2

Abelian Categories

Contents

This chapter is devoted to some of the foundations which are used throughout

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this book. The main theme is the theory of localisation for additive and abelian categories. We begin with a brief introduction into additive and exact categories. Then we describe specific constructions for localising additive and abelian categories. We provide many examples. For instance, we study localisations of module categories, and it is shown that Grothendieck categories are precisely the abelian categories arising from localising a module category. Also, for the category of modules over a commutative noetherian ring the localising subcategories are classified in terms of support.

2.1 Exact Categories

We introduce the notion of an exact category and begin with the more fundamental notions of additive and abelian categories. An exact category is by definition an additive category together with an extra structure given by a distinguished class of short exact sequences. Extreme cases arise either from additive categories by taking all split exact sequences as distinguished sequences, or from abelian categories by taking any possible short exact sequence as a distinguished sequence. Categories of finitely presented functors are a useful tool.

Additive and Abelian Categories

A category $\mathcal{A}$ is *additive* if it admits finite products, including the product indexed over the empty set, for each pair of objects X, Y the set $\operatorname{Hom}_{\mathcal{A}}(X, Y)$ is an abelian group, and the composition maps

$$\operatorname{Hom}_{\mathcal{A}}(Y, Z) \times \operatorname{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(X, Z)$$

sending a pair (ψ, ϕ) to the composite $\psi \circ \phi$ are biadditive.

Lemma 2.1.1. *In an additive category finite coproducts also exist. Moreover, finite products and coproducts coincide.*

Proof For a pair of objects X, Y the product $X \times Y$ together with the morphisms $(\mathrm{id}_X, 0)\colon X \to X \times Y$ and $(0, \mathrm{id}_Y)\colon Y \to X \times Y$ represents the coproduct of X and Y, since any pair of morphisms $\phi\colon X \to A$ and $\psi\colon Y \to A$ induces the morphism

$$X \times Y \xrightarrow{\phi \times \psi} A \times A \xrightarrow{\nabla} A$$

where ∇ denotes the sum of both projections $A \times A \to A$. □

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We write $X \oplus Y$ for the (co)product of objects X, Y in $\mathcal{A}$ and note that the group structure on $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is determined by the following commuting diagram for any pair $\phi, \psi \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$:

$$\begin{array}{ccc} X \oplus X & \xrightarrow{\phi \oplus \psi} & Y \oplus Y \\ \uparrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \nabla} \\ X & \xrightarrow{\phi + \psi} & Y \end{array} \tag{2.1.2}$$

A functor $F \colon \mathcal{A} \to \mathcal{B}$ between additive categories is *additive* if it preserves finite products. An equivalent condition is that the induced map

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(FX, FY)$$

is additive for every pair of objects X, Y in $\mathcal{A}$.

The *kernel* $\mathrm{Ker}\, F$ of an additive functor $F \colon \mathcal{A} \to \mathcal{B}$ is the full subcategory of objects X in $\mathcal{A}$ such that $F(X) = 0$.

An additive category $\mathcal{A}$ is *abelian* if every morphism $\phi \colon X \to Y$ has a kernel and a cokernel, and if the canonical factorisation

$$\begin{array}{ccccccc} \mathrm{Ker}\,\phi & \xrightarrow{\phi'} & X & \xrightarrow{\phi} & Y & \xrightarrow{\phi''} & \mathrm{Coker}\,\phi \\ & & \downarrow & & \uparrow & & \\ & & \mathrm{Coker}\,\phi' & \xrightarrow{\bar{\phi}} & \mathrm{Ker}\,\phi'' & & \end{array}$$

of ϕ induces an isomorphism $\bar{\phi}$.

Remark 2.1.3. An additive category may be characterised as follows. It is a category with finite products and coproducts (including the (co)product indexed over the empty set) such that products and coproducts coincide, and the monoid structure on $\mathrm{Hom}(X, Y)$ given by (2.1.2) yields a group structure for all objects X, Y.

Example 2.1.4. (1) Let $\mathcal{A}$ be an additive category and X an object. Set $\Lambda = \mathrm{End}_{\mathcal{A}}(X)$. Then $\mathrm{Hom}_{\mathcal{A}}(X, -) \colon \mathcal{A} \to \mathrm{Mod}\,\Lambda$ induces a fully faithful functor $\mathrm{add}\, X \to \mathrm{proj}\,\Lambda$. This functor is an equivalence if $\mathcal{A}$ is idempotent complete.

(2) The category of modules over an associative ring is an abelian category.

Finitely Presented Functors

Let $\mathcal{C}$ be an additive category. We consider additive functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$. Morphisms between such functors are the natural transformations. This gives a category which is denoted by $\mathrm{Mod}\,\mathcal{C}$. For F and G in $\mathrm{Mod}\,\mathcal{C}$, we write

$\operatorname{Hom}_{\mathcal{C}}(F,G)$ for the class of morphisms $F \to G$. Note that $\operatorname{Hom}_{\mathcal{C}}(F,G)$ is a set when $\mathcal{C}$ is essentially small.

