(\mathfrak{c}, S)_l^{-1}g^{-1})$ is isomorphic to $\mathcal{O}_{F,l}^\times\mathcal{O}_{K,\lambda,r}^\times/\mathcal{O}_{F,l}^\times\mathcal{O}_{K,\lambda,r+1}^\times$, which is isomorphic to $k(\lambda)^\times/k(l)^\times$. Hence $Z(\mathfrak{c}, S)/Z(\mathfrak{c}l, S)$ is isomorphic to $k(\lambda)^\times/k(l)^\times$ and (4.2.3) is established.

By definition of $T(l)$, the divisor $T(l) \cdot x(\mathfrak{c}, S)$ contains the orbit of $x(\mathfrak{c}l, S)$ under $G_{\mathfrak{c}\,l/\mathfrak{c}}$. According to (4.2.3), these divisors contain the same number of points, and hence coincide.

Because $Z_0(\mathfrak{c}, S)$ contains $\mathcal{O}_{F,v}^\times$ for all finite v, statements and proofs about $K_0(\mathfrak{c}, S)$ do not depend on whether $U = U_1$ or U_1^1. Likewise, the double coset $T(S' - S)$ admits an explicit set of representative which is independent of this choice. Moreover, to prove (4.2.5) and (4.2.6), it is enough to consider the case where $S' - S$ is non-zero at only one finite place v above p and with $s'_v - s_v$ equal to one. There then exists g_1 and g_2 with $\mathrm{ord}_{\varpi_v}\det(g_1^{-1}g_2) = 1$ such that:

$$Z_0(\mathfrak{c}, S)_v/Z_0(\mathfrak{c}, S')_v = q_v^{-1}(g_1U_vg_1^{-1})/q_v^{-1}(g_2U_vg_2^{-1}).$$

Conjugating q_v by an inner automorphism does not change $\mathrm{ord}_{\varpi_v}\det(g_1^{-1}g_2)$. Hence, as above, in order to compute this quotient, we can assume that q_v is an optimal embedding of $\mathcal{O}_{K,w}$ inside $\mathrm{GL}_2(\mathcal{O}_{F,v})$ for $w|v$. Then $q_v^{-1}(g_1U_vg_1^{-1})$ is the order $1 + \varpi_v^{s_v}\mathcal{O}_{K,w}$ and $q_v^{-1}(g_2U_vg_2^{-1})$ is the order $1 + \varpi_v^{s'_v}\mathcal{O}_{K,w}$. So:

$$Z_0(\mathfrak{c}, S)_v/Z_0(\mathfrak{c}, S')_v \xrightarrow{\sim} \mathcal{O}_{F,v}/\varpi_v.$$

By Lemma 4.4, this proves (4.2.5).

The computation

$$\begin{aligned} T(S' - S) \cdot x(\mathfrak{c}, S) &= \sum_{x \in X} \left[z, b(x)b(\mathfrak{c}, S)\begin{pmatrix} 1 & x\varpi_v^{-1} \\ 0 & \varpi_v^{-1}\end{pmatrix}\right] \\ &= \pi_{S'/S}\left(\sum_{x\in X}\left[z, \begin{pmatrix} 1 & x\varpi_v^{-s_{0,v}+t_v+s_v} \\ 0 & 1\end{pmatrix} b(x)b(\mathfrak{c}, S')\right]\right) \end{aligned}$$

then proves (4.2.6). □

PROPOSITION 4.6. *Let $\mathfrak{c}$ be in $\mathscr{L}(x)$ and S a level. The following short sequence is exact:*

$$1 \longrightarrow \frac{\mathcal{O}_F^\times}{\mathcal{O}_{F,1}^\times(S)} \longrightarrow \prod_{v\in S}(\mathcal{O}_{F,v}/\varpi_v^{s_v})^\times \longrightarrow \mathrm{Gal}(K(\mathfrak{c}, S)/K_0(\mathfrak{c}, S)) \longrightarrow 1. \tag{4.2.7}$$

Let σ be in G_K fixing $K_0(\mathfrak{c}, S)$. There exists an $a \in \widehat{F}^\times$ such that:

$$\sigma \cdot x(\mathfrak{c}, S) = \langle a\rangle \cdot x(\mathfrak{c}, S). \tag{4.2.8}$$

Proof. The compact open subgroups $Z_0(\mathfrak{c}, S)$ and $Z(\mathfrak{c}, S)$ differ only at places in $\operatorname{supp}(S)$. According to Lemma 4.4, the short exact sequence (4.1.1) thus induces the short exact sequence (4.2.7). The action of σ on $x(\mathfrak{c}, S)$ factors through $\operatorname{Gal}(K(\mathfrak{c}, S)/K_0(\mathfrak{c}, S))$. Thus, there exists an $a \in \widehat{F}^{\times}$ such that:

$$\sigma \cdot x(\mathfrak{c}, S) = (\operatorname{rec}_K a) \cdot x(\mathfrak{c}, S) = \langle a \rangle \cdot x(\mathfrak{c}, S). \qquad \square$$

Proposition 4.7. *Let $\mathfrak{c}$ be in $\mathscr{L}(x)$ and $K_0(\mathfrak{c}, \infty)$ be $\bigcup_{S \leqslant S'} K_0(\mathfrak{c}, S')$. Then $\operatorname{Gal}(K_0(\mathfrak{c}, \infty)/K)$ is of finite torsion and of $\mathbb{Z}_p$-rank d. The extension $K_0(\mathfrak{c}, \infty)$ does not contain a $\mathbb{Z}_p$-extension of F.*

Proof. See [CV05, § 3.3.2]. □

Let D_∞ be the $\mathbb{Z}_p^d$-extension contained in $K_0(\mathfrak{c}, \infty)$.

4.3 Equivariant classes and the p-adic Abel–Jacobi map

We consider again f as in § 3.2.3. Henceforth, Assumption 3.10 is assumed to hold for K and f.

4.3.1 *Galois cohomology and the Abel–Jacobi map.* Let E be a finite extension of F. We recall that the p-adic cycle class map and the Leray spectral sequence for the structure morphism $X(S) \longrightarrow \operatorname{Spec}(F)$ imply the existence of the class zero map:

$$cl_0 : \ CH^1(X(S) \times_F E) \longrightarrow H^0(E, H^2_{\text{et}}(X(S) \times_F \bar{F}, \mathcal{O})(1))$$

whose kernel is written $CH^1(X(S) \times_F E)_0$ and of the p-adic Abel–Jacobi map:

$$\Phi : \ CH^1(X(S) \times_F E)_0 \otimes_{\mathbb{Z}} \mathcal{O} \longrightarrow H^1(E, H^1_{\text{et}}(X(S) \times_F \bar{F}, \mathcal{O})(1)).$$

See [Wes02, § 2] for references. As in the proof of Corollary 3.2, the action of the Hecke operators $T(v)$ on the group $H^2_{\text{et}}(X(S) \times_F \bar{F}, \mathcal{O})$ is by multiplication by a power of p, and hence topologically nilpotent. Hence, the ordinary part of $H^2_{\text{et}}(X(S) \times_F \bar{F}, \mathcal{O})$ is zero. The p-adic Abel–Jacobi map thus defines a map:

$$\Phi : \ e^{\text{ord}}_{\mathfrak{m}} CH^1(X(S) \times_F E) \otimes_{\mathbb{Z}} \mathcal{O} \longrightarrow H^1(E, e^{\text{ord}}_{\mathfrak{m}} H^1_{\text{et}}(X(S) \times_F \bar{F}, \mathcal{O})(1)).$$

Applying this construction for E equal to $K(\mathfrak{c}, S)$ produces a map:

$$e^{\text{ord}}_{\mathfrak{m}} CH^1(X(S) \times_F K(\mathfrak{c}, S)) \otimes_{\mathbb{Z}} \mathcal{O} \longrightarrow H^1(K(\mathfrak{c}, S), M^{\text{ord}}_{\mathfrak{m}}(S)(1)). \tag{4.3.1}$$

Proposition 4.8. *Let $\mathfrak{c}l$ be in $\mathscr{L}(x)$ and S be a level. Let v be a place of $K(\mathfrak{c}, S)$ above l and w the unique place of $K(\mathfrak{c}l, S)$ above v. Let $\operatorname{Fr}(l) \in \operatorname{Gal}(K(\mathfrak{c}, S)/F)$ be the conjugacy class of the Frobenius morphism of l. Then:*

$$\operatorname{loc}_w \Phi(x(\mathfrak{c}l, S)) = \operatorname{Fr}(l) \operatorname{loc}_w \Phi(x(\mathfrak{c}, S)) \in H^1(K(\mathfrak{c}l, S)_w, H^1_{\text{et}}(X(S) \times_F \bar{F}, \mathcal{O})(1)).$$

Proof. Let $\mathcal{O}(\mathfrak{c}, S)$ and $\mathcal{O}(\mathfrak{c}l, S)$ be the ring of integers of $K(\mathfrak{c}, S)$ and $K(\mathfrak{c}l, S)$. Let $\mathcal{O}(\mathfrak{c}, S)_v$ and $\mathcal{O}(\mathfrak{c}l, S)_w$ be the completions of those rings at v and w and let $k(v)$ and $k(w)$ be their residue fields. The curve $X(S) \times_F K(\mathfrak{c}l, S)$ has a smooth proper model over $\mathcal{O}(\mathfrak{c}l, S)_w$. Let

$$X(S)^{\text{spe}} = (X(S) \times_F K(\mathfrak{c}l, S)) \times_{\mathcal{O}(\mathfrak{c}l, S)_w} k(w)$$

be its special fiber. The curve $X(S)$ has good reduction at l so the image of the p-adic Abel–Jacobi map localized at w is unramified:

$$\operatorname{im} \Phi \subset H^1_{ur}(K(\mathfrak{c}l, S)_w, H^1_{\text{et}}(X(S) \times_F \bar{F}, \mathcal{O})(1)).$$

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The smooth and proper base change theorem shows that the cohomology group

$$H^1_{ur}(K(\mathfrak{c}l,S)_w, H^1_{\mathrm{et}}(X(S)\times_F \bar{F},\mathcal{O})(1))$$

is isomorphic to

$$H^1(k(w), H^1_{\mathrm{et}}(X(S)^{\mathrm{spe}}\times_{k(w)} \bar{k}(w),\mathcal{O})(1)).$$

It is consequently enough to prove the result for $X(S)^{\mathrm{spe}}$. As l is inert in K, the reduction of a CM point at l is a supersingular point by [CV05, Lemma 3.1]. The description of the stabilizer of a supersingular point $[z,b]$ under the action of $\mathrm{GL}_2(F_l)\times W(F_l^{ab}/F_l)$ given in [Car86a, § 10.3] implies that the reductions modulo w of elements in the support of $T(l)[z,b]$ are all equal to the reduction modulo w of $\mathrm{Fr}(l)[z,b]$. As $x(\mathfrak{c}l,S)$ is in the support of $T(l)x(\mathfrak{c},S)$, in $CH^1(X(S)^{\mathrm{spe}})$ the equality

$$x(\mathfrak{c}l,S)^{\mathrm{spe}} = \mathrm{Fr}(l)x(\mathfrak{c},S)^{\mathrm{spe}}$$

holds. The images of $x(\mathfrak{c}l,S)^{\mathrm{spe}}$ and $\mathrm{Fr}(l)x(\mathfrak{c},S)^{\mathrm{spe}}$ are thus the same in the cohomology group:

$$H^1(k(w), H^1_{\mathrm{et}}(X(S)^{\mathrm{spe}}\times_{k(w)} \bar{k}(w),\mathcal{O})(1)).$$

This establishes the desired result by Galois equivariance of the Abel–Jacobi map. □

4.3.2 *Equivariant classes.* Applying the construction (4.3.1) for E equal to $K(\mathfrak{c},S)$ and composing with the projection from $M^{\mathrm{ord}}_{\mathfrak{m}}(S)$ to $M^{tw}_{\mathfrak{m}}(S)=e^{\mathrm{ord}}_{\mathfrak{m}}H^1_{\mathrm{et}}(X^{tw}(S)\times_F\bar{F},\mathcal{O})$ produces a map:

$$e^{\mathrm{ord}}_{\mathfrak{m}}CH^1(X(S)\times_F K(\mathfrak{c},S))\otimes_{\mathbb{Z}}\mathcal{O}\longrightarrow H^1(K(\mathfrak{c},S),M^{tw}_{\mathfrak{m}}(S)(1)). \tag{4.3.2}$$

Let σ be an element of $\mathrm{Gal}(K(\mathfrak{c},S)/K_0(\mathfrak{c},S))$. By Proposition 4.6, there is an $a\in\widehat{F}^\times$ such that $\mathrm{rec}_K\, a$ is equal to σ in $\mathrm{Gal}(K(\mathfrak{c},S)/K_0(\mathfrak{c},S))$. The image of $\mathrm{rec}_F\, a^2$ in G_F is equal to $\sigma_{|F^{ab}}$. As $\sigma\cdot x(\mathfrak{c},S)=\langle a\rangle\cdot x(\mathfrak{c},S)$ and as $\langle a\rangle$ acts trivially on $M^{tw}_{\mathfrak{m}}(S)$, the image of $e^{\mathrm{ord}}_{\mathfrak{m}}x(\mathfrak{c},S)$ under the map (4.3.2) is in the group:

$$H^0(K(\mathfrak{c},S)/K_0(\mathfrak{c},S),H^1(K(\mathfrak{c},S),M^{tw}_{\mathfrak{m}}(S)(1)).$$

By Proposition 3.3 statement (iv), the G_F-representation $M^{tw}_{\mathfrak{m}}(S)(1)$ is pure of weight -1 so has no G_{F_v}-invariants for $v\nmid p$. Hence, it has no $G_{K(\mathfrak{c},S)}$-invariants and the inflation-restriction sequence induces an isomorphism:

$$H^1(K_0(\mathfrak{c},S),M^{tw}_{\mathfrak{m}}(S)(1))\stackrel{\sim}{\longrightarrow} H^1(K(\mathfrak{c},S),M^{tw}_{\mathfrak{m}}(S)(1))^{\mathrm{Gal}(K(\mathfrak{c},S)/K_0(\mathfrak{c},S))}.$$

Let $K_0(\mathfrak{c})$ denote the extension $K_0(\mathfrak{c},1)$. The following composition of maps summarizes the construction of a class $y(\mathfrak{c},S)$ in $H^1(K_0(\mathfrak{c}),M^{tw}_{\mathfrak{m}}(S)(1))$ from $e^{\mathrm{ord}}_{\mathfrak{m}}x(\mathfrak{c},S)$.

$$\begin{array}{ccc}
e^{\mathrm{ord}}_{\mathfrak{m}}CH^1(X(S)\times_F K(\mathfrak{c},S))\otimes_{\mathbb{Z}}\mathcal{O} & \xrightarrow{\ \Phi\ } & H^1(K(\mathfrak{c},S),M^{\mathrm{ord}}_{\mathfrak{m}}(S)(1)) \\
 & & \downarrow \\
H^1(K_0(\mathfrak{c},S),M^{tw}_{\mathfrak{m}}(S)(1)) & \xleftarrow{\ \mathrm{res}^{-1}\ } & H^1(K(\mathfrak{c},S),M^{tw}_{\mathfrak{m}}(S)(1)) \\
\downarrow{\scriptstyle \mathrm{Cor}_{\mathfrak{c}S/\mathfrak{c}}} & & \\
H^1(K_0(\mathfrak{c}),M^{tw}_{\mathfrak{m}}(S)(1)) & \xrightarrow{\ T(1-S)\ } & H^1(K_0(\mathfrak{c}),M^{tw}_{\mathfrak{m}}(S)(1))
\end{array}$$

Note that the operator $T(1-S)$ is well defined on $M^{tw}_{\mathfrak{m}}$ by ordinarity. Let $y(\mathfrak{c},S)$ be the image of $x(\mathfrak{c},S)$ in $H^1(K_0(\mathfrak{c}),M^{tw}_{\mathfrak{m}}(S)(1))$.

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Proposition 4.9. *Let $S \leqslant S'$ be two levels. Then:*

$$\pi_{S'/S} y(\mathfrak{c}, S') = y(\mathfrak{c}, S). \tag{4.3.3}$$

Proof. By the statement (4.2.6) of Proposition 4.5:

$$\pi_{S'/S} \operatorname{Cor}_{\mathfrak{c}\, S'/\mathfrak{c}\, S} \Phi(x(\mathfrak{c}, S')) = T(S' - S)\Phi(x(\mathfrak{c}, S)).$$

So:

$$\pi_{S'/S} T(1 - S') \operatorname{Cor}_{\mathfrak{c}\, S'/\mathfrak{c}} \Phi(x(\mathfrak{c}, S')) = T(1 - S) \operatorname{Cor}_{\mathfrak{c}\, S/\mathfrak{c}} \Phi(x(\mathfrak{c}, S)).$$

After projection to $M^{tw}_{\mathfrak{m}}(S')$ and $M^{tw}_{\mathfrak{m}}(S)$, this is the statement (4.3.3). □

Definition 4.10 (Fundamental base fields). Let $\mathfrak{c}$ be in $\mathscr{L}_n(x)$. Let $\mathfrak{c}_p$ and $\mathfrak{c}^{\,p}$ be such that:

$$\mathfrak{c} = \mathfrak{c}_p \mathfrak{c}^{\,p} = \mathfrak{c}_p \prod_{i=1}^{n} l_i.$$

Let $K(\mathfrak{c})$ be the sub-extension of $K_0(\mathfrak{c})$ defined by:

$$K(\mathfrak{c}) = K_0(\mathfrak{c}_p) \prod_{i=1}^{n} K_0(l_i).$$

Remark. According to Lemma 4.4 and Proposition 4.5, the Galois group $\operatorname{Gal}(K(\mathfrak{c}l)/K(\mathfrak{c}))$ is isomorphic to $k(\lambda)^\times/k(l)^\times$. Let v be a finite place of F belonging to $S_B \cup \operatorname{supp}(S)$ and not above p. Let w and w' be places of $K(\mathfrak{c})$ and $K(\mathfrak{c}\mathfrak{c}_p)$ above v. The extension $K(\mathfrak{c}\mathfrak{c}_p)_{w'}/K(\mathfrak{c})_w$ is then unramified.

