1

Categories: the idea

An overview of what the point of category theory is, without formality.

I like to think of category theory as the mathematics of mathematics.

I admit this phrase sounds a bit self-important, and it comes with another problem, which is the widespread misunderstandings about what mathematics actually is. This problem is multiplied (or possibly raised to the power of infinity) here by the reference of math to itself.

Another problem is that it might make it seem like you need to understand the whole of mathematics before you could possibly understand category theory. Indeed, that is not far from what the prevailing wisdom has been about studying category theory in the past: that you have to, if not understand *all* of math, at least understand a large amount of it, say up to a graduate level, before you can tackle category theory. This is why category theory has traditionally only been taught at a graduate level, and more recently sometimes to upper level undergraduates who already have a solid background in upper level pure mathematics. The received wisdom is that all the motivating examples come from other branches of pure mathematics, so you need to understand those first before you can attempt to understand category theory.

Questioning "received wisdom" is one of my favorite pastimes. I don't advocate just blindly going against it, but the trouble with received wisdom, like "common sense", is that it too often goes unquestioned.

My experience of learning and teaching category theory has been different from that received wisdom. I did first learn category theory in the traditional way, that is, only after many undergraduate courses in pure math. However, those other subjects didn't help me to understand category theory, but the other way round: category theory was much more compelling to me and I loved and understood it in its own right, whereupon it helped me to understand all those other parts of pure math that I had never really understood before.

I eventually decided to start teaching category theory directly as well, to students with essentially no background in pure mathematics. I am convinced that the ideas are interesting in their own right and that examples illustrating

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those ideas can be found in life, not just in pure math. That's why I'm starting this book with a chapter about those ideas.

I think we can sometimes unintentionally fall into an educational scheme of believing that we need to learn and teach math in the order in which it was developed historically, because surely that is the logical order in which ideas develop. This idea is summed up in the phrase "ontogeny recapitulates phylogeny", although that is really talking about biological development rather than learning.† I think this has merit at some levels. The way in which children grasp concepts of numbers probably does follow the history of how numbers developed, starting with the counting numbers 1, 2, 3, and so on, then zero, then negative numbers and fractions (maybe the other way round) and eventually irrational numbers. However, some parts of math developed because of a lack of technology, and are now somewhat redundant. It is no longer important to know how to use a slide rule. I know very few ruler and compass constructions, but this has not hindered my ability to do category theory, just like my poor skills in horse riding have not hindered my ability to drive a car. Of course, horse riding can be enjoyable, and even crucial in some walks of life, and by the same token there are some specific situations in which mental arithmetic and long division might be useful. Indeed some people simply enjoy multiplying large numbers together. However, none of those things is truly a prerequisite for meeting and benefiting from category theory.

Crucially, I think we can benefit from the ideas and techniques of category theory even outside research math and aside from direct technical applications. Mathematics is among other things a field of research, a language, and a set of specific tools for solving specific problems. But it is also a way of thinking. Category theory is a way of thinking about mathematics, thus it is a way of thinking about thinking. Thinking about how we think might sound a bit like convoluted navel-gazing, but I believe it's a good way of working out how to think better. And in a world of fake news, catchy but contentless memes, and short attention spans, I think it's rather important for those who do want to think better to find better and better ways of doing it, and share them as widely as possible rather than keeping people under a mistaken belief that you have to learn a huge quantity of pure math first.

I have gradually realized that I use the ideas and principles of category theory in all my thinking about the world, far beyond my research, and in areas that probably wouldn't be officially considered to be applications. It is these ideas and principles that I want to describe in this first chapter, before starting to delve into how category theory implements those ideas and how it trains us

† Also the phrase was coined by Ernst Haeckel who had some repugnant views on race and eugenics, so I'm reluctant to quote him, but technically obliged to credit him for this phrase.

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in the discipline of using them all the time. This chapter is in a way an informal overview of the entire book; it might seem a little vague but I hope the ideas will become clearer as the book progresses.

We are going to build up to the definitions very gradually, so if you're feeling impatient you might want to glance at Chapter 8 in advance, but I urge you to read the chapters of build-up to get into the spirit of this way of thinking. In the Epilogue I will come back to the ideas and spirit of category theory, but from a more technical point of view after we have met and explored the formalism.

1.1 Abstraction and analogies

Mathematics relies heavily on abstraction to get it going. Its arguments are all based on rigorous logic, and rigorous logic only works properly in abstract settings. We can try and use rigorous logic in less abstract settings, but we will probably always[†] run into problems of ambiguity: ambiguity of definitions, ambiguity of interpretations, ambiguity of behavior, and so on.

In normal life situations there is always the possibility that something will get in the way of logic working perfectly. We might think that, logically, one plus one is always two, but in real life some aspect of the objects in question might get in the way. If someone gives you one cookie and then another cookie you might have two cookies but it depends if you ate them. If you had one flower and you buy another then you might have two, but perhaps you bought another because the first one died.