For each object X in $\mathcal{C}$ there is the *representable functor*

$$h_X = \operatorname{Hom}_{\mathcal{C}}(-,X)\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Ab}.$$

An important tool is *Yoneda's lemma*.

Lemma 2.1.5 (Yoneda)**.** *For objects F in* $\operatorname{Mod}\mathcal{C}$ *and X in $\mathcal{C}$, the map*

$$\operatorname{Hom}_{\mathcal{C}}(h_X,F) \longrightarrow F(X), \quad \phi \mapsto \phi_X(\mathrm{id}_X)$$

is an isomorphism of abelian groups.

Proof The inverse map sends $x \in F(X)$ to $\psi\colon h_X \to F$ given by $\psi_C(\alpha) = F(\alpha)(x)$ for $C \in \mathcal{C}$ and $\alpha \in \operatorname{Hom}_{\mathcal{C}}(C,X)$. □

It follows from this lemma that the *Yoneda functor*

$$\mathcal{C} \longrightarrow \operatorname{Mod}\mathcal{C}, \quad X \mapsto h_X$$

is fully faithful. Also, one sees for $F,G \in \operatorname{Mod}\mathcal{C}$ that $\operatorname{Hom}_{\mathcal{C}}(F,G)$ is a set when there is an epimorphism $h_X \to F$ for some object $X \in \mathcal{C}$.

(Co)kernels and (co)products in $\operatorname{Mod}\mathcal{C}$ are computed *pointwise*, and it follows that $\operatorname{Mod}\mathcal{C}$ is an abelian category which has set-indexed products and coproducts. A sequence $F \to G \to H$ of morphisms in $\operatorname{Mod}\mathcal{C}$ is exact if and only if the sequence $F(X) \to G(X) \to H(X)$ is exact for all X in $\mathcal{C}$.

Let $\operatorname{mod}\mathcal{C}$ denote the category of finitely presented functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$, where a functor F is *finitely presented* if it fits into an exact sequence

$$\operatorname{Hom}_{\mathcal{C}}(-,X) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(-,Y) \longrightarrow F \longrightarrow 0.$$

The morphisms in $\operatorname{mod}\mathcal{C}$ are given by the natural transformations.

A morphism $X \to Y$ in $\mathcal{C}$ is a *weak kernel* of a morphism $Y \to Z$ if the induced sequence

$$\operatorname{Hom}_{\mathcal{C}}(-,X) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(-,Y) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(-,Z)$$

is exact. Let $\mathcal{D}$ be an abelian category. Then an additive functor $F\colon \mathcal{C} \to \mathcal{D}$ is *weakly left exact* if it sends each weak kernel sequence $X \to Y \to Z$ in $\mathcal{C}$ to an exact sequence $F(X) \to F(Y) \to F(Z)$ in $\mathcal{D}$.

Lemma 2.1.6. *The category* $\operatorname{mod}\mathcal{C}$ *is additive and all morphisms in* $\operatorname{mod}\mathcal{C}$ *have cokernels. The category is abelian if and only if all morphisms in $\mathcal{C}$ have weak kernels.*

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Proof We fix a pair of functors with finite presentations

$$\mathrm{Hom}(-, X_i) \longrightarrow \mathrm{Hom}(-, Y_i) \longrightarrow F_i \longrightarrow 0 \qquad (i = 1, 2).$$

A morphism $\phi\colon F_1 \to F_2$ gives rise to a commutative diagram

$$\begin{array}{ccccccc} \mathrm{Hom}(-, X_1) & \longrightarrow & \mathrm{Hom}(-, Y_1) & \longrightarrow & F_1 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \phi} & & \\ \mathrm{Hom}(-, X_2) & \longrightarrow & \mathrm{Hom}(-, Y_2) & \longrightarrow & F_2 & \longrightarrow & 0 \end{array}$$

in $\operatorname{mod} \mathcal{C}$. We obtain presentations

$$\mathrm{Hom}(-, X_1 \oplus X_2) \longrightarrow \mathrm{Hom}(-, Y_1 \oplus Y_2) \longrightarrow F_1 \oplus F_2 \longrightarrow 0$$

and

$$\mathrm{Hom}(-, X_2 \oplus Y_1) \longrightarrow \mathrm{Hom}(-, Y_2) \longrightarrow \operatorname{Coker} \phi \longrightarrow 0.$$

It follows that $\operatorname{mod} \mathcal{C}$ is an additive category with cokernels.

Now suppose that $\mathcal{C}$ has weak kernels. Choose weak kernel sequences

$$Y_0 \longrightarrow X_2 \oplus Y_1 \longrightarrow Y_2 \qquad \text{and} \qquad X_0 \longrightarrow X_1 \oplus Y_0 \longrightarrow Y_1.$$

This gives rise to a commutative diagram

$$\begin{array}{ccccccc} \mathrm{Hom}(-, X_0) & \longrightarrow & \mathrm{Hom}(-, Y_0) & \longrightarrow & \operatorname{Ker} \phi & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Hom}(-, X_1) & \longrightarrow & \mathrm{Hom}(-, Y_1) & \longrightarrow & F_1 & \longrightarrow & 0 \end{array}$$

in $\operatorname{mod} \mathcal{C}$. Thus $\operatorname{mod} \mathcal{C}$ has kernels and it follows that $\operatorname{mod} \mathcal{C}$ is abelian.