Definition 4.11 (Equivariant classes). Let $\mathfrak{c}$ be in $\mathscr{L}_n(x)$ and S be a level. Let $S_{\mathfrak{c}}$ be the level with the same support as S and such that $s_v = \operatorname{ord}_v \mathfrak{c}$ for v in $\operatorname{supp}(S)$. Define:

$$z(\mathfrak{c}, S) = T(-S_{\mathfrak{c}}) \operatorname{Cor}_{K_0(\mathfrak{c})/K(\mathfrak{c})} y(\mathfrak{c}, S).$$

Remark. By definition of $\mathscr{L}_1(x)$, if v belongs to $\operatorname{supp}(S)$ and $v \nmid p$, then $\operatorname{ord}_v \mathfrak{c} = 0$. Hence, $S_{\mathfrak{c}}$ is equal to $S_{\mathfrak{c}_p}$.

Proposition 4.12. *Let $\mathfrak{c}l$ be in $\mathscr{L}(x)$ and $S \leqslant S'$ be two levels. Then:*

$$\pi_{S'/S} z(\mathfrak{c}, S') = z(\mathfrak{c}, S), \tag{4.3.4}$$

$$\operatorname{Cor}_{K(\mathfrak{c}l)/K(\mathfrak{c})} z(\mathfrak{c}l, S) = T(l) z(\mathfrak{c}, S), \tag{4.3.5}$$

$$\operatorname{Cor}_{K(\mathfrak{c})/K(\mathfrak{c}^{\,p})} z(\mathfrak{c}, S) = z(\mathfrak{c}^{\,p}, S). \tag{4.3.6}$$

Proof. Because $z(\mathfrak{c}, S)$ and $z(\mathfrak{c}, S')$ have the same conductor, the assertion (4.3.4) is a restatement of (4.3.3). For the same reason, the assertion (4.3.5) is a restatement of the assertion (4.2.4) of Proposition 4.5.

Let S' be the level $S + S_{\mathfrak{c}}$. According to Definition 4.3 item (iii), the points $x(\mathfrak{c}, S)$ and $x(\mathfrak{c}^{\,p}, S')$ satisfy the relation:

$$x(\mathfrak{c}, S) = \pi_{S'/S} x(\mathfrak{c}^{\,p}, S').$$

So:

$$\operatorname{Cor}_{K(\mathfrak{c})/K(\mathfrak{c}^{\,p})} z(\mathfrak{c}, S) = T(1 - S') \operatorname{Cor}_{K_0(\mathfrak{c}, S)/K(\mathfrak{c}^{\,p})} \pi_{S'/S} x(\mathfrak{c}^{\,p}, S') = z(\mathfrak{c}^{\,p}, S).$$

The last equality follows from Proposition 4.5 assertion (4.2.6). □

Remark. Proposition 4.12 can be rephrased in the following way. The collection $\{z(\mathfrak{c}, S)\}_{\mathfrak{c}, S}$ with $\mathfrak{c}$ with constant $\mathfrak{c}^{\,p}$ and varying levels S and $\mathfrak{c}_p$ is a projective system for corestriction and change of levels. Therefore, for all $\mathfrak{c} \in \mathscr{L}(x)$ the element $\varprojlim_{\mathfrak{c}_p, S} z(\mathfrak{c}_p \mathfrak{c}^p, S)$ is a well-defined element of

$$\varprojlim_{\mathfrak{c}_p, S} H^1(K(\mathfrak{c}_p \mathfrak{c}^p), M_{\mathfrak{m}}^{tw}(S)(1)).$$

Repeating the proof of Proposition 4.7, it is easily seen that the extension $K(\mathfrak{c}^p p^\infty)$ contains the $\mathbb{Z}_p^d$-extension D_∞. Let $D_\infty(\mathfrak{c})$ be the composite extension of D_∞ and $K(\mathfrak{c})$.

Definition 4.13 (Iwasawa cohomology). Let G be a pro-finite group and H be a closed normal subgroup of G. Let A be a local complete Noetherian ring with residual field finite of characteristic p and M a finitely generated A-module. Let $H^i_{\mathrm{Iw}}(G, H; M)$ be the $A[[\mathrm{Gal}(G/H)]]$-module:

$$H^i_{\mathrm{Iw}}(G, H; M) = \varprojlim_{U, \mathrm{Cor}} H^i(U, M), \quad U \in \{\text{open subgroup of } G \text{ containing } H\}.$$

When $G = G_F$ and $H = \mathrm{Gal}(\bar{F}/L)$ for L an extension of F, we write $H^i_{\mathrm{Iw}}(F, L; M)$ for $H^i_{\mathrm{Iw}}(G, H; M)$.

Let $C^\bullet$ be the complex $\varprojlim_U C^\bullet_{\mathrm{cont}}(G, M \otimes_A A[G/U])$ viewed in the derived category. The $A[[G/H]]$-module $H^i_{\mathrm{Iw}}(G, H; M)$ is then isomorphic to $H^i(C^\bullet)$. This might not remain true without the hypothesis that the residual field of A is finite.

Definition 4.14 (Universal norms). For $\mathfrak{c} \in \mathscr{L}(x)$, let

$$z(\mathfrak{c}) \in H^1(K(\mathfrak{c}), \mathcal{T})$$

be the image of $\varprojlim_S z(\mathfrak{c}, S)$ in $H^1(K(\mathfrak{c}), \mathcal{T})$. Let z be the corestriction of $z(1)$ from $K(\mathcal{O}_F)$ to K.

Let

$$z_\infty(\mathfrak{c}) \in H^1_{\mathrm{Iw}}(K, D_\infty(\mathfrak{c}); \mathcal{T})$$

be the image of $\varprojlim_{\mathfrak{c}, S} z(\mathfrak{c}, S)$ in $H^1_{\mathrm{Iw}}(K, K(\mathfrak{c}^p p^\infty); \mathcal{T})$ under the corestriction from $K(\mathfrak{c} p^\infty)$ to $D_\infty(\mathfrak{c})$. Let z_∞ be the corestriction of $z_\infty(1)$ from $D_\infty(\mathcal{O}_F)$ to D_∞. Then z_∞ belongs to $H^1_{\mathrm{Iw}}(K, D_\infty; \mathcal{T})$. Let R_{Iw} be $R[[\mathrm{Gal}(D_\infty/K)]]$ and $\mathcal{T}_{\mathrm{Iw}}$ be $\mathcal{T} \otimes_R R_{\mathrm{Iw}}$. By Shapiro's lemma, the cohomology class z_∞ satisfies:

$$z_\infty \in H^1(K, \mathcal{T}_{\mathrm{Iw}}).$$

5. Euler systems for $\mathcal{T}_{\mathrm{Iw}}$

5.1 Selmer structures

5.1.1 *Cohomology of R-modules.* We review briefly the theory of Selmer complexes and Selmer groups referring to [Nek06] for details. Throughout 5.1, the letters A and T denote respectively a local complete Noetherian ring of residual characteristic 0 or p and a finitely generated A-module. Whenever a group G acts on T, the group G is assumed to be profinite and the action is assumed to be continuous. All Galois representations are supposed to be unramified outside a finite set of places.

Let L be a finite extension of F and T be an $A[G_L]$-module. Let Σ denote the finite set of finite places of L containing all places above p and all places where T is ramified. Let L_Σ be the maximal extension of L unramified outside Σ and $G_{L,\Sigma}$ be $\mathrm{Gal}(L_\Sigma/L)$. For v a finite place of L, let G be G_{L_v} or $G_{L,\Sigma}$. Let $C^\bullet_{\mathrm{cont}}(G, \cdot)$ be the functor of continuous cochains from

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the category of $A[G]$-modules to the category of bounded below complexes of A-modules and $\mathrm{R}\,\Gamma(G,\cdot)$ its image in the derived categories. We write $\mathrm{R}\,\Gamma(L_v,\cdot)$ and $\mathrm{R}\,\Gamma(L_\Sigma/L,\cdot)$ for $\mathrm{R}\,\Gamma(G_{L_v},\cdot)$ and $\mathrm{R}\,\Gamma(G_{L,\Sigma},\cdot)$ respectively.

Lemma 5.1. *The complex $\mathrm{R}\,\Gamma(G,T)$ is a bounded below complex of A-modules of finite type. If $v\nmid p$, the complex $\mathrm{R}\,\Gamma(I_v,T)$ is a bounded below complex of A-modules of finite type.*

Proof. See [NSW00, Theorem 7.1.8] and [Nek06, Proposition 4.2.3]. □

Proposition 5.2. *Let T be an $A[G]$-module which is a flat A-module. Let $\mathbf{x}$ be a regular A-sequence. Then:*

$$\mathrm{R}\,\Gamma(G,T)\overset{\mathrm{L}}{\otimes}_A A/\mathbf{x}\overset{\sim}{\longrightarrow}\mathrm{R}\,\Gamma(G,T\otimes_A A/\mathbf{x}). \tag{5.1.1}$$

Assume that A and A' have finite residual fields of characteristic p. Let $\phi: A\longrightarrow A'$ be a ring morphism. Then:

$$\mathrm{R}\,\Gamma(G,T)\overset{\mathrm{L}}{\otimes}_{A,\phi} A'\overset{\sim}{\longrightarrow}\mathrm{R}\,\Gamma(G,T\otimes_A A'). \tag{5.1.2}$$

Proof. The presumably well-known proof follows from the three following facts about the functor $\mathrm{R}\,\Gamma(G,-)$: it satisfies the Mittag-Leffler condition, it is triangulated and way-out. For the convenience of the reader, we mention the following references: [SGA4, Exposé XVII, Théorème 4.3.1] or [FK06, Proposition 1.6.5]. □

A complex satisfying the property (5.1.1) (respectively property (5.1.2)) is said to descend perfectly with respect to $\mathbf{x}$ (respectively ϕ).

Proposition 5.3. *Let $v\nmid p$ be a finite place of F. Let L_w be a finite extension of F_v. Let β be a finite order character of G_{L_w} with values in $\bar{\mathbb{Q}}_p^\times$. Let $\mathcal{P}\in\mathrm{Spec}^{\mathrm{arith}}(R)$ be an arithmetic prime of parallel even weight k. Then the complexes $\mathrm{R}\,\Gamma(L_w,V(f_{\mathcal{P}})(k/2)\otimes\beta)$ and $\mathrm{R}\,\Gamma(L_w,\mathcal{T}_{\mathcal{P}})$ are acyclic and the cohomology group $H^0(L_w,\mathcal{T})$ is trivial.*

Proof. The first two assertions are [Nek06, Propositions 12.4.8.4 and 12.7.13.3]. The R-module $H^0(L_w,\mathcal{T})$ is zero after tensor product with $R_{\mathcal{P}}/\mathcal{P}$ for all $\mathcal{P}$ in a dense subset of $\mathrm{Spec}(R)$ so is torsion by Lemma 3.8, and thus trivial. □

Proposition 5.4. *Let Σ be a finite set of places of F or K containing all places above places in S_B and all places above the places in the support of the tower of levels S. Then:*

$$H^1(F_\Sigma/F,\mathcal{T})=H^1(F,\mathcal{T})$$

and

$$H^1(K_\Sigma/K,\mathcal{T}_{\mathrm{Iw}})=H^1(K,\mathcal{T}_{\mathrm{Iw}}).$$

Proof. Let v be a finite place of F outside Σ and L_w be a finite extension of F_v. By Theorem 3.1, the G_{L_w}-representation $\mathcal{T}$ is unramified. We first show that $H^1(L_w,\mathcal{T})$ is equal to $H^1_{ur}(L_w,\mathcal{T})$. The group $\langle\mathrm{Fr}(w)\rangle$ is of cohomological dimension 1 so the short sequence

$$0\longrightarrow H^1_{ur}(L_w,\mathcal{T})\longrightarrow H^1(L_w,\mathcal{T})\longrightarrow H^0(\langle\mathrm{Fr}(w)\rangle,H^1(I_w,\mathcal{T}))\longrightarrow 0$$

is exact. As $\mathcal{T}$ is an unramified G_{L_w}-representation:

$$H^1(I_w,\mathcal{T})=\mathrm{Hom}(I_w,\mathcal{T}).$$

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The finite place w is prime to p so morphisms from I_w to $\mathcal{T}$ factor through a cyclic pro-p group $\langle\sigma\rangle$ with σ satisfying the monodromy relation:

$$\operatorname{Fr}(w)^{-1}\sigma\operatorname{Fr}(w)=\sigma^{N_{F/\mathbb{Q}}w}.$$

Consequently:

$$H^0(\langle\operatorname{Fr}(w)\rangle,\operatorname{Hom}(I_w,\mathcal{T}))\xrightarrow{\sim}H^0(\langle\operatorname{Fr}(w)\rangle,\mathcal{T}(-1)).$$

According to Proposition 5.3, this last group is zero.

Let L be the finite unramified extension of K equal to the intersection of D_∞ with $K(1)$. Let w be a place of L above v. The complex $\mathrm{R}\,\Gamma(K_w,\mathcal{T}_{\mathrm{Iw}})$ is isomorphic to $\mathrm{R}\,\Gamma(L_w,\mathcal{T}\otimes_R R[[X_1,\dots,X_d]])$. According to Proposition 5.2, there is an isomorphism of complexes:

$$\mathrm{R}\,\Gamma(L_w,\mathcal{T}\otimes_R R[[X_1,\dots,X_d]])\overset{\mathrm{L}}{\otimes}_{R[[X_1,\dots,X_d]]}R\xrightarrow{\sim}\mathrm{R}\,\Gamma(L_w,\mathcal{T}).$$

By a repeated application of NAK, the cohomology group $H^0(\langle\operatorname{Fr}(w)\rangle,\mathcal{T}_{\mathrm{Iw}}(-1))$ is thus zero and $H^1_{ur}(L_w,\mathcal{T}_{\mathrm{Iw}})$ is equal to $H^1(L_w,\mathcal{T}_{\mathrm{Iw}})$. As L_w/K_w is unramified, $H^1_{ur}(K_w,\mathcal{T}_{\mathrm{Iw}})$ is equal to $H^1(K_w,\mathcal{T}_{\mathrm{Iw}})$. □

5.1.2 *Local conditions.* Let T be an $A[G_L]$-module and v be a place in Σ. A local condition at v is a complex $C_f(G_{L_v},T)$ and a morphism of complexes:

$$C_f(G_{L_v},T)\longrightarrow C^\bullet_{\mathrm{cont}}(L_v,T).$$

The Selmer complex $\mathrm{R}\,\Gamma_f(L_\Sigma/L,T)$ is the object in the derived category corresponding to:

$$\operatorname{Cone}\left(C^\bullet_{\mathrm{cont}}(L_\Sigma/L,T)\oplus\bigoplus_{v\in\Sigma}C_f(G_{L_v},T)\longrightarrow\bigoplus_{v\in\Sigma}C^\bullet_{\mathrm{cont}}(L_v,T)\right)[-1].$$

Let $\tilde{H}^i_f(L_\Sigma/L,T)$ be its ith cohomology groups.

Let v be a place of K dividing p. The G_{F_v}-representation $\mathcal{T}(f)$ is reducible. There exist two $R(\mathfrak{a})[G_{F_v}]$-modules $U_v^+(\mathcal{T}(f))$ such that

$$0\longrightarrow U_v^+(\mathcal{T}(f))\longrightarrow\mathcal{T}(f)\longrightarrow U_v^-(\mathcal{T}(f))\longrightarrow 0$$

is exact. By Assumption 3.5, the $R(\mathfrak{a})$-module $U_v^\pm(\mathcal{T}(f))$ are free of rank 1. The action of $G_{F_v}^{ab}$ identified with $F_v^\times$ on $U_v^+(\mathcal{T}(f))$ is described as follows: the inertia $\mathcal{O}_{F,v}^\times$ acts through the character $\lambda_{\mathfrak{m}}$ whereas ϖ_v acts as $T(\varpi_v)$.

By taking quotients, localization and specialization, this defines $U_v^+(T)$ and $U_v^-(T)$ for specializations and localizations of $\mathcal{T}_{\mathrm{Iw}}$. We sometimes denote $U_v^+(T)$ and $U_v^-(T)$ by T_v^+ and T_v^-.

DEFINITION 5.5 (Greenberg's local condition). Let L be a finite extension of K. Let T be a specialization of R_{Iw}. The Greenberg local condition for complexes is given by the following choices of $\mathrm{R}\,\Gamma_f(L_v,T)$ for $v\in\Sigma$. If $v|p$, define:

$$C_f(L_v,T)=C^\bullet_{\mathrm{cont}}(L_v,U_v^+(T))$$

else define:

$$C_f(L_v,T)=C^\bullet_{\mathrm{cont}}(L_v^{nr}/L_v,T^{I_v}).$$

Let $\mathrm{R}\,\Gamma_f(L_\Sigma/L,T)$ be the corresponding Selmer complexes.

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Define the Greenberg Selmer group $H^1_{Gr}(L_\Sigma/L,T)$ by:

$$H^1_{Gr}(L_\Sigma/L,T)=\ker\left(H^1(L_\Sigma/L,T)\longrightarrow\bigoplus_{v|p}H^1(L_v,U_v^-(T))\oplus\bigoplus_{v\in\Sigma\setminus\{v|p\}}H^1(I_v,T)\right).$$

Remark that by functoriality of restriction, if $x\in H^1(L_\Sigma/L,T)$ belongs to $H^1_{Gr}(L_\Sigma/L,T)$ and if there is a morphism $f:T\longrightarrow T'$, then $f(x)\in H^1(L_\Sigma/L,T')$ belongs to $H^1_{Gr}(L_\Sigma/L,T')$.

5.1.3 *Exceptional specializations.* A specialization T of $\mathcal{T}$ is said to be exceptional at $v|p$ if there exists a finite extension L_w of F_v such that:

$$H^0(L_w,U_v^-(T))\neq 0. \tag{5.1.3}$$

A specialization is said to be exceptional if it is exceptional at some places above p. According to [Nek06, Lemma 12.5.4 and Proposition 12.5.8], an arithmetic specialization of weight k can be exceptional at v only if k is parallel equal to 2 and if $\pi(f_{\mathcal{P}})_v$ is a Steinberg representation. This implies that non-exceptional arithmetic points form a Zariski-dense and $\mathfrak{m}_R$-adically dense subset of $\operatorname{Spec}R$. The specializations $\mathcal{T}$ and $\mathcal{T}_{\mathrm{Iw}}$ themselves are not exceptional. By [Nek06, (6.1.3.2)], the short sequence

$$0\longrightarrow\bigoplus_{v|p}H^0(L_w,U_v(T^-))\longrightarrow\tilde{H}^1_f(L_\Sigma/L,T)\longrightarrow H^1_f(L_\Sigma/L,T)\longrightarrow 0 \tag{5.1.4}$$

is exact.