Abstraction is the process of deciding on some details to ignore in order to ensure that our logic does work perfectly. In the situations above this might consist of specifying that we don't eat the cookies, or that the flowers don't die (or reproduce). This is an important part of the process of doing mathematics because one of the aims is to eliminate ambiguity from our arguments. This doesn't mean that ambiguity is bad; indeed ambiguity is one of the things that can make human life rich and beautiful. However, it can also make arguments frustrating and unproductive. Math is a world in which one of the aims is to make arguments unambiguous in order to reach agreement on something. We will go into detail about how abstraction works and what its advantages and disadvantages are in the next chapter. The idea is that abstraction itself has the potential to be ambiguous, and category theory provides a secure framework for performing abstractions.

[†] I am tempted to say "always" but my precise mathematical brain prevents me from making absolute statements without some sort of qualification such as "probably" or "I believe" or "it is almost certainly true that".

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1.2 Connections and unification

One of the aims and advantages of abstraction is to make connections between different situations that might previously have seemed very different. It might seem that abstraction takes us further away from "real" situations. This is superficially true, but at the same time abstraction enables us to make connections between situations that are further apart from one another. This is one of the ways in which math helps us understand more things in a more powerful way, by making connections between different situations so that we can study them all at once instead of having to do the work over and over again. Once we've understood that one plus one is two (abstractly) we don't have to keep asking ourselves that question for different objects. Spotting similarities in our thought processes enables us to make more efficient use of our brain power.

One way in which this arises is in pattern spotting. A pattern can arise as a connection within a single situation, such as when we use a repeating pattern to tile a floor or wall. Or it can arise as a connection between different situations, such as when we see a pattern of certain types of people dominating conversations or belittling others, whether it's at work or in our personal lives, in "real life" or online.

Making connections between different situations is a step in the direction of unification. In math this doesn't mean making everything the same, but it is more about making an abstract theory that can encompass and illuminate many different things. Category theory is a unifying theory that can simultaneously encompass a broad range of topics and also a broad range of scales by zooming in and out, as we'll see. Chapter 3 will be about patterns, and how this gives us a start at recognizing abstract structures.

1.3 Context

One of the starting points of category theory is the idea that we should always study things in context rather than in isolation. It's a bit like always setting a frame of reference first. This is one crucial way to eliminate ambiguity right from the start, because things can take on very different meanings and different characteristics in different contexts. Our example of one plus one giving different results was really a case of context getting in the way of our logical outcomes. One plus one does always equal two provided we are in a context of things behaving like ordinary numbers and not like some other kind of number. But there are plenty of contexts in which things behave differently, as we'll see in Chapter 4. One of the disciplines and driving principles of category theory

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is to make sure we are always aware of and specific about what context we're considering. This is relevant in all aspects of life as well. For example, the context of someone's life situation, how they grew up, what is going on for them in their personal life, and so on, has a big effect on how they behave, and what their achievements represent. The same achievement is much more impressive to me when someone has struggled against many obstructions in life, because of race, gender, gender expression, sexual orientation, poverty, family circumstance, or any number of other struggles. Sometimes this is controversially referred to as "positive discrimination" but I prefer to think of it as contextual evaluation.

1.4 Relationships

One of the crucial ways in which category theory specifies and defines context is via relationships. It takes the view that what is important in a given context is the ways in which things are related to one another, not their intrinsic characteristics. The types of relationship we consider are often key to determining what context we're in or should be in. For example, in some contexts it matters how old people are relative to one another, but in other contexts it matters what their family relationships are, or how much they earn. But if we're thinking about, say, how good different people will be at running a country, then it might not seem relevant how much money they have relative to one another. Except that in some political systems (notably the US) being very rich seems quite important in getting elected to political office.

There can also be different types of relationship between the same things in mathematics, and we might only want to focus on certain types of relationship at any given moment. It doesn't mean that the others are useless, it just means that we don't think they are relevant to the situation at hand. Or perhaps we want to study something else for now, in something a bit like a controlled experiment. Numbers themselves have various types of relationship with each other. The most obvious relationship between numbers is about size, and so we put numbers on a number line in order of size. But we could put numbers in a different diagram by indicating which numbers are divisible by others. In category theory those are two different ways of putting a category structure on the same set of numbers, by using a different type of relationship. We will go into more detail about this in Chapter 5.

The relationships used in category theory can essentially be anything, as long as they satisfy some basic principles ensuring that they can be organized in a mildly tractable way. This will guide us to the formal definition of a cat-

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egory. To build up to that we will look at the idea of formalism in Chapter 6, to ease into this aspect of mathematics that can sometimes be so offputting. In Chapter 7 we'll look at a particular type of relationship called equivalence relations, which satisfy many good properties making them exceedingly tractable. In fact, they satisfy too many good properties, so they are too restrictive to be broadly expressive in the way that category theory seeks.