Finally, suppose $\operatorname{mod} \mathcal{C}$ is abelian and fix a morphism $Y \to Z$ in $\mathcal{C}$. Let F denote the kernel of the induced morphism $\mathrm{Hom}(-, Y) \to \mathrm{Hom}(-, Z)$ in $\operatorname{mod} \mathcal{C}$. Then there is an epimorphism $\mathrm{Hom}(-, X) \to F$, and the composite $\mathrm{Hom}(-, X) \to F \to \mathrm{Hom}(-, Y)$ induces a weak kernel $X \to Y$ for the morphism $Y \to Z$. □

If $\mathcal{D}$ is an additive category with cokernels, then every additive functor $F\colon \mathcal{C} \to \mathcal{D}$ extends essentially uniquely to a right exact functor $\bar{F}\colon \operatorname{mod} \mathcal{C} \to \mathcal{D}$ such that $\bar{F}(h_X) = F(X)$ for all $X \in \mathcal{C}$. To be precise, set $\bar{F}(\operatorname{Coker} h_\phi) = \operatorname{Coker} F(\phi)$ for an object $\operatorname{Coker} h_\phi$ in $\operatorname{mod} \mathcal{C}$ given by a morphism ϕ in $\mathcal{C}$. This universal property of the Yoneda functor $h\colon \mathcal{C} \to \operatorname{mod} \mathcal{C}$ can be reformulated as follows.

Lemma 2.1.7. *For any additive category $\mathcal{D}$ with cokernels, composition with the Yoneda functor induces a functor*

$$\mathcal{H}om(\operatorname{mod}\mathcal{C},\mathcal{D}) \longrightarrow \mathcal{H}om(\mathcal{C},\mathcal{D}), \qquad F \mapsto F \circ h$$

that yields an equivalence when restricted to the full subcategory of right exact functors in $\mathcal{H}om(\operatorname{mod}\mathcal{C},\mathcal{D})$ *and the full subcategory of additive functors in* $\mathcal{H}om(\mathcal{C},\mathcal{D})$. □

The above lemma has an analogue for functors $\operatorname{mod}\mathcal{C} \to \mathcal{D}$ that are exact. Thus we suppose that all morphisms in $\mathcal{C}$ have weak kernels so that $\operatorname{mod}\mathcal{C}$ is abelian.

Lemma 2.1.8. *For any abelian category $\mathcal{D}$, composition with the Yoneda functor induces a functor*

$$\mathcal{H}om(\operatorname{mod}\mathcal{C},\mathcal{D}) \longrightarrow \mathcal{H}om(\mathcal{C},\mathcal{D}), \qquad F \mapsto F \circ h$$

that yields an equivalence when restricted to the full subcategory of exact functors in $\mathcal{H}om(\operatorname{mod}\mathcal{C},\mathcal{D})$ *and the full subcategory of additive functors in* $\mathcal{H}om(\mathcal{C},\mathcal{D})$ *that are weakly left exact.*

Proof Fix an additive functor $F\colon \mathcal{C} \to \mathcal{D}$ and its right exact extension $\bar{F}\colon \operatorname{mod}\mathcal{C} \to \mathcal{D}$ satisfying $\bar{F}(h_X) = F(X)$ for all $X \in \mathcal{C}$. We claim that $\bar{F}$ is exact if and only if F is weakly left exact. One direction is clear. So suppose that F is weakly left exact. Choose an exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$ in $\operatorname{mod}\mathcal{C}$. Then we may choose presentations

$$h_{X_{i2}} \longrightarrow h_{X_{i1}} \longrightarrow h_{X_{i0}} \longrightarrow X_i \longrightarrow 0 \qquad (i = 1,2,3)$$

that induce a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & h_{X_{12}} & \longrightarrow & h_{X_{22}} & \longrightarrow & h_{X_{32}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & h_{X_{11}} & \longrightarrow & h_{X_{21}} & \longrightarrow & h_{X_{31}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & h_{X_{10}} & \longrightarrow & h_{X_{20}} & \longrightarrow & h_{X_{30}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

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such that each sequence

$$0 \longrightarrow h_{X_{1j}} \longrightarrow h_{X_{2j}} \longrightarrow h_{X_{3j}} \longrightarrow 0 \qquad (j = 0, 1, 2)$$

is split exact. It follows that each sequence

$$F(X_{i2}) \longrightarrow F(X_{i1}) \longrightarrow F(X_{i0}) \longrightarrow \bar{F}(X_i) \longrightarrow 0 \qquad (i = 1, 2, 3)$$

is exact since F is weakly left exact and $\bar{F}$ is right exact. Thus the snake lemma implies that $0 \to \bar{F}(X_1) \to \bar{F}(X_2) \to \bar{F}(X_3) \to 0$ is exact. □

Remark 2.1.9. Let $\mathcal{C}$ be an additive category with kernels. Then left exact functors and weakly left exact functors $F\colon \mathcal{C} \to \mathcal{D}$ agree.