5.2 Local properties of $z(\mathfrak{c})$and $z_\infty(\mathfrak{c})$

5.2.1 *Local properties outside p.*

DEFINITION 5.6 (Minimally ramified module). Let T be an $A[G_{L_v}]$-module. When the natural map from T^{I_v} to $(T/\mathfrak{m}_AT)^{I_v}$ is a surjection, we say that T is minimally ramified. Let T be an $A[G_L]$-module. We say that T is minimally ramified at v if T is minimally ramified as $A[G_{L_v}]$-module.

If an A-module T is minimally ramified (at v) and if B is a quotient of A, then the B-module $T\otimes_A B$ is minimally ramified (at v). If (T,ρ,A) is a minimally ramified G_{L_v}-representation of rank 2, then T^I is a direct summand of T. Indeed, either T^I is of rank zero or it contains a basis $(e_i)_i$ such that not all the coordinates of e_i in a basis of T are in $\mathfrak{m}_R$. In this eventuality T^{I_v}/xT^{I_v} is isomorphic to $H^0(I_v,T/xT)$ through the natural map for all $x\in A$ and thus $H^1(I_v,T)_{\mathrm{tors}}$ vanishes.

PROPOSITION 5.7. *Let v be a finite place of K not dividing p. If $v\notin\Sigma$ or if v has an infinite decomposition group in $D_\infty(\mathfrak{c})$ or if $\mathcal{T}$ is minimally ramified at v, then $\operatorname{loc}_v z(\mathfrak{c})$ belongs to $H^1_{ur}(K(\mathfrak{c})_v,\mathcal{T})$ for all $\mathfrak{c}$. Moreover, if $v\in\Sigma$, then there exists a non-zero $\alpha\in R$ such that $\alpha z(\mathfrak{c})$ belongs to $H^1_{ur}(K(\mathfrak{c})_v,\mathcal{T})$ for all $\mathfrak{c}$.*

Proof. If $v\notin\Sigma$, this is the statement of Proposition 5.4 so we assume that v belongs to Σ. It is enough to prove that the cohomology classes $z(\mathfrak{c}\mathfrak{c}_p,S)$ are all unramified at v. If v has an infinite decomposition group in $D_\infty(\mathfrak{c})$, then this is the statement of [Rub00, Corollary B.3.5]. Now assume that v has a finite decomposition group in $D_\infty(\mathfrak{c})$. Fix a conductor $\mathfrak{c}$, places w and w' above v in $K(\mathfrak{c}_0)$ and $K(\mathfrak{c})$, and an arithmetic prime $\mathcal{P}$ of parallel even weight. As v does not divide $\mathfrak{c}$, the extension $K(\mathfrak{c})_{w'}/K(\mathfrak{c}_0)_w$ is unramified so I_w is equal to $I_{w'}$. We are interested in the image of $z(\mathfrak{c})$ in $H^1(I_w,\mathcal{T})$. There exists a finite order character β such that $\mathcal{T}\otimes_R R_{\mathcal{P}}/\mathcal{P}$ is isomorphic to $V(f_{\mathcal{P}})(1)\otimes\beta$. The complex $\mathrm{R}\,\Gamma(K(\mathfrak{c})_{w'},V(f_{\mathcal{P}})(1)\otimes\beta)$ is acyclic by Proposition 5.3. Thus the commutative diagram

$$\begin{array}{ccc} H^1(K(\mathfrak{c})_{w'},\mathcal{T}) & \longrightarrow & H^1(I_w,\mathcal{T}) \\ \big\downarrow{\scriptstyle \cdot\otimes_R R_{\mathcal{P}}/\mathcal{P}} & & \big\downarrow{\scriptstyle \cdot\otimes_R R_{\mathcal{P}}/\mathcal{P}} \\ H^1(K(\mathfrak{c})_{w'},V(f_{\mathcal{P}})(1)\otimes\beta) & \longrightarrow & H^1(I_w,V(f_{\mathcal{P}})(1)\otimes\beta) \end{array}$$

shows that the image of $H^1(K(\mathfrak{c})_{w'},\mathcal{T})$ in $H^1(I_w,\mathcal{T}\otimes_R R_{\mathcal{P}})$ is in $\mathcal{P}H^1(I_w,\mathcal{T}\otimes_R R_{\mathcal{P}})$. The R-module $H^1(I_w,\mathcal{T})$ is of finite type by Lemma 5.1. By Lemma 3.8, the images of the $z(\mathfrak{c})$ in $H^1(I_w,\mathcal{T})$ therefore lie in $H^1(I_w,\mathcal{T})_{\mathrm{tors}}$. Hence, there exists a non-zero $\alpha\in R$ such that $\alpha z(\mathfrak{c})$ is zero in $H^1(I_w,\mathcal{T})$ for all $\mathfrak{c}$. If $\mathcal{T}$ is minimally ramified, then $\mathcal{T}^{I_w}/x\mathcal{T}^{I_w}$ is isomorphic to $H^0(I_w,\mathcal{T}/x)$ for all $x\in R$ so $H^1(I_w,\mathcal{T})_{\mathrm{tors}}$ vanishes. ☐

5.2.2 *Local properties at p.*

PROPOSITION 5.8. *Let v be a place of F above p. Then $z(\mathfrak{c})$ is zero in $H^1(K(\mathfrak{c})_v,U_v^-(\mathcal{T}))$.*

Proof. Fix a place w of $K(\mathfrak{c})$ over v and $\mathcal{P}$ an arithmetic prime of weight 2. Let V be the G_{F_v}-representation $V(f_{\mathcal{P}})$. By [BK90, Example 3.11], the image of the p-adic Abel–Jacobi map is in $H^1_f(K(\mathfrak{c})_w,V)$ so $z(\mathfrak{c},S)$ belongs to this group. Combined with Lemma 3.8, the commutative diagram

$$\begin{array}{ccc} H^1(K(\mathfrak{c})_w,\mathcal{T}) & \longrightarrow & H^1(K(\mathfrak{c})_w,U_v^-(\mathcal{T})) \\ \big\downarrow{\scriptstyle \cdot\otimes_R R_{\mathcal{P}}/\mathcal{P}} & & \big\downarrow{\scriptstyle \cdot\otimes_R R_{\mathcal{P}}/\mathcal{P}} \\ H^1(K(\mathfrak{c})_w,V) & \longrightarrow & H^1(K(\mathfrak{c})_w,U_v^-(V)) \end{array}$$

shows that the image of $z(\mathfrak{c})$ in $H^1(K(\mathfrak{c})_w,U_w^-(\mathcal{T}))$ belongs to $H^1(K(\mathfrak{c})_w,U_w^-(\mathcal{T}))_{\mathrm{tors}}$. According to Lemma 5.1, this last R-module is of finite type so there is a $\mathfrak{c}_p$ such that $H^1(K(\mathfrak{c}\mathfrak{c}_p)_w,U_w^-(\mathcal{T}))_{\mathrm{tors}}$ is equal to $H^1(K(\mathfrak{c}p^\infty)_w,U_w^-(\mathcal{T}))_{\mathrm{tors}}$. For greater conductors, the corestriction map is thus multiplication by $[K(\mathfrak{c}\mathfrak{c}_p'):K(\mathfrak{c}\mathfrak{c}_p)]$. Since $z(\mathfrak{c})$ is in the image of all corestriction maps, it is p-divisible and hence zero. ☐

5.2.3 *Euler systems for $\mathcal{T}$ and $\mathcal{T}_{\mathrm{Iw}}$.*

DEFINITION 5.9 (Euler system). Let T be $\mathcal{T}$ or $\mathcal{T}_{\mathrm{Iw}}$. A system of classes $\{\mathfrak{z}(\mathfrak{c})\in H^1_{Gr}(K(\mathfrak{c}),T)\mid \mathfrak{c}\in\mathscr{C}\}$ is an Euler system if $\mathrm{Cor}_{\mathfrak{c}\,l/\mathfrak{c}}\ \mathfrak{z}(\mathfrak{c}l)=T(l)\mathfrak{z}(\mathfrak{c})$.

Assumption 5.10. Assume that all primes $v\in\Sigma$ at which $\mathcal{T}$ is not minimally ramified have an infinite decomposition group in D_∞.

THEOREM 5.1. *Let α be a non-zero element annihilating $H^1(I_v,\mathcal{T})^{\mathrm{Fr}(v)=1}_{\mathrm{tors}}$ for all primes of ramification v with a finite decomposition group in D_∞/K and at which $\mathcal{T}$ is not minimally ramified. Denote by z_α and $z_{\infty,\alpha}$ the classes αz and αz_∞. Then z_α belongs to $H^1_{Gr}(K_\Sigma/K,\mathcal{T})$ and $z_{\infty,\alpha}$ belongs to $H^1_{Gr}(K_\Sigma/K,\mathcal{T}_{\mathrm{Iw}})$. The system of classes $\{z_\alpha(\mathfrak{c})\}_{\mathfrak{c}}=\{\alpha z(\mathfrak{c})\}_{\mathfrak{c}}$ is an Euler system for $\mathcal{T}$. The system of classes $\{z_{\infty,\alpha}(\mathfrak{c})\}_{\mathfrak{c}}=\{\alpha z_\infty(\mathfrak{c})\}_{\mathfrak{c}}$ is an Euler system for $\mathcal{T}_{\mathrm{Iw}}$. Under Assumption 5.10, these statements are true with α equal to one.*

Proof. This is a reformulation of the Propositions 4.8, 4.12, 5.7 and 5.8. ☐

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5.3 Kolyvagin systems for specializations of $\mathcal{T}_{\mathrm{Iw}}$

We construct Kolyvagin systems in the sense of [How04a, MR04] for specializations of $\mathcal{T}_{\mathrm{Iw}}$ with values in $\mathbb{Z}_p$-flat discrete valuation rings.

5.3.1 *One-dimensional specializations of $\mathcal{T}$.*

5.3.1.1 Let S be a discrete valuation ring finite over $\mathcal{O}$ with maximal ideal $\mathfrak{m}_S$ and uniformizing parameter ϖ_S. If $\mathbf{Sp}$ is an S-valued specialization, we recall that $T_{\mathbf{Sp}}$ is the G_K-representation with coefficients in S defined by

$$T_{\mathbf{Sp}} = \mathcal{T} \otimes_{R,\mathbf{Sp}} S$$

with trivial action of G_K on S. Let $V_{\mathbf{Sp}}$, $A_{\mathbf{Sp}}$, $U_v^{\pm}(V_{\mathbf{Sp}})$ and $U_v^{\pm}(A_{\mathbf{Sp}})$ be $T_{\mathbf{Sp}} \otimes_S \mathrm{Frac}(S)$, $T_{\mathbf{Sp}} \otimes_S \mathrm{Frac}(S)/S$, $U_v^{\pm}(T_{\mathbf{Sp}}) \otimes_S \mathrm{Frac}(S)$ and $U_v^{\pm}(T_{\mathbf{Sp}}) \otimes_S \mathrm{Frac}(S)/S$ respectively. We sometimes write $X^{\pm}$ for $U_v^{\pm}(X)$ when v is clear from the context.

Lemma 5.11. *Let $\mathbf{Sp}$ be a non-exceptional specialization and v be a place above p. Let $\mathfrak{c}$ be in $\mathscr{L}(x)$ and w be a place of $K(\mathfrak{c})$ above v. Then:*

$$X = \varprojlim_{\mathfrak{c}'_p \in \mathscr{C}(p)} H^0(K(\mathfrak{c}\mathfrak{c}'_p)_w, U_v^-(A_{\mathbf{Sp}})) = 0.$$

Proof. The specialization $\mathbf{Sp}$ being non-exceptional, for all finite L_w/K_v, the S-module $H^0(L_w, A^-_{\mathbf{Sp}})$ embeds into $H^1(L_w, T^-_{\mathbf{Sp}})$ so is of finite type. The increasing sequence $H^0(K(\mathfrak{c}\mathfrak{c}_p)_w, U_v^-(A_{\mathbf{Sp}}))$ thus stabilizes for $\mathfrak{c}'_p$ large enough, after which corestriction becomes multiplication by a power of p. Consequently, the S-module X is p-divisible and of finite type over S, so trivial. □

Let $\mathbf{Sp}$ be an S-valued specialization. The system of equivariant classes $\{z(\mathfrak{c}) \in H^1(K(\mathfrak{c}), \mathcal{T})\}$ defines a system $\{\mathbf{Sp}(z(\mathfrak{c})) \in H^1(K(\mathfrak{c}), T_{\mathbf{Sp}})\}$ of equivariant classes for $T_{\mathbf{Sp}}$.

5.3.2 *Kolyvagin's derivative.*

5.3.2.1 Let $\mathbf{Sp}$ be a specialization with values in S.

Definition 5.12 (Kolyvagin's derivative). Let $\mathscr{S}_1$ be the set of primes of $l \in \mathscr{L}_1(x)$ such that the character χ is trivial on G_{F_l}. Let $\mathscr{S}_r$ be the product of r distinct elements of $\mathscr{S}_1$ and $\mathscr{S}$ be the union of $\mathscr{S}_r$ for all integers r. For $l \in \mathscr{S}$, let G_l be the cyclic groups $\mathrm{Gal}(K(l)/K(1))$ and σ_l be a fixed choice of generator. Let D_l be Kolyvagin's derivative:

$$D_l = \sum_{i=0}^{|G_l|-1} i\sigma_l^i.$$

For $\mathfrak{c} \in \mathscr{S}$, let $G_{\mathfrak{c}}$ be the group $\mathrm{Gal}(K(\mathfrak{c})/K(1))$ and $D_{\mathfrak{c}}$ be the products of D_l for $l|\mathfrak{c}$. Let $I_{\mathfrak{c}}$ be the smallest ideal of S containing the images of $(N_{F/\mathbb{Q}}l + 1)$, of $\lambda_{\mathfrak{m}}(T(l))$ and of $\chi_\Gamma(\mathrm{Fr}(l)) - 1$ in S. Let $\phi_{\mathfrak{c}}$ be the natural map from the cohomology of $T_{\mathbf{Sp}}$ to $T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}}$.

Remark. Though this does not appear in the notation, the ideals $I_{\mathfrak{c}}$ depend on $\mathbf{Sp}$. By Cebotarev density theorem, the set $\mathscr{S}_1$ is of positive density. In the absence of the assumption that $Z(x) \subset \widehat{\mathcal{O}}^{\times}_{\mathfrak{c}_0}$ of Definition 4.3, one has to replace $(N_{F/\mathbb{Q}}l + 1)$ by $(N_{F/\mathbb{Q}}l + 1)/|\mathcal{O}_K^{\times}/\mathcal{O}_F^{\times}|$.

5.3.2.2 Let l be in $\mathscr{S}_1$ and λ the unique place of K above l. The residue field $k(\lambda)$ of λ is of cardinal $(N_{F/\mathbb{Q}}l)^2$. The place λ splits completely in $K(1)$ and is totally tamely ramified in $K(l)$. Let L be the maximal p-extension of K_λ in $K(l)_\lambda$. Let $H^1_{tr}(K_\lambda, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$ be the kernel of $H^1(K_\lambda, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$ in $H^1(K(l)_\lambda, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$.

Let $\mathfrak{c}$ be in $\mathscr{S}_r$ with $l|\mathfrak{c}$. According to Theorem 3.1, the characteristic polynomial of the Frobenius morphism $\mathrm{Fr}(l)$ on $T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}}$ is X^2-1 so $\mathrm{Fr}(\lambda)$ acts trivially on $T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}}$. The ideal I_l annihilates $T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}}$ so the same is true for $N_{F/\mathbb{Q}}l+1$ and consequently for $|k(\lambda)^\times|$. The representation $T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}}$ thus satisfies the hypotheses of [MR04, Lemmas 1.2.1, 1.2.3 and 1.2.4]. Accordingly, there exists a finite-singular isomorphism

$$\phi_l^{fs}:\ H^1_{ur}(K_\lambda, T_{\mathbf{Sp}}/I_{\mathfrak{c}l}T_{\mathbf{Sp}}) \xrightarrow{\sim} H^1_{tr}(K_\lambda, T_{\mathbf{Sp}}/I_{\mathfrak{c}l}T_{\mathbf{Sp}})$$

coming from

$$H^1_{nr}(K_\lambda, T_{\mathbf{Sp}}/I_{\mathfrak{c}l}T_{\mathbf{Sp}}) \xrightarrow{\sim} T_{\mathbf{Sp}}/I_{\mathfrak{c}l}T_{\mathbf{Sp}} \xleftarrow{\sim} H^1_{tr}(K_\lambda, T_{\mathbf{Sp}}/I_{\mathfrak{c}l}T_{\mathbf{Sp}}).$$

Assumption 5.13. Assume that $\bar{\rho}$ restricted to $G_{K(\mathcal{O}_F)}$ is irreducible.

This is equivalent to requiring that $\pi(f)$ has not residual CM. According to [Dim05], if $\pi(f)$ does not acquire CM by any quadratic extension of F contained in $K(\mathcal{O}_F)$, then Assumption 5.13 is satisfied for sufficiently large p.

LEMMA 5.14. *Let r be a positive integer, $\mathfrak{c}$ an element of $\mathscr{S}_r$ and $\mathfrak{c}'_p$ an element of $\mathscr{C}(p)$. Then $H^1(K(\mathfrak{c}\mathfrak{c}'_p), T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})^{G_{\mathfrak{c}}}$ is isomorphic to $H^1(K(\mathfrak{c}'_p), T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$ and $\phi_{\mathfrak{c}\mathfrak{c}'_p}(D_{\mathfrak{c}}z(\mathfrak{c}\mathfrak{c}'_p))$ belongs to this group.*

Proof. The Assumption 5.13 implies that $H^0(K(\mathfrak{c}), T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$ is trivial for all $\mathfrak{c}$. The restriction map

$$\mathrm{res}:\ H^1(K(\mathfrak{c}'_p), T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}}) \longrightarrow H^1(K(\mathfrak{c}\mathfrak{c}'_p), T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})^{G_{\mathfrak{c}}}$$

is thus an isomorphism. That $D_{\mathfrak{c}}z(\mathfrak{c}\mathfrak{c}'_p)$ is invariant under the action of $G_{\mathfrak{c}}$ is the standard property of Kolyvagin derived classes. □

DEFINITION 5.15 (Kolyvagin classes). For $\mathfrak{c}\in\mathscr{S}$, let $\kappa(\mathfrak{c})\in H^1(K, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$ (respectively $\kappa_\alpha(\mathfrak{c})$) be the corestriction to K of the canonical pre-image of $\phi_{\mathfrak{c}}(D_{\mathfrak{c}}z(\mathfrak{c}))$ (respectively $\phi_{\mathfrak{c}}(D_{\mathfrak{c}}z_\alpha(\mathfrak{c}))$) in $H^1(K(\mathfrak{c}_p), T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$.

