A NOTE ON A GROUP DEFINED BY A QUADRATIC FORM

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1. Introduction. In a recent series of papers [3, 4, 5], H. Zassenhaus considered the structure of those linear transformations T on real 4-space, R_4, into itself that preserve the quadratic form $f(x) = x_1^2 + x_2^2 - x_3^2 - x_4^2$. That is,

(1.1) $$f(T(x)) = f(x) \text{ for all } x \in R_4.$$

Define a function ϕ on R_4 to the space M_2 of 2-square matrices over the complex numbers as follows:

(1.2) $$\phi(x) = \phi(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ x_3 - ix_4 & x_1 - ix_2 \end{pmatrix}.$$

Let G_2 be the vector space of matrices generated by all real linear combinations of

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

It is easy to check that (i) G_2 is an algebra over the real numbers; (ii) ϕ is an isomorphism of R_4 onto the additive group of G_2 over the reals; (iii) $d(\phi(x)) = f(x)$ for each $x \in R_4$, where d denotes determinant. It is also simple to verify that

(1.3) $$G_2 = \{A \mid A^* = PA'P\}$$

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where A^* is the conjugate transpose of A, A' is the transpose of A and $P = g_3$. Let Ω_2 denote the set of T satisfying (1.1). In view of (iii) it is clear that the structure of Ω_2 will be completely known if we determine the structure of those S which are linear mappings of G_2 into G_2 such that $d(S(A)) = d(A)$ for all $A \in G_2$. In other words, if we denote this class of S by Γ_2 then $\Omega_2 = \phi \Gamma_2 \phi^{-1}$.

We are thus led for general n to defining a class G_n in the space M_n of n-square matrices over the complex numbers by

$$(1.4) \qquad G_n = \{A \mid A^* = PA'P\}$$

where P is the n-square matrix with 1 in positions $n - j$, $j + 1$, $j = 0, \ldots, n - 1$ and 0 elsewhere. We define Γ_n to be the set of all linear transformations on G_n to G_n satisfying

$$(1.5) \qquad d(S(A)) = d(A) \quad \text{for all } A \in G_n .$$

2. Results. Our main result is contained in the following

THEOREM. $S \in \Gamma_n$ if and only if there exist U and V in G_n such that either

$$(2.1) \qquad S(A) = UAV \text{ for all } A \in G_n$$

or

$$(2.2) \qquad S(A) = UA'V \text{ for all } A \in G_n$$

where $d(UV) = 1$.

Consider the set of matrices $\mathcal{E}$

$$(2.3) \quad E_{st} + E_{n-s+1,n-t+1}, \; i(E_{st} - E_{n-s+1,n-t+1}), \quad 1 \leq s < t \leq n$$

$$E_{ss} + E_{n-s+1,n-s+1}, \; i(E_{ss} - E_{n-s+1,n-s+1}), \quad 1 \leq s \leq k'$$

where $k' = k$ if $n = 2k$ and $k' = k+1$ if $n = 2k + 1$. It is simple to verify that the elements of $\mathcal{E}$ are linearly independent over the complex numbers. Now let $A \in G_n$. Then, from (1.4),

$$A^* = PA'P,$$

$$\bar{a}_{st} = a_{n-s+1,n-t+1}, \quad s,t = 1, \ldots, n,$$

and we check easily that A is in the linear closure of $\mathcal{E}$ over the reals.

Since $\mathcal{E}$ generates M_n over the complex numbers as well, S may be extended linearly to a linear map of M_n into itself. We denote the extended map by S also.

We next observe that

$$d(S(X)) = d(X) \tag{2.4}$$

for all $X \epsilon M_n$. To see this, let $z_1, \ldots, z_{n^2}$ be indeterminates over the complex numbers, and let $e_1, \ldots, e_{n^2}$ be the elements of $\mathcal{E}$ arranged in some order. Define the polynomial p by

$$p(z_1, \ldots, z_{n^2}) = d\left(\sum_{t=1}^{n^2} z_t S(e_t)\right) - d\left(\sum_{t=1}^{n^2} z_t e_t\right) .$$

Since G_n is generated over the reals by $\mathcal{E}$ and moreover $d(S(A)) = d(A)$ for all $A \epsilon G_n$, we conclude that p is identically zero for all real values of $z_1, \ldots, z_{n^2}$. Hence p is identically zero for all complex values of $z_1, \ldots, z_{n^2}$. However, M_n is the linear closure of $\mathcal{E}$ over the complex numbers and (2.4) follows.