We end our discussion of finitely presented functors with an equivalent description. Let $\mathcal{C}$ be an additive category and denote by $\mathcal{C}^{\mathbf{2}}$ the *category of morphisms* in $\mathcal{C}$. The objects are morphisms $x\colon X_1 \to X_0$ in $\mathcal{C}$, and for an object $y\colon Y_1 \to Y_0$ the morphisms $\phi\colon x \to y$ are given by pairs of morphisms (ϕ_0, ϕ_1) making the following square commutative.

$$\begin{array}{ccc} X_1 & \xrightarrow{x} & X_0 \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} \\ Y_1 & \xrightarrow{y} & Y_0 \end{array}$$

Such a morphism ϕ is called *null-homotopic* if there is a morphism $\rho\colon X_0 \to Y_1$ satisfying $y \circ \rho = \phi_0$. Let us denote by $\mathcal{C}^{\mathbf{2}}/\mathrm{htp}$ the category which is obtained from $\mathcal{C}^{\mathbf{2}}$ by identifying parallel morphisms ϕ and ψ if $\phi - \psi$ is null-homotopic.

Lemma 2.1.10. *Taking an object $x\colon X_1 \to X_0$ in $\mathcal{C}^{\mathbf{2}}$ to the functor F_x with presentation*

$$\mathrm{Hom}_{\mathcal{C}}(-, X_1) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(-, X_0) \longrightarrow F_x \longrightarrow 0$$

yields an equivalence $\mathcal{C}^{\mathbf{2}}/\mathrm{htp} \xrightarrow{\sim} \operatorname{mod} \mathcal{C}$. □

We give an application. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between additive categories. We write $f_!\colon \operatorname{mod} \mathcal{C} \to \operatorname{mod} \mathcal{D}$ for the right exact functor sending h_X to $h_{f(X)}$ for each $X \in \mathcal{C}$.

Lemma 2.1.11. *$f_!\colon \operatorname{mod} \mathcal{C} \to \operatorname{mod} \mathcal{D}$ is fully faithful if and only if $f\colon \mathcal{C} \to \mathcal{D}$ is fully faithful.*

Proof Clearly, f is fully faithful if and only if the induced functor $\mathcal{C}^{\mathbf{2}}/\mathrm{htp} \to \mathcal{D}^{\mathbf{2}}/\mathrm{htp}$ is fully faithful. □

Exact Categories

Let $\mathcal{A}$ be an additive category. A sequence

$$0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0$$

of morphisms in $\mathcal{A}$ is *exact* if α is a kernel of β and β is a cokernel of α. An *exact category* is a pair $(\mathcal{A}, \mathcal{E})$ consisting of an additive category $\mathcal{A}$ and a class $\mathcal{E}$ of exact sequences in $\mathcal{A}$ (called *admissible* and given by an *admissible monomorphism* followed by an *admissible epimorphism*) which is closed under isomorphisms and satisfies the following axioms.

(Ex1) The identity morphism of each object is an admissible monomorphism and an admissible epimorphism.

(Ex2) The composite of two admissible monomorphisms is an admissible monomorphism, and the composite of two admissible epimorphisms is an admissible epimorphism.

(Ex3) Each pair of morphisms $X' \xleftarrow{\phi} X \xrightarrow{\alpha} Y$ with α an admissible monomorphism can be completed to a pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow \\ X' & \xrightarrow{\alpha'} & Y' \end{array}$$

such that α' is an admissible monomorphism. And each pair of morphisms $Y \xrightarrow{\beta} Z \xleftarrow{\psi} Z'$ with β an admissible epimorphism can be completed to a pullback diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\beta'} & Z' \\ \downarrow & & \downarrow{\scriptstyle\psi} \\ Y & \xrightarrow{\beta} & Z \end{array}$$

such that β' is an admissible epimorphism.

Observe that in (Ex3) the morphism ϕ induces an isomorphism $\operatorname{Coker}\alpha \xrightarrow{\sim} \operatorname{Coker}\alpha'$, while ψ induces an isomorphism $\operatorname{Ker}\beta' \xrightarrow{\sim} \operatorname{Ker}\beta$.

A pair of admissible exact sequences ξ and ξ' is called *equivalent* if there is a commutative diagram of the following form.

$$\begin{array}{lccccccccc} \xi: & 0 & \longrightarrow & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle\phi} & & \| & & \\ \xi': & 0 & \longrightarrow & X & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z & \longrightarrow & 0 \end{array}$$

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In this case ϕ is an isomorphism. We write $\operatorname{Ext}^1_{\mathcal{A}}(Z,X)$ for the set of equivalence classes of such *extensions* and note that it is an abelian group via the *Baer sum*, which is given by the following diagram.

$$\begin{array}{lccccccccc}
\xi_1 \oplus \xi_2\colon & 0 & \longrightarrow & X\oplus X & \longrightarrow & Y_1\oplus Y_2 & \longrightarrow & Z\oplus Z & \longrightarrow & 0 \\
 & & & \| & & \uparrow & & \uparrow{\scriptstyle\Delta} & & \\
 & 0 & \longrightarrow & X\oplus X & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle\nabla} & & \downarrow & & \| & & \\
\xi_1 + \xi_2\colon & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0
\end{array}$$

We obtain a functor

$$\operatorname{Ext}^1_{\mathcal{A}}(-,-)\colon \mathcal{A}^{\mathrm{op}}\times\mathcal{A}\longrightarrow \mathrm{Ab}$$

which is given on morphisms by taking pullbacks (in the first argument) and pushouts (in the second argument).