5.3.3 *Local properties of the derived classes.*

5.3.3.1 For $\mathfrak{c}\in\mathscr{S}$ and $\mathbf{Sp}$ a non-exceptional specialization, let $H^1_{f(\mathfrak{c})}(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$ be the Selmer group associated with the following local conditions.

(i) For v dividing p, let $T^+_{\mathbf{Sp}}$ be $U^+_v(T_{\mathbf{Sp}})$. Then $H^1_{f(\mathfrak{c})}(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$ is the image in $H^1(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$ of $H^1(K_v, T^+_{\mathbf{Sp}})$.

(ii) For v in Σ, let $H^1_{f(\mathfrak{c})}(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$ be the image of $H^1_{ur}(K_v, T_{\mathbf{Sp}})$ in $H^1(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$.

(iii) For v dividing $\mathfrak{c}$:

$$H^1_{f(\mathfrak{c})}(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}}) = H^1_{tr}(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}}).$$

LEMMA 5.16. *Let v be a place of K above p and $\mathbf{Sp}$ be a non-exceptional specialization. Let $\mathfrak{c}$ be in $\mathscr{S}$. Then $\mathrm{loc}_v\,\kappa(\mathfrak{c})$ belongs to $H^1_{f(\mathfrak{c})}(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$.*

Proof. Let G be the absolute Galois group of K_v and let H be its inertia subgroup. Let w be a place of $K(\mathfrak{c})$ above v and let $G_{\mathfrak{c}}$ be the absolute Galois group of $K(\mathfrak{c})_w$. Then H is also the inertia subgroup of $G_{\mathfrak{c}}$. Let M be $H^0(H, A^-_{\mathbf{Sp}})$. The module M is divisible so

$$H^1(G/H, M) = M(\mathrm{Fr}(v)-1)M = 0, \quad H^1(G_{\mathfrak{c}}/H, M) = M/(\mathrm{Fr}(w)-1)M = 0.$$

The Herbrand quotients of the cyclic groups G/H and $G_{\mathfrak{c}}/H$ acting on M are both equal to one so $H^0(G, A^-_{\mathbf{Sp}})$ and $H^0(G_{\mathfrak{c}}, A^-_{\mathbf{Sp}})$ are equal respectively to $N_G A^-_{\mathbf{Sp}}$ and $N_{G_{\mathfrak{c}}} A^-_{\mathbf{Sp}}$. Hence the norm map from $H^0(G_{\mathfrak{c}}, A^-_{\mathbf{Sp}})$ to $H^0(G, A^-_{\mathbf{Sp}})$ is onto. By Tate local duality, $H^2(K_v, T^+_{\mathbf{Sp}})$ thus injects into $H^2(K(\mathfrak{c})_w, T^+_{\mathbf{Sp}})$. Let $\kappa(\mathfrak{c})_v$ be the localization at v of $\kappa(\mathfrak{c})$ and let $\mathrm{res}\,\kappa(\mathfrak{c})_v$ be the restriction of $\kappa(\mathfrak{c})_v$ to $H^1(K(\mathfrak{c})_w, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$. In order to prove that $\kappa(\mathfrak{c})_v$ is in $H^1_{f(c)}(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$, it is enough by the above to show that $\mathrm{res}\,\kappa(\mathfrak{c})_v$ is in the image of $H^1(K(\mathfrak{c})_w, T^+_{\mathbf{Sp}})$. This is true by construction of $\kappa(\mathfrak{c})$ and Proposition 5.8. □

Remark. As in [How04a, Lemma 2.3.4], the proof above even shows that $\kappa(\mathfrak{c})_v$ is in the image of $H^1_{\mathrm{Iw}}(K_v, K(p^\infty)_w; T^+_{\mathbf{Sp}})$ but we will not need this fact.

Lemma 5.17. *Let $v \nmid p$ be a place of K. If $v \notin \Sigma$, then $\mathrm{loc}_v\,\kappa(\mathfrak{c})$ belongs to $H^1_{f(\mathfrak{c})}(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$ for all $\mathbf{Sp}$ and all $\mathfrak{c}$ not divided by v. If v belongs to Σ, let α be the non-zero element of Proposition 5.7. For all $\mathbf{Sp}$ such that $\mathbf{Sp}(\alpha) \neq 0$, there exists a non-zero β in S such that for all $\mathfrak{c} \in \mathscr{L}(x)$, the classes $\beta\kappa(\mathfrak{c})$ belong to $H^1_{f(\mathfrak{c})}(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$. If v has an infinite decomposition group in D_∞ or if $\mathcal{T}$ is minimally ramified at v, then $\mathrm{loc}_v\,\kappa(\mathfrak{c})$ belongs to $H^1_{f(\mathfrak{c})}(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$ for all $\mathfrak{c}$.*

Proof. Assume first that $v \notin \Sigma$ and let $\mathfrak{c}$ be in $\mathscr{S}$ with $v \nmid \mathfrak{c}$. The extension $K(\mathfrak{c})/K$ is unramified at v so the restriction of $\kappa(\mathfrak{c})$ to I_v is equal to the restriction of $D_{\mathfrak{c}}\,\mathfrak{z}(\mathfrak{c})$ which is zero by Proposition 5.7. Thus $\kappa(\mathfrak{c})$ belongs to $H^1_{ur}(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$. As $T_{\mathbf{Sp}}$ is unramified at v, the group $H^1_{ur}(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$ is equal to the image of $H^1_{ur}(K_v, T_{\mathbf{Sp}})$ inside $H^1(K_v, T_{\mathbf{Sp}})$ by [MR04, Lemma 1.1.9].

We assume henceforth that v belongs to Σ. The short exact sequence

$$0 \longrightarrow T^{I_v}_{\mathbf{Sp}} \longrightarrow T^{I_v}_{\mathbf{Sp}} \longrightarrow T^{I_v}_{\mathbf{Sp}}/I_{\mathfrak{c}}T^{I_v}_{\mathbf{Sp}} \longrightarrow 0$$

induces as in the following diagram.

$$H^1(K^{ur}_v/K_v, T^{I_v}_{\mathbf{Sp}}) \longrightarrow H^1(K^{ur}_v/K_v, T^{I_v}_{\mathbf{Sp}}/I_{\mathfrak{c}}T^{I_v}_{\mathbf{Sp}}) \longrightarrow H^2(K^{ur}_v/K_v, T^{I_v}_{\mathbf{Sp}})$$

As the last group is zero, in order to prove that $\mathrm{loc}_v\,\kappa(\mathfrak{c})$ is in the image of $H^1_{ur}(K_v, T_{\mathbf{Sp}})$, it is enough to prove that $\mathrm{loc}_v\,\kappa(\mathfrak{c})$ is in $H^1(K^{ur}_v/K_v, T^{I_v}_{\mathbf{Sp}}/I_{\mathfrak{c}}T^{I_v}_{\mathbf{Sp}})$. Let w be a place of $K(\mathfrak{c})$ above v and $z(\mathfrak{c})_w$ be the restriction of $z(\mathfrak{c})$ to $G_{K(\mathfrak{c})_w}$.

First assume that v has an infinite decomposition group in D_∞ or that $\mathcal{T}$ is minimally ramified at v. According to Proposition 5.7, the class $z(\mathfrak{c})_w$ is then unramified. The class $D_{\mathfrak{c}}z(\mathfrak{c})_w$ then belongs to the group $H^1(K^{nr}_v/K(\mathfrak{c})_w, T^{I_v}_{\mathbf{Sp}})$ and thus the class $\phi_{\mathfrak{c}}(D_{\mathfrak{c}}z(\mathfrak{c})_w)$ belongs to $H^1(K^{nr}_v/K(\mathfrak{c})_w, T^{I_v}_{\mathbf{Sp}}/I_{\mathfrak{c}}T^{I_v}_{\mathbf{Sp}})$. Viewing this last group as a subgroup of $H^1(K(\mathfrak{c})_w, T^{I_v}_{\mathbf{Sp}}/I_{\mathfrak{c}}T^{I_v}_{\mathbf{Sp}})$ establishes:

$$\phi_{\mathfrak{c}}(D_{\mathfrak{c}}z(\mathfrak{c})_w) \in H^1(K(\mathfrak{c})_w, T^{I_v}_{\mathbf{Sp}}/I_{\mathfrak{c}}T^{I_v}_{\mathbf{Sp}}).$$

The natural map

$$H^1(K(\mathfrak{c})_w, T^{I_v}_{\mathbf{Sp}}/I_{\mathfrak{c}}T^{I_v}_{\mathbf{Sp}}) \longrightarrow H^1(K(\mathfrak{c})_w, T/I_{\mathfrak{c}}T)$$

allows us to consider the image of $\phi_{\mathfrak{c}}(D_{\mathfrak{c}}z_w(\mathfrak{c}))$ in $H^1(K(\mathfrak{c})_w, T/I_{\mathfrak{c}}T)$. The following diagram is commutative.

$$\begin{array}{ccccc} H^1(K, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}}) & \xrightarrow{\text{loc}} & H^1(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}}) & & \\ \downarrow{\scriptstyle\text{res}} & & \downarrow{\scriptstyle\text{res}} & & \\ H^1(K(\mathfrak{c}), T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}}) & \xrightarrow{\text{loc}} & H^1(K(\mathfrak{c})_w, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}}) & \longleftarrow & H^1(K(\mathfrak{c})_w, T_{\mathbf{Sp}}^{I_v}/I_{\mathfrak{c}}T_{\mathbf{Sp}}^{I_v}) \end{array}$$

The image of $\kappa(\mathfrak{c})$ under localization at v followed by restriction coincides by definition with the image of $\phi_{\mathfrak{c}}(D_{\mathfrak{c}}z_w(\mathfrak{c}))$. Thus, it is in $H^1(K_v^{nr}/K(\mathfrak{c})_w, T_{\mathbf{Sp}}^{I_v}/I_{\mathfrak{c}}T_{\mathbf{Sp}}^{I_v})$ and its restriction to I_v is zero. It follows that $\text{loc}_v\,\kappa(\mathfrak{c})$ belongs to $H^1(K_v^{nr}/K_v, T_{\mathbf{Sp}}^{I_v}/I_{\mathfrak{c}}T_{\mathbf{Sp}}^{I_v})$ and consequently to $H^1_f(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$.

We now remove supplementary assumptions on v. According to Proposition 5.8, there exists a non-zero α in R such that $\alpha z(\mathfrak{c})_w$ is unramified for all $\mathfrak{c}$. The same proof with $z(\mathfrak{c})$ replaced by $\alpha z(\mathfrak{c})$ shows that $\mathbf{Sp}(\alpha)\kappa(\mathfrak{c})$ is in $H^1_f(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$. □

Lemma 5.18. *Let $\mathfrak{c}$ be in $\mathscr{S}$. Let λ be the unique place above l a place of F dividing $\mathfrak{c}$ and let* $\mathbf{Sp}$ *be a specialization. Then $\text{loc}_\lambda\,\kappa(\mathfrak{c})$ belongs to $H^1_{tr}(K_\lambda, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$.*

Proof. This is a very classical computation due to Kolyvagin. In this context, see for instance [Fou07, Proposition 2.2.18]. □

5.3.3.2 Let $\mathfrak{c}l$ be in $\mathscr{S}$ and λ be the unique place of K above l. Let $\mathbf{Sp}$ be a specialization with values in S. According to Lemmas 5.17 and 5.18, the classes $\kappa(\mathfrak{c}l)_\lambda$ and $\kappa(\mathfrak{c})_\lambda$ satisfy:

$$\text{loc}_\lambda\,\kappa(\mathfrak{c}) \in H^1_f(K_\lambda, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}}), \quad \text{loc}_\lambda\,\kappa(\mathfrak{c}l) \in H^1_{tr}(K_\lambda, T_{\mathbf{Sp}}/I_{\mathfrak{c}l}T_{\mathbf{Sp}}).$$

The various isomorphisms of § 5.3.2.2 induce:

$$H^1_f(K_\lambda, T_{\mathbf{Sp}}/I_{\mathfrak{c}l}T_{\mathbf{Sp}}) \xrightarrow{\sim} T_{\mathbf{Sp}}/I_{\mathfrak{c}l}T_{\mathbf{Sp}} \xleftarrow{\sim} H^1_{tr}(K_\lambda, T_{\mathbf{Sp}}/I_{\mathfrak{c}l}T_{\mathbf{Sp}}).$$

The first isomorphism is given by evaluation at the Frobenius morphism $\text{Fr}(l)$ while the second is evaluation at σ_l.

Lemma 5.19. *The image of $\kappa(\mathfrak{c})$ in $T_{\mathbf{Sp}}/I_{\mathfrak{c}l}T_{\mathbf{Sp}}$ is equal to the image of $-\text{Fr}(l)\kappa(\mathfrak{c}l)$.*

Proof. This results from an examination of Kolyvagin's derivative, as conducted for instance in [Nek92, § 7]. □

5.3.4 *Kolyvagin systems for $T_{\mathbf{Sp}}$.*

Definition 5.20 (Bloch–Kato local condition). Let L be a finite extension of K and $\mathbf{Sp}$ a non-exceptional S-specialization. Define the local compact (respectively usual, respectively discrete) Bloch–Kato Selmer group $H^1_{BK}(L_v, T_{\mathbf{Sp}})$ (respectively $H^1_{BK}(L_v, V_{\mathbf{Sp}})$, respectively $H^1_{BK}(L_v, A_{\mathbf{Sp}})$) to be the pre-image (respectively the image, respectively the image) in $H^1(L_v, T_{\mathbf{Sp}})$ (respectively in $H^1(L_v, V_{\mathbf{Sp}})$, respectively in $H^1(L_v, A_{\mathbf{Sp}})$) of $H^1(L_v, U_v^+(V_{\mathbf{Sp}}))$ if $v|p$ and to be the pre-image (respectively the image, respectively the image) of $H^1_{ur}(L_v, V_{\mathbf{Sp}})$ in $H^1(L_v, T_{\mathbf{Sp}})$ (respectively in $H^1(L_v, V_{\mathbf{Sp}})$, respectively in $H^1(L_v, A_{\mathbf{Sp}})$) if $v \in \Sigma\backslash\{p\}$. For $X = T_{\mathbf{Sp}}, V_{\mathbf{Sp}}, A_{\mathbf{Sp}}$, define the Bloch–Kato Selmer group by:

$$H^1_{BK}(L_\Sigma/L, X) = \ker\left(H^1(L_\Sigma/L, X) \longrightarrow \bigoplus_{v\in\Sigma} H^1(L_v, X)/H^1_{BK}(L_v, X)\right).$$

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When $\mathbf{Sp}$ is arithmetic, [BK90] defines subgroups $H^1_f(L_\Sigma/L, T_{\mathbf{Sp}})$ and $H^1_f(L_\Sigma/L, V_{\mathbf{Sp}})$ with a seemingly different condition at $v|p$ coming from p-adic Hodge theory. When the subgroups they define and those of Definition 5.20 are both defined, they are equal, for instance by [Nek06, Proposition 12.5.9.2].

For $\mathfrak{c}\in\mathscr{S}$, let $BK(\mathfrak{c})$ be the condition on $H^1(K, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$ locally defined to be equal to the propagation of BK from $H^1(K_v, T_{\mathbf{Sp}})$ to $H^1(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$ for $v\nmid\mathfrak{c}$ and equal to $H^1_{tr}(K_v, T_{\mathbf{Sp}}/I_{\mathfrak{c}}T_{\mathbf{Sp}})$ for $v|\mathfrak{c}$. The following theorem and corollary are the generalization to our context of the results of [How04a, How04b], themselves generalizing the classical results of [Kol90, Theorem 10].

Theorem 5.2. *Let $\alpha\in R$ be the element of Theorem 5.1. Let $\mathbf{Sp}$ be a non-exceptional specialization with values in a discrete valuation ring flat over $\mathbb{Z}_p$. Then the system of classes $\{\kappa_\alpha(\mathfrak{c})\}_{\mathfrak{c}\in\mathscr{S}}$ is a Kolyvagin system for $T_{\mathbf{Sp}}$ and the condition BK. Under Assumption 5.10, the system of classes $\{\kappa(\mathfrak{c})\}_{\mathfrak{c}\in\mathscr{S}}$ is a Kolyvagin system for $T_{\mathbf{Sp}}$ and the condition BK.*

Proof. This is a restatement of Lemmas 5.16–5.19. □

Corollary 5.21. *Let $\mathbf{Sp}$ be a non-exceptional specialization with values in a discrete valuation ring S flat over $\mathbb{Z}_p$. Let $z_{S,\alpha}$ be the class of the Kolyvagin system of Theorem 5.2 of conductor 1. If $z_{S,\alpha}$ is not zero, then $H^1_{BK}(K, T_{\mathbf{Sp}})$ is free of rank 1 and there exists a torsion module M of finite type over S with*

$$\ell_S M \leqslant \ell_S(H^1_{BK}(K, T_{\mathbf{Sp}})/z_{S,\alpha}) \tag{5.3.1}$$

such that:

$$H^1_{BK}(K, A_{\mathbf{Sp}}) = \mathrm{Frac}(S)/S \oplus M \oplus M.$$

Proof. We show that $T_{\mathbf{Sp}}$ and the condition BK satisfy the five hypotheses of [How04b, § 2.2]. Hypotheses 3 to 5 are satisfied for exactly the same reasons as in [How04b]. Hypothesis 2 is satisfied thanks to Assumption 3.10. Let us show that Hypothesis 1 holds. Let $G_{\bar\rho}$ be the image of $\bar\rho$ inside $\mathrm{GL}_2(S/\mathfrak{m}_S)$. According to [How04b, Proof of Theorem 2.3.7], it is enough to show that $H^1(G_{\bar\rho}, T_{\mathbf{Sp}}/\mathfrak{m}_S)$ vanishes. This is the case if $p\nmid|G_{\bar\rho}|$ so we can assume that $p||G_{\bar\rho}|$. According to Assumptions 3.4 and 5.13, the group $G_{\bar\rho}$ acts irreducibly on $T_{\mathbf{Sp}}/\mathfrak{m}_S$ and so is not contained in a Borel subgroup. Dickson's classification of subgroups of $\mathrm{GL}_2(\mathbb{F}_q)$, as given for instance in [Suz82, Theorem 6.21], then shows that the group $G_{\bar\rho}$ contains a non-trivial homothety. This shows that $H^1(G_{\bar\rho}, T_{\mathbf{Sp}}/\mathfrak{m}_S)$ vanishes by Sah's lemma [Sah68, Proposition 2.7 (b)]. Thanks to the axiomatic system of [How04a, § 1] and [How04b, § 2], we conclude by [How04b, Theorem 2.2.2]. □

6. Iwasawa theory

6.1 Determinants of Selmer complexes for specializations of R_{Iw}

6.1.1 *Determinants.* We briefly review the formalism of determinants as described first in [KM76], see for instance [Kat93, § 2.1] and [BG03, § 2] for definitions and further properties. The reader is advised that we use the normalization of [Kat93, § 2.1] and not that of [BG03, § 2]. We also recall that there is a notorious misprint on [KM76, p. 20] regarding signs. Let S be a complete local reduced Noetherian ring. The determinant Det_S is then defined to be the functor

$$\mathrm{Det}_S P = \left(\bigwedge_S^{\mathrm{rank}_S P} P, \mathrm{rank}_S P\right)$$

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from the category of finite free S-modules to the category of graded invertible S-modules. We write S for the graded invertible module $(S,0)$ in what follows.