Proceeding to the proof of the theorem we use a result in [1] or [2] that states that if T is any linear transformation on M_n to M_n such that $d(T(X)) = d(X)$ for all $X \epsilon M_n$ then $T(X) = UXV$ or $T(X) = UX'V$ where $d(UV) = 1$. Actually, Dieudonné [1] shows that if T is assumed to be non-singular as well this result follows. But the non-singularity of T is a consequence of the fact that T is linear and preserves all determinants as shown in [2]. The theorem then follows from the

LEMMA. If $UAV \epsilon G_n$ for all $A \epsilon G_n$ and U and V are non-singular, then non-singular U_1 and V_1 may be chosen in G_n such that

$$UXV = U_1XV_1 \text{ for all } X \epsilon M_n . \tag{2.5}$$

A similar statement holds if $UA'V \epsilon G_n$ for all $A \epsilon G_n$.

Proof. We have that

$$(UAV)^* = P(UAV)'P \text{ for all } A \epsilon G_n$$

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and hence

$$(V')^{-1}PV^* A^*U^*P(U')^{-1} = A' ,$$

(2.6) $$[(V')^{-1}PV^*P] A' [PU^*P(U')^{-1}] = A'$$

for all $A \in G_n$. Since $A \in G_n$ if and only if $A' \in G_n$, we conclude from (2.6) that $CAD = A$ for all $A \in G_n$, $C = (V')^{-1}PV^*P$, $D = PU^*P(U')^{-1}$. It follows that $CXD = X$ for all $X \in M_n$ and thus $C = \lambda I$, $D = \lambda^{-1}I$, where I is the n-square identity matrix.

Thus

(2.7) $$V^* = \lambda PV'P, \quad U^* = \lambda^{-1}PU'P .$$

From (2.7) and the fact that V is non-singular, we have $\lambda = \overline{d(V)}/d(V)$ and thus $\lambda = e^{i\theta}$, $0 \leq \theta < 2\pi$. Now choose a complex number ω such that $|\omega| = 1$ and $\bar{\omega}/\omega = e^{-i\theta}$ and set $V_1 = \omega V$, $U_1 = \bar{\omega}U$. Then $UAV = |\omega|^{-2}U_1AV_1 = U_1AV_1$ and moreover

$$V_1^* = \bar{\omega}V^* = \bar{\omega}/\omega \;\; e^{i\theta} PV_1'P = PV_1'P,$$

$$U_1^* = \omega U^* = \omega/\bar{\omega} \;\; e^{-i\theta} PU_1'P = PU_1'P$$

and the proof of the lemma is complete.

We remark that the transformation $S(A) = UAV$ has the matrix representation $U \otimes V'$ with respect to the doubly lexicographically ordered basis E_{ij} in M_n, and the matrix representation of $\sigma(A) = A'$ with respect to this ordered basis is the n^2-square matrix σ_1 whose (i, j) n-square block is E_{ji} for $i, j = 1, \ldots, n$. Here $\otimes$ indicates Kronecker product.

Hence we have

COROLLARY 1. If $S \in \Gamma_n$ then there exists a basis of M_n such that the matrix representation of S is either

$$U \otimes V$$

or

$$(U \otimes V)\sigma_1$$

where U and V are in G_n.

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COROLLARY 2. If $S \epsilon \Gamma_n$ then there exists a basis of M_n such that the matrix representation of S with respect to this basis is in G_{n^2}.

Proof. From corollary 1 it suffices to show that if U and V are in G_n then $U \otimes V \epsilon G_{n^2}$ and $\sigma_1 \epsilon G_{n^2}$ (since G_{n^2} is closed under multiplication). We note first that the n^2-square matrix Q with 1 in the position $n^2 - j, j + 1, j = 0, \ldots, n^2 - 1$ and 0 elsewhere is given by

$$Q = P \otimes P.$$

Then

$$(U \otimes V)^* = U^* \otimes V^* = (PU'P) \otimes (PV'P)$$

$$= (P \otimes P)(U' \otimes V')(P \otimes P)$$

$$= Q(U \otimes V)'Q,$$

and hence $(U \otimes V) \epsilon G_{n^2}$. Now $\sigma_1 \epsilon G_{n^2}$ if it commutes with Q. To see this without multiplying matrices simply note that Q is the matrix representation with respect to the E_{ij} basis of the transformation R defined by

$$R(A) = PAP.$$

Then, since σ_1 is the matrix representation of σ with respect to the same basis, it suffices to show that $R\sigma = \sigma R$. But

$$R\sigma(A) = PA'P = (PAP)' = \sigma R(A),$$

and the proof is complete.

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