Given exact categories $\mathcal{A}$ and $\mathcal{B}$, a functor $\mathcal{A}\to\mathcal{B}$ is *exact* if it is additive and takes admissible exact sequences in $\mathcal{A}$ to admissible exact sequences in $\mathcal{B}$. A *full exact subcategory* of an exact category $\mathcal{A}$ is a full additive subcategory $\mathcal{B}\subseteq\mathcal{A}$ that is *extension closed*, which means that for an admissible exact sequence in $\mathcal{A}$ with end terms in $\mathcal{B}$ the middle term is also in $\mathcal{B}$.

Example 2.1.12. (1) An additive category endowed with all split exact sequences is an exact category.

(2) An abelian category endowed with all short exact sequences is an exact category. Conversely, an exact category is an abelian category, if each morphism ϕ admits a factorisation $\phi=\phi''\phi'$ such that ϕ' is an admissible epimorphism and ϕ'' is an admissible monomorphism.

(3) Let $\mathcal{B}$ be an exact category and let $\mathcal{A}\subseteq\mathcal{B}$ be a full exact subcategory. Then $\mathcal{A}$ becomes an exact category by taking as admissible exact sequences those which are admissible in $\mathcal{B}$.

(4) Any essentially small exact category $\mathcal{A}$ can be embedded into an abelian category $\mathcal{B}$ such that it identifies with a full extension closed subcategory. For instance, take for $\mathcal{B}$ the category of left exact functors $F\colon\mathcal{A}^{\mathrm{op}}\to\mathrm{Ab}$; see Proposition 2.3.7. This yields an alternative definition for essentially small categories: an exact category is a full extension closed subcategory $\mathcal{A}\subseteq\mathcal{B}$ of an abelian category $\mathcal{B}$, endowed with all sequences which are short exact in $\mathcal{B}$.

(5) Let $\mathcal{C}$ be an additive category. Then $\operatorname{mod}\mathcal{C}$ is an exact category, if one chooses as admissible exact sequences the sequences that are pointwise exact.

Projective and Injective Objects

Let $\mathcal{A}$ be an exact category. An object P in $\mathcal{A}$ is *projective* if every admissible epimorphism $X \to Y$ induces a surjective map $\mathrm{Hom}_{\mathcal{A}}(P, X) \to \mathrm{Hom}_{\mathcal{A}}(P, Y)$. Dually, an object I is *injective* if every admissible monomorphism $X \to Y$ induces a surjective map $\mathrm{Hom}_{\mathcal{A}}(Y, I) \to \mathrm{Hom}_{\mathcal{A}}(X, I)$.

An exact category $\mathcal{A}$ has *enough projective objects* if every object X in $\mathcal{A}$ admits an admissible epimorphism $P \to X$ such that P is projective, and $\mathcal{A}$ has *enough injective objects* if every object X in $\mathcal{A}$ admits an admissible monomorphism $X \to I$ such that I is injective.

Example 2.1.13. (1) The category of modules over a ring Λ has enough projective objects, because every free module is projective. We write $\mathrm{Proj}\,\Lambda$ for the full subcategory of projective Λ-modules.

(2) The category of modules over a ring Λ has enough injective objects, and we write $\mathrm{Inj}\,\Lambda$ for the full subcategory of injective Λ-modules. More generally, any Grothendieck category has enough injective objects; cf. Corollary 2.5.4.

(3) Let $\mathcal{C}$ be an additive category and view $\mathrm{mod}\,\mathcal{C}$ as an exact category, with exact structure given by all pointwise exact sequences. Then each representable functor $\mathrm{Hom}_{\mathcal{C}}(-, X)$ is a projective object in $\mathrm{mod}\,\mathcal{C}$ by Yoneda's lemma.

Let $\mathcal{A}$ be an exact category and write $\mathcal{C} := \mathrm{Proj}\,\mathcal{A}$ for the full subcategory of projective objects in $\mathcal{A}$. Suppose that $\mathcal{A}$ has enough projective objects. Then every object $X \in \mathcal{A}$ admits a *projective presentation*

$$P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0,$$

that is, an exact sequence such that each P_i is projective. This yields an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(-, P_1) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(-, P_0) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(-, X)|_{\mathcal{C}} \longrightarrow 0$$

and therefore the functor

$$F \colon \mathcal{A} \longrightarrow \mathrm{mod}\,\mathcal{C}, \quad X \mapsto \mathrm{Hom}_{\mathcal{A}}(-, X)|_{\mathcal{C}}$$

is well defined.