Assume that a in the total ring of fraction of S belongs to $S_{\mathcal{P}}$ after localization at all primes $\mathcal{P}$ of grade 1. Then the ideal $I_a=\{s\in S\mid sa\in S\}$ is not contained in a prime ideal of grade 1 and hence is of grade at least 2. Then $\mathrm{Ext}^1_S(S/I_a,S)$ vanishes so the natural map from $\mathrm{Hom}_S(S,S)$ to $\mathrm{Hom}(I_a,S)$ is onto. As multiplication by a is in $\mathrm{Hom}(I_a,S)$, this implies that a belongs to S. Invertible ideals are thus determined by their localization at grade 1 primes. Consequently, invertible ideals are determined by their localizations at height 1 primes when S is Cohen–Macaulay.

A complex of S-modules C is said to be perfect if there exists a quasi-isomorphism between C and a bounded complex of projective (hence free) S-modules of finite type. If C is a perfect complex of S-modules, the S-module $\mathrm{Det}_S\,C$ is defined to be

$$\mathrm{Det}_S\,C=\bigotimes_{i\in\mathbb{Z}}\mathrm{Det}_S^{(-1)^i}\,C^i \tag{6.1.1}$$

where the C^i are chosen finite and free. The determinant functor extends in this way to a functor from the derived category of perfect complexes of S-modules with morphisms restricted to quasi-isomorphisms to the category of graded invertible S-modules. If

$$C_1\longrightarrow C_2\longrightarrow C_3$$

is a distinguished triangle of complexes in the derived category, then $\mathrm{Det}_S\,C_2$ is canonically and functorially isomorphic to $\mathrm{Det}_S\,C_1\otimes\mathrm{Det}_S\,C_3$ by [KM76, Proposition 7] (as was noticed first by D. Ferrand, this property is not true if S is not reduced). If an S-module is quasi-isomorphic to a perfect complex, we say it is perfect. If $H^i(C)$ is perfect for all i, for instance if S is regular, then there is a canonical isomorphism

$$\mathrm{Det}_S\,C\overset{\sim}{\longrightarrow}\bigotimes_{i\in\mathbb{Z}}\mathrm{Det}_S^{(-1)^i}\,H^i(C) \tag{6.1.2}$$

which we take to be an identification.

A good map between perfect complexes of S-modules in the sense of Knudsen–Mumford [KM76, p. 47] is a map between complexes which becomes an isomorphism after localization at all minimal prime ideals.

6.1.2 *Trivialization.* Let S be a domain and $\mathcal{K}$ its field of fraction.

Let $C=[M_0\overset{\phi_0}{\longrightarrow}M_1\overset{\phi_1}{\longrightarrow}M_2]$ be a perfect complex of S-modules concentrated in degree $[0,3]$. Assume that the cohomology of C is non-zero only in degree 1 and 2 and that $H^1(C)$ and $H^2(C)$ have the same rank d. Then $C\overset{\mathrm{L}}{\otimes}_S\mathcal{K}$ can be represented by a complex $[V\longrightarrow V]$ and its determinant is thus isomorphic to $\mathcal{K}$ through the isomorphism (6.1.1). We call this identification the Id-trivialization of $\mathrm{Det}_{\mathcal{K}}(C\overset{\mathrm{L}}{\otimes}_S\mathcal{K})$. Inside $\mathrm{Det}_{\mathcal{K}}(C\overset{\mathrm{L}}{\otimes}_S\mathcal{K})$, we identify $\mathrm{Det}_S\,0$ with S. Because Det_S factors through the derived category, if M is a free module of finite rank and ψ is a surjective morphism such that $\mathrm{Det}_S[M\overset{\psi}{\longrightarrow}M]\subset\mathrm{Det}_{\mathcal{K}}(C\overset{\mathrm{L}}{\otimes}_S\mathcal{K})$, the identification of $\mathrm{Det}_S\,0$ with S also identifies the determinant of the complex $[M\overset{\psi}{\longrightarrow}M]$ inside $\mathcal{K}$ with S. If C can be represented by a complex $C=[M\overset{\psi}{\longrightarrow}M]$ with $\ker\psi$ a direct summand of M as $S[\psi]$-module, then choosing an S-basis of $\ker\psi$ and $\operatorname{coker}\psi$ produces a map $[S^d\overset{\mathrm{Id}}{\longrightarrow}S^d]\longrightarrow C$ which becomes an isomorphism of complexes after tensor product with $\mathcal{K}$, so is good in the sense of Knudsen–Mumford. To this map is associated a Cartier divisor $\mathrm{Div}(C)$ on $\mathrm{Spec}\, S$ by the method of [KM76, p. 47].

Assume now that S is a discrete valuation ring with uniformizing parameter ϖ_S. Then the complex C can be represented by $[S^d \xrightarrow{\psi_1} S^d] \oplus [N \xrightarrow{\psi_2} N]$ with ψ_1 an isomorphism and ψ_2 an injection with torsion cokernel. Consequently, the image of $\mathrm{Det}_S\, C$ inside $\mathcal{K}$ through Id-trivialization is equal to $\varpi_S^{-\ell_S(\mathrm{coker}\,\psi_2)}S$. We thus see that the image of $\mathrm{Det}_S\, C$ in $\mathcal{K}$ through Id-trivialization coincides with the invertible ideal generated by the inverse of $\mathrm{Div}(C)$. This implies that the image of the determinant of the S-module $\varpi_S^a S$ seen as a complex placed in degree 0 is equal to $\varpi_S^a S$ for all $a \in \mathbb{N}$.

Assume more generally that S is a normal Cohen–Macaulay domain; for instance that S is a regular ring. Then height 1 and grade 1 primes coincide. If M is a torsion S-module, the ideal $\mathrm{char}_S\, M$ is defined to be:

$$\mathrm{char}_S\, M = \prod_{\mathrm{ht}\,\mathcal{P}=1} \mathcal{P}^{\ell_{S_\mathcal{P}} M_\mathcal{P}}. \tag{6.1.3}$$

The image of $\mathrm{Det}_S\, C$ inside $\mathcal{K}$ through Id-trivialization is equal to $(\mathrm{char}_S\, H^2(C)_{\mathrm{tors}})^{-1}S$. Indeed, both the image of $\mathrm{Det}_S\, C$ inside $\mathcal{K}$ through Id-trivialization and $(\mathrm{char}_S\, H^2(C)_{\mathrm{tors}})^{-1}S$ are uniquely determined by their localizations at height 1 primes, after which S can be assumed to be a discrete valuation ring. In this case, the results of the previous paragraph apply.

We now drop our extra assumptions on S but assume that C can be represented by a complex $[M \xrightarrow{\psi} M]$ in degree 1,2 with M free of finite rank and such that $\ker\psi$ is a direct summand of M as $S[\psi]$-module. Then C can be represented by a complex $[\ker\psi \longrightarrow \ker\psi] \oplus [N \xrightarrow{\psi^*} N]$ with ψ^* injective so the image of $\mathrm{Det}_S\, C$ inside $\mathcal{K}$ through Id-trivialization is equal to $(\det\psi^*)^{-1}S$. This holds in particular if M is of rank at most 1 or if ψ is an injection.

6.1.3 *Selmer complexes for specializations of R_{Iw}.* Let Λ_a^{tw} be $\Lambda^{tw}[[\mathrm{Gal}(D_\infty/K)]]$. Let S be a specialization of R_{Iw} or of Λ_a^{tw}. Let T be the corresponding S-specialization of $\mathcal{T}_{\mathrm{Iw}}$ (seen either as R_{Iw}-module or as Λ_a^{tw}-module) and $\mathcal{K}$ the fraction field of S. In what follows, the determinant $\mathrm{Det}_S\, \mathrm{R}\,\Gamma_f(K_\Sigma/K, T)$ plays a crucial role. When the complex $\mathrm{R}\,\Gamma_f(K_\Sigma/K, T)$ is not perfect, we therefore modify it.

Definition 6.1 (Modified Selmer complexes). Let L be a finite extension of K. The modified local condition at $v \nmid p$ for complexes is given by the choice of the zero complex. Let $\mathrm{R}\,\Gamma_c(L_\Sigma/L, T)$ be the corresponding Selmer complexes and $\tilde{H}^i_{f,c}(L_\Sigma/L, T)$ its cohomology groups.

We collect necessary facts about $\mathrm{R}\,\Gamma_f(K_\Sigma/K, T)$ and $\mathrm{R}\,\Gamma_c(K_\Sigma/K, T)$.

Proposition 6.2. (i) *The complexes $\mathrm{R}\,\Gamma_f(K_\Sigma/K, T)$ and $\mathrm{R}\,\Gamma_c(K_\Sigma/K, T)$ are acyclic outside degree* $1, 2$.

(ii) *The S-modules $\tilde{H}^1_f(K_\Sigma/K, T)$ and $\tilde{H}^1_{f,c}(K_\Sigma/K, T)$ are torsion-free.*

(iii) *The complex $\mathrm{R}\,\Gamma_c(K_\Sigma/K, T)$ is perfect with amplitude $[0, 2]$. If T^{I_v} is a free S-module for all $v \nmid p$ or if S is regular, the complex $\mathrm{R}\,\Gamma_f(K_\Sigma/K, T)$ is perfect with amplitude $[0, 2]$.*

(iv) *The S-modules $\tilde{H}^1_f(K_\Sigma/K, T)$ and $\tilde{H}^2_f(K_\Sigma/K, T)$ have the same S-ranks. The same is true of the S-modules $\tilde{H}^1_{f,c}(K_\Sigma/K, T)$ and $\tilde{H}^2_{f,c}(K_\Sigma/K, T)$.*

(v) *The complexes $\mathrm{R}\,\Gamma_f(K_\Sigma/K, T \otimes_S \mathcal{K})$ and $\mathrm{R}\,\Gamma_c(K_\Sigma/K, T \otimes_S \mathcal{K})$ are isomorphic except possibly if there is a $v \in \Sigma$ such that $T^{I_v} \neq 0$ and $\mathrm{Fr}(v)$ has an eigenvalue equal to* 1.

Proof. (i) The p-cohomological dimension of G_{K_v} and $G_{K,\Sigma}$ is 2 so $\mathrm{R}\,\Gamma_f(K_\Sigma/K,T)$ is acyclic outside $[0,3]$. The S-module $\tilde{H}^0_f(K_\Sigma/K,T)$ is included in the $G_{K,\Sigma}$-invariants of T so is zero. As $\bar{\rho}_f$ is absolutely irreducible, the Pontryagin dual $D(T)$ of T also has no $G_{K,\Sigma}$-invariants and thus $\tilde{H}^3_f(K_\Sigma/K,T)$ is zero by [Nek06, (8.9.6.1)]. The complex $\mathrm{R}\,\Gamma_f(K_\Sigma/K,T)$ is thus acyclic outside $i=1,2$. The exact same proof with [Nek06, (8.9.6.1)] replaced by [Nek06, Theorem 5.4.5] establishes the same assertions for $\mathrm{R}\,\Gamma_c(K_\Sigma/K,T)$.

(ii) By (5.1.4), it is enough to show that $H^1_{Gr}(K_\Sigma/K,T)$ is torsion-free, and so to show that $H^1(K_\Sigma/K,T)$ is torsion-free. Let $x\in S$ be a non-zero element. The complex $\mathrm{R}\,\Gamma(K_\Sigma/K,T)$ descends perfectly with respect to x so $H^1(K_\Sigma/K,T)[x]$ is the cokernel of the map:

$$H^0(K_\Sigma/K,T)/xH^0(K_\Sigma/K,T)\longrightarrow H^0(K_\Sigma/K,T/x)$$

and so vanishes by irreducibility of T and T/x.

(iii) When G has bounded cohomological dimension, the functor $\mathrm{R}\,\Gamma(G,-)$ takes perfect complexes to perfect complexes, see for instance [Nek06, Proof of Proposition 4.2.9] or [Kat93, Theorem 3.1.3]. By construction, the complex $\mathrm{R}\,\Gamma_c(K_\Sigma/K,T)$ is the cone of a morphism between complexes of the form $\mathrm{R}\,\Gamma(G,N)$ with N a free module. Thus, it is perfect. If T^{I_v} is free for all $v\nmid p$, the complex $\mathrm{R}\,\Gamma_f(K_\Sigma/K,T)$ is likewise the cone of a morphism between complexes of the form $\mathrm{R}\,\Gamma(G,N)$ with N a free module. If S is regular, the complex $\mathrm{R}\,\Gamma_f(K_\Sigma/K,T)$ is perfect by the theorem of Auslander–Buchsbaum and Serre. The statement about amplitudes follows from (i).

(iv) The equality of $\operatorname{rank}\tilde{H}^1_f(K_\Sigma/K,T)$ and $\operatorname{rank}\tilde{H}^2_f(K_\Sigma/K,T)$ is [Nek06, (8.9.6.2) and (8.9.6.4.3)] together with self-duality of T. The equality of $\operatorname{rank}\tilde{H}^1_{f,c}(K_\Sigma/K,T)$ and $\operatorname{rank}\tilde{H}^2_{f,c}(K_\Sigma/K,T)$ is [Nek06, Corollary 7.8.7] together with the self-duality and the oddness of T.

(v) If for all $v\nmid p$, either T^{I_v} is zero or $\mathrm{Fr}(v)-1$ is injective, then $\mathrm{R}\,\Gamma_f(K_v,T\otimes_S\mathcal{K})$ is the zero complex for all v so $\mathrm{R}\,\Gamma_f(K_\Sigma/K,T\otimes_S\mathcal{K})$ and $\mathrm{R}\,\Gamma_c(K_\Sigma/K,T\otimes_S\mathcal{K})$ are equal. □

For ϕ a morphism of $\mathcal{O}$-algebras between S and S', we say that $\mathrm{R}\,\Gamma_c(K_\Sigma/K,T)$ descends perfectly with respect to ϕ if $\mathrm{R}\,\Gamma_c(K_\Sigma/K,T)\overset{\mathrm{L}}{\otimes}_{S,\phi}S'$ is isomorphic to $\mathrm{R}\,\Gamma_c(K_\Sigma/K,T')$.

LEMMA 6.3. *Let $\phi:S\longrightarrow S'$ be a morphism between two specializations of R_{Iw} or of Λ^{tw}_a. Let T' be $T\otimes_{S,\phi}S'$.*

(i) *The complex $\mathrm{R}\,\Gamma_c(K_\Sigma/K,T)$ descends perfectly with respect to ϕ.*

(ii) *If the complex $\mathrm{R}\,\Gamma_f(K_v,T)$ descends perfectly with respect to ϕ for all $v\nmid p$, then the complex $\mathrm{R}\,\Gamma_f(K_\Sigma/K,T)$ descends perfectly with respect to ϕ. This is the case if $T^{I_v}\otimes_{S,\phi}S'$ is equal to $(T')^{I_v}$.*

Proof. (i) The complex $\mathrm{R}\,\Gamma_c(K_\Sigma/K,T)$ is the cone of a morphism of complexes between complexes which all descend perfectly with respect to ϕ by (5.1.2). Hence, it descends perfectly with respect to ϕ.

(ii) Assume that $\mathrm{R}\,\Gamma_f(K_v,T)$ descends perfectly with respect to ϕ for all $v\nmid p$. Then $\mathrm{R}\,\Gamma_f(K_\Sigma/K,T)$ descends perfectly with respect to ϕ by the same arguments as in the proof of the first assertion. If $T^{I_v}\otimes_{S,\phi}S'$ is equal to $(T')^{I_v}$, then $\mathrm{R}\,\Gamma_f(K_v,T)$ descends perfectly with respect to ϕ by (5.1.2). □

PROPOSITION 6.4. *Let α be such that $z_{\infty,\alpha}$ belongs to $H^1_{Gr}(K_\Sigma/K,\mathcal{T}_{\mathrm{Iw}})$. Let $z_{S,\alpha}$ be the image of $z_{\infty,\alpha}$ under the map from $H^1_{Gr}(K_\Sigma/K,\mathcal{T}_{\mathrm{Iw}})$ to $H^1(K_\Sigma/K,T)$. There exists a*

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unique $\tilde{z}_{S,\alpha}\in\tilde{H}^1_f(K_\Sigma/K,T)$ *such that the natural projection from* $\tilde{H}^1_f(K_\Sigma/K,T)$ *to* $H^1_f(K_\Sigma/K,T)$ *sends* $\tilde{z}_{S,\alpha}$ *to* $z_{S,\alpha}$. *Under Assumption 5.10, this assertion is true with* α *equal to* 1.

Proof. As remarked after Definition 5.5, the class $z_{S,\alpha}$ belongs to $H^1_{Gr}(K_\Sigma/K,T)$. The short exact sequence

$$0\longrightarrow\bigoplus_{v|p}H^0(K_v,U_v(T^-))\longrightarrow\tilde{H}^1_f(K_\Sigma/K,T)\longrightarrow H^1_{Gr}(K_\Sigma/K,T)\longrightarrow 0$$

shows that $z_{S,\alpha}$ belongs to the image of $\tilde{H}^1_f(K_\Sigma/K,T)$ inside $H^1_{Gr}(K_\Sigma/K,T)$. Because $z_{S,\alpha}$ is a universal norm under corestriction in the extension D_∞/K and because $H^0_{\mathrm{Iw}}(K,D_\infty;U_v^-(T))$ vanishes for all $v|p$, there is a unique pre-image $\tilde{z}_{S,\alpha}$ to $z_{S,\alpha}$. Under Assumption 5.10, z_∞ belongs to $H^1_{Gr}(K_\Sigma/K,\mathcal{T}_{\mathrm{Iw}})$ so these assertions hold with α equal to 1. □

The definition of $z_{S,\alpha}$ given in the proposition coincides with the definition given in Corollary 5.21 when S is a discrete valuation ring flat over $\mathbb{Z}_p$. Because $\tilde{z}_{S,\alpha}$ is uniquely determined by $z_{S,\alpha}$, we denote both classes by the same symbol in the following. Proposition 6.4 applies in particular for α as in Theorem 5.1.