We will see that category theory is a framework that achieves a remarkable trade-off between good behavior and expressive possibilities. If a framework demands too much good behavior then expressivity is limited, as in a totalitarian state with very strict laws. On the other hand if there are too *few* demands, then there is great potential for expressivity, but also for chaos and anarchy. Category theory achieves a productive balance between those, in the way it specifies what type of relationship it is going to study.

Part One of the book will build up to the formal definition of a category. We will then take an Interlude which will be a tour of mathematics, presenting various mathematical structures as examples of categories. The usual way of doing this is to assume that a student of category theory is already familiar with these examples and that this will help them feel comfortable with the definition of category theory. I will not do that, but will introduce those examples from scratch, taking the ideas of category theory as a starting point for introducing these mathematical topics instead. In Part Two of the book we will then look more deeply into the sorts of things we do with category theory.

1.5 Sameness

One of the main principles and aims of category theory is to have more nuanced ways of describing sameness. Sameness is a key concept in math and at a basic level this arises as equality, along with the concept of equations. Indeed, many people get the impression that math is *all* about numbers and equations. This is very far from true, especially for a category theorist. First of all, while numbers are an example of something that can be organized into a category, the whole point is to be able to study a much, much broader range of things than numbers. Secondly, category theory specifically does not deal in equations because equality is much too strong a notion of sameness in category theory.

The point is, many things that we write with an equals sign in basic math aren't really equal deep down. For example when we say $\boxed{5 + 1 = 1 + 5}$ we really mean that the outcomes are the same, not that the two sides of the equation are actually completely the same. Indeed, if the two sides were completely the same there would be no point writing down the equation. The whole point

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is that there is a sense in which the two sides are different and a sense in which the two sides are the same, and we use the sense in which they're the same to pivot between the senses in which they're different in order to make progress and build up more complex thoughts. We will go into this in Chapter 14.

Numbers and equations go together because numbers are quite straightforward concepts,[†] so equality is an appropriate notion of "sameness" for them. However, when we study ideas that are more complex than numbers, much more subtle notions of sameness are possible. To take a very far opposite extreme, if we are thinking about people then the notion of "equality" becomes rather complicated. When we talk about equality of people we don't mean that any two people are actually the same person (which would make no sense) but we mean something more subtle about how they should be treated, or what opportunities they deserve, or how much say they should have in our democracy. Arguments often become heated around what different people mean by "equality" for people, as there are so many possible interpretations.

Math is about trying to iron out ambiguity and have more sensible arguments. Category theory seeks to study notions of sameness that are more subtle and complex than direct equality, but still unambiguous enough to be discussed in rigorous logical arguments. Sometimes a much better question isn't to ask whether two things are equal or not, but *in what ways* they are and aren't equal, and furthermore, if we look at some way in which they're not equal, how much and in what ways do they fail to be equal? This is a level of subtlety provided by category theory which we sorely need in life too.

1.6 Characterizing things by the role they play

Category theory seeks to characterize things by the role they play in context rather than by some intrinsic characteristics. This is related to the idea of context and relationships being so important. Once we understand that objects take on very different characteristics in different contexts it becomes clearer that the whole idea of intrinsic characteristics is rather shaky.

I think this applies to people as well. I don't think I have an intrinsic personality because I behave very differently depending on what sort of situation I'm in. In some situations I'm confident and talkative, and in other situations I'm nervous and shy. Even mathematical objects do something similar, although in that case the characteristics we're thinking about aren't personality traits, but mathematical behaviors.

[†] Actually they're very profound, but once they're defined there's not much nuance to them.

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For example, we might think the number 5 is prime "because it's only divisible by 1 and itself", but we really ought to point out that the context we're thinking of here is the whole numbers, because if we allow fractions then 5 is divisible by everything really (except 0).[†]

In normal life we often mix up when we're characterizing things by role and by property in the way that we use language. For example "pumpkin spice" is named after the role that this spice combination plays in classic American pumpkin pie, but it has now come to be used as a flavoring in its own right in any number of things that are not actually pumpkin pie, but it's still called pumpkin spice, which is quite confusing for non-Americans. Conversely "pound cake" is named after the fact that it's a recipe consisting of a pound each of basic cake ingredients. So it's named after an intrinsic property, and it's still called pound cake even if you change the quantity that you use. I, personally, have never made such an enormous cake.

One of the advantages of characterizing things by the role they play in context is that you can then make comparisons across different contexts, by finding things that play analogous roles in other contexts. We will talk about this when we discuss universal properties in Chapter 16. This might sound like the opposite of what I just described, as it sounds a bit like properties that are universal regardless of context, but what it actually refers to is the property of being somehow extreme or canonical within a context. This can tell us something about the objects with that property, but it can also tell us something about the context itself. If we go round looking at the highest and lowest paid employees in different companies, that tells us something about those companies, not just about the employees. It is only one piece of information (as opposed to a whole distribution of salaries across the company) but it still tells us something.