Lemma 2.1.14. *The functor F is fully faithful; it is an equivalence when $\mathcal{A}$ is abelian.*

Proof For the first assertion fix objects X, Y in $\mathcal{A}$ and choose projective presentations

$$P_1 \xrightarrow{p} P_0 \to X \to 0 \qquad \text{and} \qquad Q_1 \xrightarrow{q} Q_0 \to Y \to 0.$$

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Then the morphisms $X \to Y$ in $\mathcal{A}$ correspond to equivalence classes of commutative squares in $\mathcal{C}$

$$\begin{array}{ccc} P_1 & \xrightarrow{p} & P_0 \\ \downarrow & & \downarrow \\ Q_1 & \xrightarrow{q} & Q_0 \end{array}$$

which in turn correspond to morphisms $\operatorname{Hom}_{\mathcal{A}}(-, X)|_{\mathcal{C}} \to \operatorname{Hom}_{\mathcal{A}}(-, Y)|_{\mathcal{C}}$, by Lemma 2.1.10.

Now suppose that $\mathcal{A}$ is abelian. Then the inclusion $\mathcal{C} \to \mathcal{A}$ extends to a quasi-inverse $\operatorname{mod} \mathcal{C} \to \mathcal{A}$ for F. □

We obtain the following correspondence; it provides a useful principle when dealing with abelian categories having enough projectives.

Proposition 2.1.15. *The assignments $\mathcal{C} \mapsto \operatorname{mod} \mathcal{C}$ and $\mathcal{A} \mapsto \operatorname{Proj} \mathcal{A}$ induce (up to equivalence) mutually inverse bijections between*

- *additive categories that are idempotent complete such that each morphism admits a weak kernel, and*
- *abelian categories with enough projective objects.* □

An application of this correspondence is the following criterion.

Corollary 2.1.16. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then a right exact functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is exact if and only if for each exact sequence $X_2 \to X_1 \to X_0$ in $\mathcal{A}$ with each $X_i \in \operatorname{Proj} \mathcal{A}$ the sequence $FX_2 \to FX_1 \to FX_0$ is exact.*

Proof Set $\mathcal{C} = \operatorname{Proj} \mathcal{A}$ and identify $\mathcal{A} = \operatorname{mod} \mathcal{C}$. Then apply Lemma 2.1.8. □

There is a dual version of the above proposition for abelian categories with enough injective objects.

Proposition 2.1.17. *The assignments $\mathcal{C} \mapsto (\operatorname{mod}(\mathcal{C}^{\mathrm{op}}))^{\mathrm{op}}$ and $\mathcal{A} \mapsto \operatorname{Inj} \mathcal{A}$ induce (up to equivalence) mutually inverse bijections between*

- *additive categories that are idempotent complete such that each morphism admits a weak cokernel, and*
- *abelian categories with enough injective objects.* □

We end our discussion of projectives and injectives with a basic fact that will be used throughout without further reference.

Lemma 2.1.18. *The left adjoint of an exact functor takes projective objects to projective objects. Dually, the right adjoint of an exact functor takes injective objects to injective objects.* □

Projective Covers and Injective Envelopes

Let $\mathcal{A}$ be an abelian category. An epimorphism $\phi\colon X \to Y$ is *essential* if any morphism $\alpha\colon X' \to X$ is an epimorphism provided that the composite $\phi\alpha$ is an epimorphism. This condition can be rephrased as follows: if $U \subseteq X$ is a subobject with $U + \operatorname{Ker}\phi = X$, then $U = X$. An epimorphism $\phi\colon P \to X$ is a *projective cover* of X if P is projective and ϕ is essential.

There are the following dual notions. A monomorphism $\phi\colon X \to Y$ is *essential* if any morphism $\alpha\colon Y \to Y'$ is a monomorphism provided that the composite $\alpha\phi$ is a monomorphism. This condition can be rephrased as follows: if $U \subseteq Y$ is a subobject with $U \cap \operatorname{Im}\phi = 0$, then $U = 0$. A monomorphism $\phi\colon X \to I$ is an *injective envelope* of X if I is injective and ϕ is essential.

We collect some basic properties of projective covers and injective envelopes. In most cases we provide only one formulation (say, about injective envelopes) and leave the dual result (about projective covers) to the reader.

Lemma 2.1.19. *Let I be an injective object. Then the following are equivalent for a monomorphism $\phi\colon X \to I$.*

(1) *The morphism ϕ is an injective envelope of X.*
(2) *Every endomorphism $\alpha\colon I \to I$ satisfying $\alpha\phi = \phi$ is an isomorphism.*

Proof (1) $\Rightarrow$ (2): Let $\alpha\colon I \to I$ be an endomorphism satisfying $\alpha\phi = \phi$. Then α is a monomorphism since ϕ is essential. Thus there exists $\alpha'\colon I \to I$ satisfying $\alpha'\alpha = \mathrm{id}_I$ since I is injective. It follows that $\alpha'\phi = \phi$ and therefore α' is a monomorphism. On the other hand, α' is an epimorphism. Thus α' and α are isomorphisms.