Proposition 6.5. *Let* r *be the number of place* $v|p$ *such that* $U_v^-(T)$ *is invariant under the action of* G_{K_v}. *If* S *is a discrete valuation ring or of dimension at least* 2 *and if* $z_{S,\alpha}$ *is not zero, then the common rank of* $\tilde{H}^1_f(K_\Sigma/K,T)$ *and* $\tilde{H}^2_f(K_\Sigma/K,T)$ *is equal to* $1+r$.

Proof. Assume S to be a discrete valuation ring, then $\tilde{H}^1_f(K_\Sigma/K,T)$ is free of rank $1+r$ by Corollary 5.21 and Proposition 6.2.

Now assume S is of dimension $d\geqslant 2$. The ring S is a complete local Noetherian domain. Consequently, the regular locus of S is a non-empty open set by [Gro65, Lemme (6.12.4.1)], and hence is Zariski-dense. From this statement and the Hauptidealsatz, there exists a prime ideal $\mathcal{P}$ of height $d-1$ such that $S_\mathcal{P}$ is regular, such that the rank of T^{I_v} is equal to the rank of $(T_\mathcal{P}/\mathcal{P})^{I_v}$ for all $v\nmid p$, such that $pz_{S,\alpha}\notin\mathcal{P}$ and such that $\mathcal{P}$ does not belong to the support of $\tilde{H}^2_f(K_\Sigma/K,T)_{\mathrm{tors}}$. The complex $\mathrm{R}\,\Gamma_f(K_\Sigma/K,T_\mathcal{P})$ then descends perfectly with respect to $\mathcal{P}$ by Lemma 6.3. Hence $\tilde{H}^2_f(K_\Sigma/K,T_\mathcal{P})/\mathcal{P}$ is isomorphic to $\tilde{H}^2_f(K_\Sigma/K,T_\mathcal{P}/\mathcal{P})$. Because $S_\mathcal{P}/\mathcal{P}$ is a finite extension of $\mathbb{Q}_p$, the dimension of $H^1_{Gr}(K_\Sigma/K,T_\mathcal{P}/\mathcal{P})$ is equal to 1 by Corollary 5.21. Because $U_v^-(T_\mathcal{P}/\mathcal{P})$ is of dimension 1, its G_{K_v}-invariants are either 0 or itself. Hence, the dimension of $\tilde{H}^1_f(K_\Sigma/K,T_\mathcal{P}/\mathcal{P})$ is equal to $1+r$ by (5.1.4). Consequently, the $S_\mathcal{P}$-module $\tilde{H}^1_f(K_\Sigma/K,T)_\mathcal{P}$ is generated by $1+r$ elements. As it is torsion-free, it is free of rank $1+r$. Hence, the rank of $\tilde{H}^i_f(K_\Sigma/K,T)$ is equal to $1+r$ for $i=1,2$. □

6.1.4 *Integral structures.* We say that T or S is a good specialization if $\mathrm{R}\,\Gamma_f(K_\Sigma/K,T\otimes_S\mathcal{K})$ and $\mathrm{R}\,\Gamma_c(K_\Sigma/K,T\otimes_S\mathcal{K})$ are isomorphic. According to the last assertion of Propositions 6.2 and 5.3, arithmetic specializations are good specializations. Let S be a good specialization. According to Proposition 6.2, there exist free S-modules C_0,C_1,C_2 such that the complex $\mathrm{R}\,\Gamma_c(K_\Sigma/K,T)$ can be represented by $[C_0\xrightarrow{\phi}C_1\xrightarrow{\psi}C_2]$ in degree $0,1,2$. Let V be $C_2\otimes_S\mathcal{K}$. Then $C_1\otimes_S\mathcal{K}$ splits in a direct summand isomorphic to the $\mathcal{K}$-vector space generated by $\operatorname{im}\phi$ and a vector space V' of the same dimension as V. A choice of a basis b of V induces a choice

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of a basis of V' by taking the dual basis for the duality:

$$\mathrm{R}\,\Gamma_c(K_\Sigma/K, T\otimes_S\mathcal{K}) \xrightarrow{\sim} D(\mathrm{R}\,\Gamma_c(K_\Sigma/K, T\otimes_S\mathcal{K}))[-3].$$

Here, we use the isomorphism between $\mathrm{R}\,\Gamma_f(K_\Sigma/K, T\otimes_S\mathcal{K})$ and $\mathrm{R}\,\Gamma_c(K_\Sigma/K, T\otimes_S\mathcal{K})$. Hence, we can identify V and V' and the discussion of § 6.1.2 applies.

Definition 6.6 (Integral structure). An integral structure X on $\mathrm{Det}_{\mathcal{K}}\,\mathrm{R}\,\Gamma_f(K_\Sigma/K, T\otimes_S\mathcal{K})$ is a couple (M, triv) where triv is an identification of $\mathrm{Det}_{\mathcal{K}}\,\mathrm{R}\,\Gamma_f(K_\Sigma/K, T\otimes_S\mathcal{K})$ with $\mathcal{K}$ and M is a S-module of rank 1 inside $\mathrm{Det}_{\mathcal{K}}\,\mathrm{R}\,\Gamma_f(K_\Sigma/K, T\otimes_S\mathcal{K})$ identified with $\mathcal{K}$. To emphasize the dependency on T, we sometimes write $X_T = (M_T, \mathrm{triv}_T)$.

We consider the three following integral structures. For each $v\in\Sigma$ not dividing p, let M_v be a free sub-module of T^{I_v} of maximal rank. The $\mathcal{K}$-vector space

$$\mathrm{Det}_{\mathcal{K}}\,\mathrm{R}\,\Gamma_f(K_\Sigma/K, T\otimes_S\mathcal{K})$$

is canonically isomorphic to

$$\mathrm{Det}_{\mathcal{K}}\,\mathrm{R}\,\Gamma_c(K_\Sigma/K, T\otimes_S\mathcal{K}) \otimes \bigotimes_{v\in\Sigma\setminus\{p\}} \mathrm{Det}_{\mathcal{K}}\,\mathrm{R}\,\Gamma(\mathrm{Fr}(v), T^{I_v}\otimes_S\mathcal{K}).$$

Let Id be the identification of

$$\mathrm{Det}_{\mathcal{K}}\,\mathrm{R}\,\Gamma_c(K_\Sigma/K, T\otimes_S\mathcal{K}) \otimes \bigotimes_{v\in\Sigma\setminus\{p\}} \mathrm{Det}_{\mathcal{K}}\,\mathrm{R}\,\Gamma(\mathrm{Fr}(v), T^{I_v}\otimes_S\mathcal{K})$$

with $\mathcal{K}$ given by Id-trivialization on each term of the tensor product.

Definition 6.7 (Canonical structure). The canonical structure is the choice of $(M_{\mathrm{can}}, \mathrm{Id})$ where M_{can} is the image of

$$\mathrm{Det}_S\,0 \subset \mathrm{Det}_S\,\mathrm{R}\,\Gamma_c(K_\Sigma/K, T) \otimes \bigotimes_{v\in\Sigma\setminus\{p\}} \mathrm{Det}_S\,\mathrm{R}\,\Gamma(\mathrm{Fr}(v), M_v)$$

inside $\mathcal{K}$ through Id.

Definition 6.8 (Characteristic structure). The characteristic structure is the choice of $(M_{\mathrm{char}}, \mathrm{Id})$ where M_{char} is the image of

$$\mathrm{Det}_S\,\mathrm{R}\,\Gamma_c(K_\Sigma/K, T) \otimes \bigotimes_{v\in\Sigma\setminus\{p\}} \mathrm{Det}_S\,\mathrm{R}\,\Gamma(\mathrm{Fr}(v), M_v)$$

inside $\mathcal{K}$ through Id.

When $z_{S,\alpha}$ is non-zero, its existence defines a third integral structure, which we call the α-Euler structure, in the following manner. The class $z_{S,\alpha}$ is an element of $\tilde{H}^1_{f,c}(K_\Sigma/K, T\otimes_S\mathcal{K})$, so induces a morphism of complexes

$$\mathcal{K}[-1]z_{S,\alpha} \longrightarrow \mathrm{R}\,\Gamma_c(K_\Sigma/K, T\otimes_S\mathcal{K})$$

and hence a morphism

$$\mathrm{Det}_{\mathcal{K}}\,\mathcal{K}[-1]z_{S,\alpha} \longrightarrow \mathrm{Det}_{\mathcal{K}}\,\mathrm{R}\,\Gamma_c(K_\Sigma/K, T\otimes_S\mathcal{K}).$$

Let Eul_α be the image of $\mathrm{Det}_S\,S[-1]z_{S,\alpha}$ inside $\mathrm{Det}_{\mathcal{K}}\,\mathrm{R}\,\Gamma_f(K_\Sigma/K, T\otimes_S\mathcal{K})$ through this morphism.

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Definition 6.9 (Euler structure). The α-Euler structure is the choice of $(M_{\mathrm{Eul}}, \mathrm{Id})$ where M_{Eul} is the tensor product of the image of Eul_α inside $\mathcal{K}$ through Id with M_{char}^{-1}. We refer to the 1-Euler structure simply as the Euler structure.

Note that the α-Euler structure is defined even when $z_{S,\alpha}$ is not known to be an element of $\tilde{H}^1_{f,c}(T,S)$.

Lemma 6.10. *The three integral structures do not depend on the choices of the M_v.*

Proof. Let X be one of the three integral structures under discussion. Let $v\nmid p$ be a finite place. The choice of M_v impacts X only because it impacts the Id-trivialization of $\mathrm{R}\,\Gamma(\mathrm{Fr}(v), M_v)$. Because M_v is free of rank at most 1, the discussion of § 6.1.2 applies. The Id-trivialization of $\mathrm{R}\,\Gamma(\mathrm{Fr}(v), M_v) = [M_v \xrightarrow{\mathrm{Fr}(v)-1} M_v]$ thus does not depend on the choice of M_v. □

Henceforth, all specializations are assumed to be good specializations, even when this is not explicitly mentioned.

6.1.5 *Comparisons.* Let (X,t) and (Y,s) be two integral structures on $\mathrm{Det}_{\mathcal{K}}\,\mathrm{R}\,\Gamma_f(K_\Sigma/K, T\otimes_S\mathcal{K})$. Let $y\in\mathcal{K}$ be a generator of Y as S-module and let Z be the invertible ideal y^2S inside $\mathcal{K}$. If Z contains X, or equivalently if $X\otimes_S Z^{-1}\subset S$, we say that the square of Y contains X. If there exists a multiplicative set E such that $Z\otimes_S E^{-1}S$ contains $X\otimes_S E^{-1}S$, we say that the square of Y contains X up to E.

Proposition 6.11. *Let S be a normal Cohen–Macaulay ring. Assume $z_{S,\alpha}$ to be non-zero and let $\beta\in\mathcal{K}^\times$ be such that $\beta z_{S,\alpha}$ belongs to $\tilde{H}^1_f(K_\Sigma/K,T)$. If we identify the canonical structure $M_{T,\mathrm{can}}$ with S inside $\mathcal{K}$, then the characteristic structure is equal to the invertible ideal $\mathrm{char}_S^{-1}\,\tilde{H}^2_f(K_\Sigma/K,T)_{\mathrm{tors}}$ and the α-Euler structure is equal to the invertible ideal $\beta\,\mathrm{char}_S^{-1}(\tilde{H}^1_f(K_\Sigma/K,T)/\beta z_{S,\alpha})_{\mathrm{tors}}$.*

Proof. Because invertible ideals are uniquely determined by their localization at height 1 primes and because S is normal, we can and do assume that S is a discrete valuation ring. The complex $\mathrm{R}\,\Gamma_f(K_\Sigma/K,T)$ is then perfect. By Lemma 6.10, we can choose M_v to be equal to T^{I_v} for all finite $v\nmid p$. Then the canonical structure (respectively the characteristic structure, respectively the α-Euler structure) is canonically isomorphic to the one defined by replacing $\mathrm{R}\,\Gamma_c(K_\Sigma/K,T)$ by $\mathrm{R}\,\Gamma_f(K_\Sigma/K,T)$ and by deleting the complexes $\mathrm{R}\,\Gamma(\mathrm{Fr}(v),M_v)$ in Definition 6.7 (respectively 6.8, respectively 6.9).

By Proposition 6.2 and the structure theorem for discrete valuation rings, there exists a free S-module N such that the complex $\mathrm{R}\,\Gamma_f(K_\Sigma/K,T)$ can be represented by $[N\xrightarrow{\psi}N]$ with ψ semi-simple. According to the discussion of § 6.1.2, the invertible ideal $\mathrm{char}_S\,\tilde{H}^2_f(K_\Sigma/K,T)_{\mathrm{tors}}$ is then equal to the image of $\mathrm{Det}_S\,\mathrm{R}\,\Gamma_f(K_\Sigma/K,T)$ inside $\mathcal{K}$.

Because S is a discrete valuation ring, the S-module $\tilde{H}^1_f(K_\Sigma/K,T)$ is free. The class $z_{S,\alpha}\in\tilde{H}^1_f(K_\Sigma/K,T\otimes_S\mathcal{K})$ can thus be written $z_{S,\alpha}=\lambda e$ with $\lambda\in\mathcal{K}^\times$ and with e an element of an S-basis of $\tilde{H}^1_f(K_\Sigma/K,T)$. Moreover, because $\beta\,\mathrm{char}_S^{-1}(\tilde{H}^1_f(K_\Sigma/K,T)/\beta z_{S,\alpha})_{\mathrm{tors}}$ does not depend on our choice of β, we can take β to be λ^{-1}. The S-module $\tilde{H}^1_f(K_\Sigma/K,T)/\beta z_{S,\alpha}$ is then free so we have to show that the α-Euler structure is equal to $\lambda^{-1}S$. As above, the complex $\mathrm{R}\,\Gamma_f(K_\Sigma/K,T)$ can be represented by a complex $[Se\oplus M\xrightarrow{\psi}Se\oplus M]$ with ψ semi-simple and with the first summand included in $\ker\psi$. Then, the α-Euler structure is the image of $\mathrm{Det}_S[Sz_{S,\alpha}\oplus M\longrightarrow Se\oplus M]$ inside $\mathcal{K}$ through the identification of $\mathrm{Det}_S\,\mathrm{R}\,\Gamma_f(K_\Sigma/K,T)$ with S.

Because the determinant of $[Se\oplus M\xrightarrow{\psi}Se\oplus M]$ is sent to S in this identification, the α-Euler structure is equal to the image of $\mathrm{Det}_S[Se\xrightarrow{\lambda}Se]$ seen as an S-submodule inside $\mathcal{K}$ by Id-trivialization, and so to $\lambda^{-1}S$. □

6.2 Tamagawa numbers and the Iwasawa main conjecture

6.2.1 *A formulation of the equivariant Tamagawa number conjecture (ETNC).* We are now in position to formulate a variant of the ETNC adapted to our purpose. A specialization $R_{\mathrm{Iw}}\longrightarrow S$ or $\Lambda_a^{tw}\longrightarrow S$ is said to contain a specialization ϕ with values in S' if there exists an $\mathcal{O}$-algebra map $S\longrightarrow S'$ such that the composite map is equal to ϕ. We use the same terminology for specializations of $\mathcal{T}_{\mathrm{Iw}}$. Of particular interest to us is the case of specializations containing arithmetic specializations. The specializations $\mathcal{T}_{\mathrm{Iw}}$ and $\mathcal{T}$ themselves contain arithmetic specializations. Because specializations are $\mathcal{O}$-algebra morphisms, if $\mathrm{Fr}(v)$ with $v\nmid p$ acts on T with an eigenvalue equal to 1, the same is true for a specialization of T. Hence, if T contains an arithmetic specialization, it is necessarily good.

CONJECTURE 6.12. Let T be a specialization of $\mathcal{T}_{\mathrm{Iw}}$ containing a non-exceptional arithmetic specialization and such that z_S is non-zero. Then the square of the Euler structure contains the characteristic structure.

According to Propositions 6.5 and 6.11, when S is normal and Cohen–Macaulay, Conjecture 6.12 implies that

$$\mathrm{char}_S\,\tilde{H}^2_f(K_\Sigma/K,T)_{\mathrm{tors}}|(\beta^{-1}\,\mathrm{char}_S\,\tilde{H}^1_f(K_\Sigma/K,T)/\beta z_S)^2 \tag{6.2.1}$$

where $\beta\in\mathcal{K}^\times$ is such that βz_S belongs to $\tilde{H}^1_f(K_\Sigma/K,T)$. As the left-hand side of (6.2.1) is a principal ideal of S, Conjecture 6.12 implies that the class z_S belongs to $\tilde{H}^1_f(K_\Sigma/K,T)$ and that:

$$\mathrm{char}_S\,\tilde{H}^2_f(K_\Sigma/K,T)_{\mathrm{tors}}|(\mathrm{char}_S\,\tilde{H}^1_f(K_\Sigma/K,T)/z_S)^2. \tag{6.2.2}$$

A historically important case of Conjecture 6.12 is the following.

CONJECTURE 6.13. Let T be a non-exceptional arithmetic specialization of $\mathcal{T}$ such that z_S is non-zero. Then:

$$\mathrm{char}_{\Lambda_a}\,\tilde{H}^2_f(K_\Sigma/K,T\otimes\Lambda_a)_{\mathrm{tors}}|(\mathrm{char}_{\Lambda_a}\,\tilde{H}^1_f(K_\Sigma/K,T\otimes\Lambda_a)/z_{\Lambda_a})^2. \tag{6.2.3}$$

A crucial feature of the ETNC is that it should be compatible with base-change, see for instance [Kat93, Conjecture 3.2.2 (i)]. This leads to the following conjecture, in which we relax the assumption that T should be non-exceptional.

CONJECTURE 6.14. Let T be a specialization of $\mathcal{T}_{\mathrm{Iw}}$ containing an arithmetic specialization. Let T' be a non-exceptional specialization containing T and such that $z_{S'}$ is non-zero. Let $X_{S'}$ be the characteristic structure of T' and $Z_{S'}$ the square of the Euler structure of T'. Then $(X_{S'}\otimes_{S'}Z_{S'}^{-1})\otimes_{S'}S$ is included in S.