1.7 Zooming in and out

One of the powerful aspects of category theory's level of abstraction is that it enables us to zoom in and out and look at large and small scale mathematical structures in a similar light. It's like a theory that unifies the sub-atomic level with the level of galaxies. This is one of my favorite aspects of category theory.

If we study birds then we might need to make a theory of birds in order to make our study rigorous. However, that theory of birds is not itself a bird — it's one level more abstract. On the other hand if we study mathematical objects then we similarly might need a theory of them. I find it enormously

[†] Also this is more of a characterization than a definition.

satisfying that that theory is itself also a mathematical object, which we can then study using the same theory. Category theory is a theory of mathematics, but is itself a piece of mathematics, and so it can be used to study itself. This sounds self-referential, but what ends up happening is that although we are still in category theory we find ourselves in a slightly higher dimension of category theory. Dimensions in this case refer to levels of relationship. In basic category theory our insight begins by saying we should study relationships between objects, not just the objects themselves. But what about the relationships? If we consider those to be new mathematical objects, shouldn't we also study relationships between those? This gives us one more dimension.

Then, of course, why stop there? What about relationships between relationships between relationships? This gives us a third dimension. And really there is no logically determined place to stop, so we might keep going and end up with infinite dimensions. This is essentially where my research is, in the field of higher-dimensional category theory, and we will see a glimpse of this to finish the book. To me this is the ultimate "fixed point" of theories. If category theory is a theory of mathematics, then higher-dimensional category theory is a theory of categories. But a theory of higher-dimensional category theory is still higher-dimensional category theory.

This is not just about abstraction for the sake of it, although I do find abstraction fun in its own right. It is about subtlety. Category theory is about having more subtle ways of expressing things while still maintaining rigor, and every extra dimension gives us another layer of possible subtlety.

Subtlety and nuance are aspects of thinking that I find myself missing and longing for in daily life. So much of our discourse has become black-and-white in futile attempts to be decisive, or to grab attention, or to make devastating arguments, or to shout down the opposition. Higher-dimensional category theory trains us in balancing nuance with rigor so that we don't need to resort to black-and-white, and so that we don't *want* to either.

I think mathematics is a spectacular controlled environment in which to practice this kind of thinking. The aim is that even if the theory is not directly applicable in the rest of our lives, the thinking becomes second nature. This is how I have found category theory to help me in everyday life, surprising though it may sound.

1.8 Framework and techniques

As I have described it so far, category theory might sound like a philosophy more than anything else. But the point is that it is only *guided* by these vari-

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ous philosophies. It is still entirely rigorous technical mathematics. It sets up a framework for implementing these philosophies and pursuing these goals rigorously. The framework consists of a formal definition of a category as an algebraic structure, and then techniques for studying these structures and for constructing and investigating particular features that might arise in them.

To this end, a certain amount of formal mathematics is needed if we are ever going to get very far into the theory itself, rather than poetically exploring the ideas behind it. This is one of the things that can be offputting about mathematics, and I do advocate the idea of seeing and appreciating the ideas of math even if you can't or don't want to follow the formality. However that is not the aim of this book. (It was, in a way, the aim of my book *How to Bake* π.) I do think that way of appreciating mathematics is under-rated. It is a bit like going to visit a country without learning to speak the language. I think it would be culturally limiting for us to decide we should never visit a country without learning the language first. However, I also think that if we can learn at least some of the language then even if we're not fluent we will get much more out of a visit. This is what this book is for.

Mathematics is sometimes taught as if the only valid interaction with it is to be able to do it. As I said in the Prologue, languages are taught with a "productive" and a "receptive" component (as well as a cultural component, in my experience not examinable). When we talk about basic education we sometimes talk about "reading, writing and arithmetic". Aside from the over-emphasis on boring arithmetic (for which we basically all have phone calculators now), there is again the idea that for language the skills of reading and writing are separate, but math is just math.

In this book I'm not going to expect readers to become fluent in all aspects of category theory. My aim isn't to be able to get you to be able to do research in category theory, but mainly to be able to read and appreciate it, and have some build-up into the formality of it in case you do want to go further. Tourism is sometimes used as a derogatory word, tourists thought of as superficial visitors who take selfies and then leave. But well-informed and curious tourists are a valuable part of cultural exchange. I have always appreciated living in places that are interesting enough to attract tourists from around the world. And tourists do sometimes become long-term visitors, permanent residents, or even citizens. One way to learn a language is to be deposited in a foreign country where nobody speaks your native tongue, but I want to do something more gentle than that. The next few chapters will build up to the formal language gradually.