(2) $\Rightarrow$ (1): Let $\alpha\colon I \to I'$ be a morphism such that $\alpha\phi$ is a monomorphism. Then ϕ factors through $\alpha\phi$ via a morphism $\alpha'\colon I' \to I$ since I is injective. The composite $\alpha'\alpha$ is an isomorphism and therefore α is a monomorphism. Thus ϕ is essential. □

We write $E(X) = I$ when $X \to I$ is an injective envelope. The following statement justifies this notation.

Lemma 2.1.20. *Let $\phi\colon X \to I$ and $\phi'\colon X \to I'$ be injective envelopes of an object X. Then there is an isomorphism $\alpha\colon I \to I'$ such that $\phi' = \alpha\phi$.* □

There is a close relation between projective covers and radical morphisms. We establish this in two steps: first for modules, and then for general abelian categories.

Lemma 2.1.21. *Let* Λ *be a ring and* $P_1 \xrightarrow{\phi} P_0 \xrightarrow{\psi} X \to 0$ *an exact sequence of* Λ*-modules such that each* P_i *is finitely generated projective. Then the following are equivalent.*

(1) ψ *is essential.*
(2) $\operatorname{Im}\phi \subseteq \operatorname{rad} P_0$.
(3) $\phi \in \operatorname{Rad}(P_1, P_0)$.

Proof (1) ⇒ (2): Set $U = \operatorname{Im}\phi$. Suppose that ψ is essential and let $V \subseteq P_0$ be a maximal subobject not containing U. Then $U + V = P_0$ and therefore $V = P_0$. This is a contradiction and therefore U is contained in every maximal subobject. Thus $U \subseteq \operatorname{rad} P_0$.

(2) ⇒ (1): Suppose that $U \subseteq \operatorname{rad} P_0$ and let $V \subseteq P_0$ be a subobject with $U + V = P_0$. If $V \neq P_0$, then there is a maximal subobject $V' \subseteq P_0$ containing V since P_0 is finitely generated. Thus $P_0 = U + V \subseteq V'$. This is a contradiction and therefore $V = P_0$. It follows that ψ is essential.

(2) ⇔ (3): When $P_1 = \Lambda$ we have the identification $\operatorname{Rad}(\Lambda, P_0) = \operatorname{rad} P_0$ via $\lambda \mapsto \lambda(1)$. In particular, for $\lambda\colon \Lambda \to P_0$ we have $\lambda \in \operatorname{Rad}(\Lambda, P_0)$ if and only if $\operatorname{Im}\lambda \subseteq \operatorname{rad} P_0$. More generally, for $\lambda\colon \Lambda^n \to P_0$ we have $\lambda \in \operatorname{Rad}(\Lambda^n, P_0)$ if and only if $\operatorname{Im}\lambda \subseteq \operatorname{rad} P_0$.

For the implication (3) ⇒ (2) choose an epimorphism $\pi\colon \Lambda^n \to P_1$. Then $\phi\pi$ is a radical morphism and therefore $\operatorname{Im}\phi = \operatorname{Im}\pi\phi \subseteq \operatorname{rad} P_0$.

For the implication (2) ⇒ (3) choose an epimorphism $\Lambda^n \to U$. Then $\lambda\colon \Lambda^n \to U \rightarrowtail P_0$ is a radical morphism, and therefore $\phi \in \operatorname{Rad}(P_1, P_0)$ since ϕ factors through λ. □

Proposition 2.1.22. *Let* $P_1 \xrightarrow{\phi} P_0 \xrightarrow{\psi} X \to 0$ *be an exact sequence in an abelian category such that each* P_i *is projective. Then* ψ *is a projective cover if and only if* $\phi \in \operatorname{Rad}(P_1, P_0)$.

Proof Let $\mathcal{A}$ denote the abelian category and $\mathcal{C}$ the smallest full additive subcategory which is closed under cokernels and contains $P = P_0 \oplus P_1$. Set $\Lambda = \operatorname{End}(P)$. The functor $H = \operatorname{Hom}(P, -)\colon \mathcal{A} \to \operatorname{Mod}\Lambda$ restricts to an equivalence $\mathcal{C} \xrightarrow{\sim} \operatorname{mod}\Lambda$. It follows from the dual of Lemma 2.1.19 that ψ is a projective cover if and only if $H\psi$ is a projective cover. On the other hand, $\operatorname{Rad}(P_1, P_0) \xrightarrow{\sim} \operatorname{Rad}(HP_1, HP_0)$ via H. Thus the assertion follows from Lemma 2.1.21. □

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We record the dual characterisation of injective envelopes via the radical.

Proposition 2.1.23. *Let* $0 \to X \xrightarrow{\phi} I^0 \xrightarrow{\psi} I^1$ *be an exact sequence in an abelian category such that each* I^i *is injective. Then* ϕ *is an injective envelope if and only if* $\psi \in \operatorname{Rad}(I^0, I^1)$. □

We say that a morphism $\phi\colon X \to Y$ in an additive category admits a *minimal decomposition* if ϕ can be written as a direct sum

$$X = X' \oplus X'' \xrightarrow{\phi' \oplus \phi''} Y' \oplus Y'' = Y$$

such that ϕ' is an isomorphism and ϕ'' is a radical morphism.