Proposition 6.17 below establishes that the statement of Conjecture 6.14 does not depend on the choice of T' and that Conjectures 6.12 and 6.14 are compatible when both apply. In concrete cases, it is often known that $\mathcal{T}_{\mathrm{Iw}}$ is such that $z_{R_{\mathrm{Iw}}}$ is non-zero, so that Conjecture 6.14 can be applied with $T'=\mathcal{T}_{\mathrm{Iw}}$. We also raise the following question.

Question 6.15. Let T be a specialization of $\mathcal{T}_{\mathrm{Iw}}$ containing an arithmetic specialization. Let T' be a non-exceptional specialization containing T and such that $z_{S'}$ is non-zero. Let $X_{S'}$

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be the characteristic structure of T' and $Z_{S'}$ the square of the Euler structure of T'. Is $(X_{S'} \otimes_{S'} Z_{S'}^{-1}) \otimes_{S'} S$ equal to S?

For reasons which are discussed at length below, we do not explicitly conjecture the veracity of the previous statement. Nonetheless, for simplicity, we refer henceforth to this question as a conjecture. As above, when S is regular and T non-exceptional, Question 6.15 is equivalent to:

$$\mathrm{char}_S\, \tilde{H}^2_f(K_\Sigma/K, T)_{\mathrm{tors}} = (\mathrm{char}_S\, \tilde{H}^1_f(K_\Sigma/K, T)/z_S)^2. \tag{6.2.4}$$

The remainder of this subsection is devoted to a discussion of the form and history of Conjectures 6.12, 6.13 and Question 6.15, as well as arguments and speculations in support of them.

First, the ETNC should link an integral structure in the determinant of the cohomology with special values of L-functions, here presumably with the derivatives of a p-adic L-function. Second, the resort to the square of the Euler structure seems artificial and should be eliminated. These two points would be simultaneously dealt with if we knew the p-adic height pairing on $\tilde{H}^1_f(K_\Sigma/K, T)$ to be non-degenerate, or almost equivalently if we knew the derivative of the p-adic L-function to be non-zero. In that case, the class $z_S \in \tilde{H}^1_f(K_\Sigma/K, T \otimes_S \mathcal{K})$ is also a non-zero element z_S^* of $\mathrm{Hom}_{\mathcal{K}}(\tilde{H}^1_f(K_\Sigma, T \otimes_S \mathcal{K}), \mathcal{K})$ and hence a non-zero element of $\tilde{H}^2_f(K_\Sigma/K, T \otimes_S \mathcal{K})$. Using z_S and z_S^*, we can define an integral structure Eul^* whose comparison with the canonical structure is equal to the square of the Euler structure. Beside $z_S^*(z)$ could then be conjectured to be equal to L'_p and the complex $\mathrm{R}\,\Gamma_f(K_\Sigma/K, T)$ would be semi-simple in the sense of [Bur09, Definition 5.1]. In other words, the complete apparatus needed to formulate the ETNC would be in place. Unfortunately, such a non-degeneracy statement seemingly lies in the world of transcendence and is apparently out of reach.

Historically, Question 6.15 has been first proposed in [Per87, Conjectures A, B] (though part of the idea is there attributed to B. Mazur) in the simplest possible Iwasawa-theoretic situation, that it to say when the following hypotheses are satisfied: the representation $\pi(f)$ comes from a rational elliptic curve $E/\mathbb{Q}$ with good ordinary reduction; Hypothesis 5.10, which in that case is the Heegner hypothesis, holds; the ring S is equal to Λ_a; the p-adic height pairing is known to be non-degenerate. In the same setting but with the hypothesis on the p-adic height pairing replaced by an explicit consideration of the Euler system, it is mentioned in [How04a] in a form equivalent to ours via Proposition 6.11. In [How04b], it is conjectured that, possibly up to a power of p, Question 6.15 holds for $S = \Lambda_a$ and $\pi(f)$ of parallel weight 2 with trivial central character. The first explicit version of Question 6.15 incorporating Hida theory known to this author is in [How07, Conjecture 3.3.1] under the hypothesis that $B^\times = \mathrm{GL}_2(\mathbb{Q})$, that Hypothesis 5.10 holds and that $S = R_{\mathrm{Iw}}$ is a regular ring. Question 6.15 was proposed in [LV11, Conjecture 10.8] for $S = R_{\mathrm{Iw}}$ a regular ring and B a quaternion algebra over $\mathbb{Q}$. An ubiquitous feature of all these formulations is that they require the specialization T to be non-exceptional and that they formulate the conjecture only for S regular and maximal with respect to the situation considered, that is to say for $S = \Lambda_a$ or $S = R_{\mathrm{Iw}}$.

In contrast, we only require S to contain an arithmetic specialization and we do not distinguish between the case of good ordinary reduction and the case of exceptional zero in Conjecture 6.14 and Question 6.15.

Finally, we explain why we do not propose Question 6.15 explicitly as a conjecture. Let $\mathcal{K}_{\mathrm{Iw}}$ be the fraction field of R_{Iw}. Assuming the conjectural framework of the ETNC and of Iwasawa theory, we should expect the R_{Iw}-valued height $h(z_\infty, z_\infty)$ to provide an integral structure

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on $\mathrm{Det}_{\mathcal{K}_{\mathrm{Iw}}}\,\mathrm{R}\,\Gamma_f(K_\Sigma/K,\mathcal{T}_{\mathrm{Iw}}\otimes_{R_{\mathrm{Iw}}}\mathcal{K}_{\mathrm{Iw}})$ and this integral structure to be equal to the one provided by L'_p via conjectures on special values. By definition the element L'_p is the derivative of a function interpolating the special values of the L-function divided by a suitable period so as to obtain an algebraic number. However, there are several choices of periods possible and it is conceivable that the integral structures coming from different choices of L'_p corresponding to different choices of periods are not the same. For instance, it is shown in [Vat03, Theorem 1.1] that this phenomenon occurs in classical dihedral Iwasawa theory of quaternionic automorphic forms when the sign ε of the functional equation is equal to $+1$, rather than -1 as in this text. In that case, the discrepancy between the different integral structures is equal to the product of Tamagawa numbers at primes for which Assumption 5.10 is not satisfied. In other words, the discrepancy measures the difference in ramification between $\mathcal{T}_{\mathrm{Iw}}$ and $\bar{\rho}$ at primes which do not satisfy Assumption 5.10. Because we have not investigated which choice of periods should correspond to the Euler integral structures and because we do not assume systematically Assumption 5.10, it seems to us not impossible that the Euler and characteristic structure could differ, for instance by a power of p. Nonetheless, what we prove below of Conjectures 6.12, 6.14 and Question 6.15 combined with the vanishing of the μ-invariant of the characteristic and Euler structure in the classical Iwasawa-theoretic case seems to weigh heavily in favor of Question 6.15. Alternatively, it seems that the way our Euler system was constructed forces it to record congruences with forms of lower level at primes which do not satisfy Assumption 5.10.

6.2.2 *Compatibilities.* Let $\phi: S\longrightarrow S'$ be a morphism between two good specializations of R_{Iw} and let T' be $T\otimes_{S,\phi}S'$. We say that the integral structure (M,ψ) descends perfectly with respect to ϕ if $M_T\otimes_{S,\phi}S'$ is equal to $M_{T'}$.

Proposition 6.16. *Let $\phi: S\longrightarrow S'$ be a morphism between two specializations of R_{Iw} and let T' be $T\otimes_{S,\phi}S'$. For all $v\in\Sigma$ not dividing p, let M_v and M'_v be maximal rank free submodules of T^{I_v} and $(T')^{I_v}$ respectively. Let S_T be the image of $\mathrm{Det}_S\,\mathrm{R}\,\Gamma(\mathrm{Fr}(v),M_v)$ through Id-trivialization and likewise for T'.*

(i) *The canonical structure descends perfectly with respect to ϕ.*

(ii) *Assume that $M_v\otimes_S S'$ and M'_v have the same S'-ranks. Then $S_T\otimes_{S,\phi}S'$ is equal to $S_{T'}$. If M_v and M'_v have the same ranks for all v, then the characteristic structure descends perfectly with respect to ϕ. This statement holds in particular if both T and T' contain an arithmetic specialization.*

(iii) *Let $\mathcal{P}\in\mathrm{Spec}(S)$ be such that $S_{\mathcal{P}}$ is a regular ring. Assume that $\mathrm{Fr}(v)-1$ is invertible on $T_{\mathcal{P}}/\mathcal{P}$ for all $v\nmid p$ at which T is ramified. Then the characteristic structure on $T_{\mathcal{P}}$ descends perfectly with respect to reduction modulo $\mathcal{P}$. This statement holds in particular for $T=\mathcal{T}_{\mathrm{Iw}}$ and $\mathcal{P}\in\mathrm{Spec}^{\mathrm{arith}}(R_{\mathrm{Iw}})$ an arithmetic prime of even parallel weight.*

Proof. (i) By definition, both $S_{T'}$ and $S_T\otimes_S S'$ are equal to S' inside $\mathcal{K}'$.

(ii) According to (5.1.2), the complex $\mathrm{R}\,\Gamma_c(K_\Sigma/K,T)$ descends perfectly with respect to ϕ. Assume $T^{I_v}\otimes_{S,\phi}S'$ and T'^{I_v} have the same S'-ranks. If this rank is 2, then T and T' are actually unramified at v, and so the complex $\mathrm{R}\,\Gamma_f(G_{K_v},T)$ descends perfectly with respect to ϕ. In particular, S_T and $S_{T'}$ are then equal. Otherwise, the discussion of the end of § 6.1.2 applies. Then, both $S_T\otimes_{S,\phi}S'$ and $S_{T'}$ are the module $\det^{-1}(\mathrm{Fr}(v)-1)^*S'$. So they are equal. Hence, the characteristic structure descends perfectly. If both T and T' contain arithmetic specialization $T_{\mathbf{Sp}}$ and $T'_{\mathbf{Sp}}$, then the ranks of M_v and M'_v are both less than the rank of $\mathcal{T}_{\mathrm{Iw}}^{I_v}$ and greater than the ranks of $T_{\mathbf{Sp}}^{I_v}$ and $(T'_{\mathbf{Sp}})^{I_v}$. So all these ranks are equal by Lemma 3.9.

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(iii) Localizing at $\mathcal{P}$, we can assume that S is a regular ring. Let $v\nmid p$ be a place of ramification of T. The assumption on $\mathrm{Fr}(v)$ implies that $\mathrm{Fr}(v)-1$ is invertible on T^{I_v} for all $v\nmid p$ at which T is ramified. The Id-trivialization of $\mathrm{Det}_S\,\mathrm{R}\,\Gamma_f(G_{K_v},T_{\mathcal{P}})$ is thus S. The characteristic structure is then entirely determined by $\mathrm{Det}_S\,\mathrm{R}\,\Gamma_c(K_\Sigma/K,T)$. Because $\mathfrak{m}_S$ is generated by a regular sequence, the complex $\mathrm{R}\,\Gamma_c(K_\Sigma/K,T)$ descends perfectly with respect to reduction modulo $\mathfrak{m}$ by (5.1.1). If $T=\mathcal{T}_{\mathrm{Iw}}$, so that $S=R_{\mathrm{Iw}}$, and if $\mathcal{P}$ belongs to $\mathrm{Spec}^{\mathrm{arith}}(S)$, then $S_{\mathcal{P}}$ is regular by Lemma 3.1 and $\mathrm{Fr}(v)-1$ is invertible on $T_{\mathcal{P}}/\mathcal{P}$ by purity or Proposition 5.3. The hypotheses of the statement are thus verified. □

We record the following compatibility property of Conjecture 6.12 and Question 6.15.

Proposition 6.17. *Let $\phi:S\longrightarrow S'$ be an $\mathcal{O}$-morphism between good specializations such that the diagram*

$$\begin{array}{ccc} R_{\mathrm{Iw}} & \longrightarrow & S' \\ \downarrow & \nearrow_{\phi} & \\ S & & \end{array} \tag{6.2.5}$$

commutes. Assume that the characteristic structure descends perfectly with respect to ϕ and that Conjecture 6.12 (respectively Question 6.15) holds for T. Then Conjecture 6.12 (respectively Question 6.15) holds for T'. In particular, if T is an arithmetic specialization of $\mathcal{T}$ and if Conjecture 6.12 (respectively Question 6.15) holds for $T\otimes_{\mathcal{O}}\Lambda_a$ or $\mathcal{T}_{\mathrm{Iw}}$, then it holds for T. If Conjecture 6.12 holds for $\mathcal{T}_{\mathrm{Iw}}$ and if Question 6.15 holds for an arithmetic specialization T, then Question 6.15 holds for $\mathcal{T}_{\mathrm{Iw}}$.

Proof. Because the diagram (6.2.5) commutes, $\phi(z_S)$ is equal to $z_{S'}$ and so the diagram (6.2.6) is commutative.

$$\begin{array}{ccc} \mathrm{Det}_S\,S[-1]z_S & \xrightarrow{\mathrm{Id}} & \mathcal{K} \\ {\scriptstyle -\otimes_S S'}\downarrow & & \downarrow \\ \mathrm{Det}_{S'}\,S'[-1]z_{S'} & \xrightarrow{\mathrm{Id}} & \mathcal{K}' \end{array} \tag{6.2.6}$$

This combined with the assumption that the characteristic structure descends perfectly with respect to ϕ implies that the Euler structure descends perfectly with respect to ϕ. If Conjecture 6.12 (respectively Question 6.15) holds then the characteristic structure of T in $\mathcal{K}$ is contained in (respectively is equal to) the square of the Euler structure of T in $\mathcal{K}$. The same relation then holds between the characteristic and Euler structures for T' in $\mathcal{K}'$.

In order to prove the second assertion, it is thus enough to satisfy that the pair of specializations $(T\otimes_{\mathcal{O}}\Lambda_a,T)$ and $(\mathcal{T}_{\mathrm{Iw}},T)$ satisfy the hypotheses of the first assertion. The commutativity of (6.2.5) is by construction. According to Lemma 3.9, the pair $(\mathcal{T}_{\mathrm{Iw}},T)$ satisfies the hypotheses of Proposition 6.16 and thus both integral structures descend perfectly. In order to prove the same result for the pair $(T\otimes_{\mathcal{O}}\Lambda_a,T)$, it is enough for the same reason to prove that the hypotheses of Proposition 6.16 hold. If $v\nmid p$, then Λ is unramified at v so $(T\otimes_{\mathcal{O}}\Lambda_a)^{I_v}$ is equal to $T^{I_v}\otimes_{\mathcal{O}}\Lambda_a$.

As both the characteristic and Euler structures descend perfectly with respect to arithmetic specializations, if the characteristic structure is contained in the square of the Euler structure for $\mathcal{T}_{\mathrm{Iw}}$ and if their images are equal for T, then they are equal for $\mathcal{T}_{\mathrm{Iw}}$. □

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6.3 Main results

6.3.1 *Descent.* Let $\mathfrak{p}\in\operatorname{Spec} S$ be a prime of height $r-1$. We say that a height $r-1$ prime $\mathfrak{p}'\in\operatorname{Spec} S$ is n-close to $\mathfrak{p}$ if and only if $\mathfrak{p}\neq\mathfrak{p}'$, p does not belong to $\mathfrak{p}'$ and $\ell_{\mathcal{O}}S/(\mathfrak{p},\mathfrak{p}')=n$.

Lemma 6.18. *Let S be a specialization of R_{Iw} of dimension at least 2. Let $I=aS$ and $J=bS$ be invertible ideals.*

(i) *Assume that there exists a prime $\mathfrak{p}$ such that I belongs to $\mathfrak{p}$ but b does not belong to $\mathfrak{p}$. Then there exists a specialization A of R_{Iw} which is a discrete valuation ring flat over $\mathbb{Z}_p$ and such that $I\otimes_S A$ is not contained in $J\otimes_S A$. Moreover, the set of kernels of these specializations is a Zariski-dense subset of* Spec S.

(ii) *Assume that S is regular and that $I\not\subset J$. Then, for all $N>0$ there exists a discrete valuation ring A flat over $\mathbb{Z}_p$ with uniformizing parameter ϖ_A such that $\varpi_A^N(I\otimes_S A)$ is not contained in $J\otimes_S A$. Moreover, the set of kernels of these specializations is a Zariski-dense subset of* Spec S.

Proof. (i) Under the assumptions of (i), by Krull's Hauptidealsatz, there is a height $r-1$ prime $\mathfrak{p}$ containing I, which does not contain p and such that b does not belong to $\mathfrak{p}$. For all $n>0$, let E_n be the set of height $r-1$ primes of S which do not contain p and which are at least n-close to $\mathfrak{p}$. There is an N such that for all $n\geqslant N$, if $b\in\mathfrak{r}$, then $\mathfrak{r}$ is not in E_n. Let n be greater than N. For all $\mathfrak{p}'\in E_n$ and all $\mathfrak{r}$ containing b, there exists $x\in\mathfrak{r}$ such that $x\notin\mathfrak{m}^n_{S/\mathfrak{p}'}$. Hence, there exists a Zariski-dense subset E_n' of E_n such that if $\mathfrak{p}'\in E_n'$, then $b\notin\mathfrak{m}^n_{S/\mathfrak{p}'}$. On the other hand, a belongs to $\mathfrak{p}$ so a belongs to $\mathfrak{m}^n_{S/\mathfrak{p}'}$. For all $\mathfrak{p}'\in E_n'$, the normalization $A(\mathfrak{p}')$ of $S/\mathfrak{p}'$ is a discrete valuation ring of unequal characteristic such that $I\otimes_S A$ is not included in $J\otimes_S A$.

(ii) Assume that S is regular and that $I\not\subset J$. Then, up to multiplication of I and J by a common invertible ideal if necessary, we can assume that there exists an irreducible element π dividing a such that b does not belong to $\mathfrak{p}=(\pi)S$. Let $X\in\mathfrak{m}_S$ be an element not contained in $\mathfrak{p}$. For all $n>0$, let $\phi_{X,n}$ be the surjection from S to $S_{X,n}=S/(\pi-X^n)$. Then $\phi_{X,n}(I)$ belongs to (X^n) for all $n\geqslant 0$. For n large enough, $\phi_{X,n}(b)\neq 0$. Because $b\notin\mathfrak{p}$, for n large enough $\phi_{X,n}(J)\notin(X^n)$. Consequently, for all $n_0\geqslant 0$, there exists an n_1 such that for all $n\geqslant n_1$ the following properties are true: the element $\pi-X^n$ is part of a system of generators; $\phi_{X,n}(J)$ is not zero; $X^{n_0}\subset\phi_{X,n}(I)$ and $X^{n_0}\not\subset\phi_{X,n}(J)$. Because the previous procedure can be carried on with an arbitrary $X\in\mathfrak{m}_S$, we can ensure that $S_{X,n}$ has residual characteristic p. Proceeding by descending induction on dimension if necessary, we construct in this way a discrete valuation ring A and specializations ϕ_n such that $I\otimes_{S,\phi_n}A$ is an invertible ideal and such that for all n_0, there exists an n_1 such that $\varpi_A^{n_0}I\otimes_{S,\phi_n}A$ does not belong to $J\otimes_{S,\phi_n}A$ for all $n\geqslant n_1$. The set of kernels of the ϕ_n is of codimension 0. □

6.3.2 *Results over regular rings.* We reformulate the result of Corollary 5.21. Let S be a discrete valuation ring and T be a good non-exceptional specialization of $\mathcal{T}_{\mathrm{Iw}}$. Let A be the $G_{K,\Sigma}$-representation $T\otimes_S\mathcal{K}/S$. Then $\tilde{H}^1_f(K_\Sigma/K,T)$ is free of rank 1 by Proposition 6.5. By [Nek06, (8.9.6.2)], the Pontrjagin dual of $\tilde{H}^2_f(K_\Sigma/K,T)$ is equal to $\tilde{H}^1_f(K_\Sigma/K,A)$. The short sequence

$$0\longrightarrow\bigoplus_{v|p}H^0(K_v,A_v^-)\longrightarrow\tilde{H}^1_f(K_\Sigma/K,A)\longrightarrow H^1_{Gr}(K_\Sigma/K,A)\longrightarrow 0$$

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is exact. The diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H^1_{Gr}(K_\Sigma/K, A) & \longrightarrow & H^1(K_\Sigma/K, A) & \longrightarrow & \bigoplus_{v\in\Sigma} H^1(K_\Sigma/K, A)/H^1_{Gr}(K_\Sigma/K, A) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & H^1_{BK}(K_\Sigma/K, A) & \longrightarrow & H^1(K_\Sigma/K, A) & \longrightarrow & \bigoplus_{v\in\Sigma} H^1(K_\Sigma/K, A)/H^1_{BK}(K_\Sigma/K, A) \end{array} \tag{6.3.1}$$

shows that:

$$\ell_S H^1_{Gr}(K_\Sigma/K, A)_{\mathrm{tors}} \leqslant \ell_S H^1_{BK}(K_\Sigma/K, A)_{\mathrm{tors}} + \sum_{v\nmid p} \ell_S H^1(I_v, T)^{\mathrm{Fr}(v)=1}_{\mathrm{tors}} + \sum_{v\mid p} \ell_S H^0(K_v, A_v^-).$$

Hence:

$$\ell_S \tilde{H}^2_f(K_\Sigma/K, T)_{\mathrm{tors}} \leqslant \ell_S H^1_{BK}(K_\Sigma/K, A)_{\mathrm{tors}} + \sum_{v\nmid p} \ell_S H^1(I_v, T)^{\mathrm{Fr}(v)=1}_{\mathrm{tors}} + 2\sum_{v\mid p} \ell_S H^0(K_v, A_v^-).$$

Let $\mathrm{Tam}(T)_{\max}$ be the index of p in the factorization of the cardinal of the largest of the S-modules $H^1(I_v, T)^{\mathrm{Fr}(v)=1}_{\mathrm{tors}}$ for $v \nmid p$ and $H^0(K_v, A_v^-)$ for $v|p$. Let $\mathrm{Tam}(T)^{(p)}_{\max}$ be the index of p in the factorization of the cardinal of the largest of the S-modules $H^1(I_v, T)^{\mathrm{Fr}(v)=1}_{\mathrm{tors}}$ for $v \nmid p$ and v has a finite decomposition group in D_∞. Then:

$$\ell_S H^1_{BK}(K_\Sigma/K, T)/z_S \leqslant \ell_S H^1_{Gr}(K_\Sigma/K, T)/z_S + \mathrm{Tam}(T)_{\max}.$$

Corollary 5.21 then translates to:

$$\begin{aligned} \ell_S \tilde{H}^2_f(K_\Sigma/K, T)_{\mathrm{tors}} \leqslant 2\ell_S \tilde{H}^1_f(K_\Sigma/K, T)/z_S &+ \sum_{v\nmid p} \ell_S H^1(I_v, T)^{\mathrm{Fr}(v)=1}_{\mathrm{tors}} \\ &+ 2\sum_{v\mid p} \ell_S H^0(K_v, A_v^-) + 2\,\mathrm{Tam}(T)_{\max}. \end{aligned} \tag{6.3.2}$$

We improve upon this result in the following theorem.

Theorem 6.1. *Let T be a good non-exceptional specialization of $\mathcal{T}_{\mathrm{Iw}}$ with coefficients in a discrete valuation ring S. Let $T \otimes \Lambda_a$ denote the specialization $T \otimes_S (\Lambda_a \otimes_{\mathcal{O}} S)$ and let z_{Λ_a} be the image of z_∞ under this specialization. Assume that z_{Λ_a} is not zero. Let α be $\mathrm{Tam}(T)^{(p)}_{\max}$. Then the square of the α-Euler structure contains the characteristic structure of $T \otimes \Lambda_a$. If moreover z_S is non-zero, then the square of the α-Euler structure contains the characteristic structure of T. In other words, if z_{Λ_a} is non-zero then:*

$$\mathrm{char}_{\Lambda_a} \tilde{H}^2_f(K_\Sigma/K, T \otimes \Lambda_a)_{\mathrm{tors}} | (\mathrm{char}_{\Lambda_a} \tilde{H}^1_f(K_\Sigma/K, T \otimes \Lambda_a)/z_{\alpha,\Lambda_a})^2. \tag{6.3.3}$$

If moreover z_S is non-zero, then:

$$\ell_S \tilde{H}^2_f(K_\Sigma/K, T)_{\mathrm{tors}} \leqslant 2\ell_S \tilde{H}^1_f(K_\Sigma/K, T)/z_{\alpha,\Lambda_a}. \tag{6.3.4}$$

If Assumption 5.10 is in force, then α can be taken to be 1 in (6.3.3) and (6.3.4).

Proof. Assume that the square of the α-Euler structure does not contain the characteristic integral structure of $T \otimes \Lambda_a$. By case (ii) of Lemma 6.18, for all $N > 0$, there is a specialization T' of $T \otimes \Lambda_a$ with coefficients in a discrete valuation ring S', such that $z_{S'}$ is not zero and such

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that:

$$\ell_{S'}\tilde{H}^2_f(K_\Sigma/K,T')_{\mathrm{tors}} \geqslant 2\ell_{S'}\tilde{H}^1_f(K_\Sigma/K,T')/z_{S'}+N.$$

In order to derive a contradiction with (6.3.2), it is thus enough to show that

$$\sum_{v\nmid p}\ell_{S'}H^1(I_v,T')^{\mathrm{Fr}(v)=1}_{\mathrm{tors}}+2\sum_{v|p}\ell_{S'}H^0(K_v,U^-_v(A'))+2\,\mathrm{Tam}(T')_{\max}$$

is bounded independently of T'. If $v\nmid p$, then Λ_a is not v-ramified so $\ell_{S'}H^1(I_v,T')$ is equal to $\ell_{S'}H^1(I_v,T)\otimes_S(S\otimes_{\mathcal{O}}\Lambda)\otimes_S S'$ and hence is bounded independently of T'. If $v|p$, the cardinal of $H^0(K_v,U^-_v(A'))$ is bounded unless there exists a quotient T'' of $T\otimes\Lambda_a$ such that $H^0(K_v,U^-_v(T''))$ is non-zero. This implies that the action of G_K on $U^-_v(V'')$ factors through D_∞. Because the subgroup of $\mathrm{Gal}(D_\infty/K)$ generated by the union of the inertia subgroups at places above p is of finite index, the image of the inertia group I_v is of finite index in the image of G_{K_v} in $\mathrm{End}(V'')$. Hence, a character of G_{K_v} which factors through a finitely ramified extension of K inside D_∞/K with values in $\mathrm{End}(U^-_v(V''))$ factors through a finite extension. This is in particular the case for unramified characters and for characters factoring through the $\mathbb{Z}_p$-extensions of F. Applying this to the character giving the action of G_{K_v} on $U^-_v(V'')$ and remarking that the character giving the action of G_{K_v} on $U^-_v(V)$ is a product of characters unramified in a finite extension of the maximal $\mathbb{Z}_p$-extension of F, we deduce that $\mathrm{Fr}(v)$ and I_v act on $U^-_v(V)$ through a finite group. Thus T is exceptional, contrary to our assumption.

The statement for T then follows by Proposition 6.17. If Assumption 5.10 is true, then α can already be taken as equal to 1 in Theorem 5.1. □

Remark. We explain why it is in fact likely that α can be taken to be 1 in Theorem 6.1 even in the absence of Assumption 5.10. By definition, the integer α is the largest power of p dividing the Tamagawa number of T at a place of bad reduction with a finite decomposition group in D_∞. It makes an appearance in Theorem 6.1 because z is not known to belong to $\tilde{H}^1_f(K_\Sigma/K,T\otimes\Lambda_a)$. Moreover, it impacts Conjecture 6.12 only at the primes above p, or through the μ-invariant part of the main conjecture in classical language. However, the μ-invariant of $\tilde{H}^2_f(K_\Sigma/K,T\otimes\Lambda_a)$ is actually a quantity within reach. Indeed, a computation of the respective μ-invariants of branches of Hida families as conducted in the cyclotomic case in [EPW06] shows that the μ-invariant of $\tilde{H}^2_f(K_\Sigma/K,T\otimes\Lambda_a)$ is equal to the sum of the p-adic valuation of α and of the μ-invariant of $\tilde{H}^2_f(K_\Sigma/K,T'\otimes\Lambda_a)$ with T' the G_K-representation coming from a congruent automorphic representation with minimal ramification, and in particular with level lowered at the places with a finite decomposition group in D_∞. For the level-lowered representation T', the divisibility of Theorem 6.1 holds with α equal to 1 because a minimally ramified representation satisfies Assumption 5.10. Hence the μ-invariant of $\tilde{H}^2_f(K_\Sigma/K,T'\otimes\Lambda_a)$ vanishes provided that the μ-invariant of the Euler class $z'\in\tilde{H}^1_f(K_\Sigma/K,T'\otimes\Lambda_a)$ vanishes. This in turn follows from the results of [AN10, CV04, CV05, Vat03]. Putting everything together, we see that the μ-invariant of $\tilde{H}^2_f(K_\Sigma/K,T\otimes\Lambda_a)$ is equal to α, and in particular that z belongs to $\tilde{H}^1_f(K_\Sigma/K,T\otimes\Lambda_a)$.

Theorem 6.2. *Let Λ^{tw}_a be the $1+2d$-dimensional regular ring $\Lambda^{tw}[[\mathrm{Gal}(D_\infty/K)]]$. We consider $\mathcal{T}_{\mathrm{Iw}}$ as a free Λ^{tw}_a-module. Let α be as in Theorem 5.1. Assume $z_{\infty,\alpha}$ to be non-zero. Then, the square of the α-Euler structure of $\mathcal{T}_{\mathrm{Iw}}$ contains the characteristic structure of $\mathcal{T}_{\mathrm{Iw}}$. In other words:*

$$\mathrm{char}_{\Lambda^{tw}_a}\tilde{H}^2_f(K_\Sigma/K,\mathcal{T}_{\mathrm{Iw}})_{\mathrm{tors}}\,|\,(\mathrm{char}_{\Lambda^{tw}_a}\tilde{H}^1_f(K_\Sigma/K,\mathcal{T}_{\mathrm{Iw}})/z_{\infty,\alpha})^2. \tag{6.3.5}$$

Proof. Let I be the invertible ideal of Λ_a^{tw} equal to the square of α-Euler structure and let J be the characteristic structure. Assume that J is not included in I. By case (ii) of Lemma 6.18, there then exists a Zariski-dense set of specializations of Λ_a^{tw} with values in a discrete valuation ring A such that $J\otimes_{\Lambda_a^{tw}}A$ is not contained in $I\otimes_{\Lambda_a^{tw}}A$. Among these specializations, we choose a good specialization ϕ of Λ^{tw} such that the specializations ψ of R_{Iw} inducing ϕ on Λ^{tw} all have the following properties. First, we require that if ψ with values in S_ψ is a specialization of R which induces ϕ on Λ^{tw}, then ψ is good not exceptional and $\psi(z_{\infty,\alpha})$ is not zero. Second, we require that $T_\psi^{I_v}$ has the same S_ψ-rank as $\mathcal{T}_{\mathrm{Iw}}^{I_v}\otimes_{R_{\mathrm{Iw}}}S_\psi$ for all $v\nmid p$. Then both the characteristic and α-Euler structures descend perfectly with respect to ψ. Let B be a discrete valuation ring finite and flat over $\mathcal{O}$ containing all the S_ψ. Then $(\mathcal{T}_{\mathrm{Iw}}\otimes_{\Lambda_a,\phi}A)\otimes_A B$ is the direct sum of the $B[G_K]$-modules $T_\psi\otimes_B B[[\mathrm{Gal}(D_\infty/K)]]$. The fact that $I\otimes_{\Lambda_a^{tw}}A$ does not contain $J\otimes_{\Lambda_a^{tw}}A$ implies that $(I\otimes_{\Lambda_a^{tw}}A)\otimes_A B$ does not contain $(J\otimes_{\Lambda_a^{tw}}A)\otimes_A B$, which then contradicts Theorem 6.1. □

Corollary 6.19. *Let T be an arithmetic specialization of $\mathcal{T}_{\mathrm{Iw}}$ with values in discrete valuation ring S. Assume z_∞ to be non-zero. Then Conjecture 6.14 is true up to p for $T\otimes\Lambda_a=T\otimes_S(\Lambda_s\otimes_{\mathcal{O}}S)$. If Assumption 5.10 is in force, then Conjecture 6.14 is true for $T\otimes\Lambda_a$ and for T.*

Proof. Let Z be the square of the Euler structure, Z_α the square of the α-Euler structure and X the characteristic structure of $\mathcal{T}_{\mathrm{Iw}}$ seen as Λ_a^{tw}-module. Under the assumptions of the corollary, Z_α contains X thanks to Theorem 6.2. Hence $X\otimes_{\Lambda_a^{tw}}Z^{-1}$ is included in $\Lambda_a^{tw}[1/\alpha]$. The second statement of Proposition 6.17 then shows that $(X\otimes_{\Lambda_a^{tw}}Z^{-1})\otimes_{\Lambda_a^{tw}}\Lambda_a$ is included in $\Lambda_a[1/p]$ so Conjecture 6.14 is true up to p. If Assumption 5.10 is in force, then $X\otimes_{\Lambda_a^{tw}}Z^{-1}$ is included in Λ_a^{tw} so $(X\otimes_{\Lambda_a^{tw}}Z^{-1})\otimes_{\Lambda_a^{tw}}\Lambda_a$ is included in Λ_a and $(X\otimes_{\Lambda_a^{tw}}Z^{-1})\otimes_{\Lambda_a^{tw}}S$ is included in S. □

6.3.3 *Results over Hecke algebras.*

Theorem 6.3. *Assume z_∞ to be non-zero. Let α be as in Theorem 5.1. Then the support of the characteristic structure of $\mathcal{T}_{\mathrm{Iw}}$ is contained in the support of the α-Euler structure of $\mathcal{T}_{\mathrm{Iw}}$. If moreover R_{Iw} is a regular ring, then the square of the α-Euler structure contains the characteristic structure of $\mathcal{T}_{\mathrm{Iw}}$.*

Proof. Assume the statement to be false. Let I be the invertible ideal of R_{Iw} equal to the square of α-Euler structure and let J be the characteristic structure. By case (i) of Lemma 6.18, there then exists a Zariski-dense set of specializations of R_{Iw} with values in a discrete valuation ring A such that $I\otimes_{R_{\mathrm{Iw}}}A$ does not contain $J\otimes_{R_{\mathrm{Iw}}}A$. By Zariski-density, we can choose an element ϕ of this set which is good, such that z_A is not zero, which is not exceptional and such that the associated specialization T of $\mathcal{T}_{\mathrm{Iw}}$ has the property that T^{I_v} has the same A-rank as $\mathcal{T}_{\mathrm{Iw}}^{I_v}\otimes_{R_{\mathrm{Iw}}}A$ for all $v\nmid p$. Then both the characteristic and α-Euler structure descend perfectly with respect to ϕ. The fact that $I\otimes_{R_{\mathrm{Iw}}}A$ does not contain $J\otimes_{R_{\mathrm{Iw}}}A$ then means that the square of the α-Euler structure for T does not contain the characteristic structure, which contradicts Theorem 6.1.

The proof of the second assertion is similar to the proof of Theorem 6.2 but easier. □

We recall that the assumption that z_∞ is non-zero is known to hold when $F=\mathbb{Q}$ and $B=\mathcal{M}_2(\mathbb{Q})$ by [How07, Theorem 3.3.1] and should hold for general F and B thanks to [CV04, Theorem 4.10], [CV05, Corollary 2.10] and [AN10, Theorem 4.3.1].

Corollary 6.20. *Assume that R_{Iw} is a regular ring. Assume that there exists an arithmetic specialization T such that Question 6.15 is true for T. Then Question 6.15 is true for $\mathcal{T}_{\mathrm{Iw}}$.*

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Proof. This follows from Theorem 6.3 and the last assertion of Proposition 6.17. □

Concluding remarks. Because normalization is used in an essential way in the proof of (i) of Lemma 6.18 and because normalization can destroy non-integrality, it is of course easy to manufacture abstract examples of invertible ideals which are not integral and such that all specializations to discrete valuation rings are integral. When the coefficient ring is not normal, our methods are thus intrinsically unable to prove that the characteristic structure of $\mathcal{T}_{\mathrm{Iw}}$ is contained in the square of the α-Euler structure of $\mathcal{T}_{\mathrm{Iw}}$. However, in the common situation where R_{Iw} can be written $A/\mathbf{x}$ with A a regular ring and $\mathbf{x}$ an A-regular sequence, i.e. when R_{Iw} is a complete intersection, it is likely that one can prove Conjecture 6.14 for $\mathcal{T}_{\mathrm{Iw}}$ by proving it first for a deformation of $\mathcal{T}_{\mathrm{Iw}}$ with coefficients in the regular ring A.

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