An abelian category has *injective envelopes* if every object admits an injective envelope. Dually, an abelian category has *projective covers* if every object admits a projective cover.

Corollary 2.1.24. *Let* $\mathcal{A}$ *be an abelian category with enough injective objects. Then* $\mathcal{A}$ *has injective envelopes if and only if all morphisms in* $\operatorname{Inj}\mathcal{A}$ *admit minimal decompositions.*

Proof Suppose first that $\mathcal{A}$ has injective envelopes. Let $\phi\colon X \to Y$ be a morphism in $\operatorname{Inj}\mathcal{A}$. Choose a decomposition $X = X' \oplus X''$ such that $X'' = E(\operatorname{Ker}\phi)$. Let $\phi'\colon X' \xrightarrow{\sim} Y' = \phi(X')$ be the restriction $\phi|_{X'}$. Then ϕ' is a direct summand of ϕ. Thus we get a decomposition $\phi = \phi' \oplus \phi''$ and ϕ'' is radical by Proposition 2.1.23.

For the converse let $A \in \mathcal{A}$ and choose an exact sequence $0 \to A \to X \xrightarrow{\phi} Y$ with $\phi \in \operatorname{Inj}\mathcal{A}$. Decomposing $\phi = \phi' \oplus \phi''$ yields an injective envelope $A \to X''$, again by Proposition 2.1.23. □

Example 2.1.25. Let $\mathcal{A}$ be a Krull–Schmidt category. Then every morphism $\phi\colon X \to Y$ in $\mathcal{A}$ admits a minimal decomposition.

To see this, choose decompositions $X = \bigoplus_i X_i$ and $Y = \bigoplus_j Y_j$ into indecomposables. Then $\phi = (\phi_{ij})$ belongs to $\operatorname{Rad}(X, Y)$ if and only if $\phi_{ij} \in \operatorname{Rad}(X_i, Y_j)$ for all i, j. Suppose $\phi_{i_0 j_0}$ is not radical. Then $\phi_{i_0 j_0}$ is an isomorphism and we may decompose $X = X_{i_0} \oplus \bar{X}$ and $Y = Y_{j_0} \oplus \bar{Y}$ such that $\phi = \phi_{i_0 j_0} \oplus \bar{\phi}$. Removing successively summands ϕ_{ij} that are not radical we obtain the decomposition $\phi = \phi' \oplus \phi''$ as required.

Stable Categories

Let $\mathcal{A}$ be an exact category and suppose that $\mathcal{A}$ has enough injective objects. Thus for each object $X \in \mathcal{A}$ we can choose an exact sequence $0 \to X \to I_X \to X' \to 0$ such that I_X is injective.

The *injectively stable category* $\operatorname{St}\mathcal{A}$ has by definition the same objects as $\mathcal{A}$ while the morphisms for objects X, Y are given by the quotient

$$\operatorname{Hom}_{\operatorname{St}\mathcal{A}}(X,Y) = \operatorname{Hom}_{\mathcal{A}}(X,Y)/\{\phi \mid \phi \text{ factors through an injective object}\}.$$

Lemma 2.1.26. *The assignment* $X \mapsto \operatorname{Ext}^1(-,X)$ *induces a fully faithful functor* $\operatorname{St}\mathcal{A} \to \operatorname{mod}\mathcal{A}$.

Proof For each object $X \in \mathcal{A}$ the sequence $0 \to X \to I_X \to X' \to 0$ induces a presentation

$$0 \to \operatorname{Hom}(-,X) \to \operatorname{Hom}(-,I_X) \to \operatorname{Hom}(-,X') \to \operatorname{Ext}^1(-,X) \longrightarrow 0.$$

Given a morphism $\phi\colon X \to Y$ in $\mathcal{A}$, we have $\operatorname{Ext}^1(-,\phi) = 0$ if and only if ϕ factors through $X \to I_X$. On the other hand, given a morphism of functors, $\psi\colon \operatorname{Ext}^1(-,X) \to \operatorname{Ext}^1(-,Y)$, we use Yoneda's lemma and obtain from the above presentation a morphism $\operatorname{Hom}(-,X) \to \operatorname{Hom}(-,Y)$ which corresponds to a morphism $\bar{\psi}\colon X \to Y$ in $\mathcal{A}$. Clearly, $\operatorname{Ext}^1(-,\bar{\psi}) = \psi$. □

We call a pair of objects X, Y in $\mathcal{A}$ *stably equivalent* if the equivalent conditions in the following lemma are satisfied.

Lemma 2.1.27. *For objects $X, Y \in \mathcal{A}$ the following are equivalent.*

(1) $\operatorname{Ext}^1(-,X) \cong \operatorname{Ext}^1(-,Y)$ *in* $\operatorname{mod}\mathcal{A}$.
(2) $X \cong Y$ *in* $\operatorname{St}\mathcal{A}$.
(3) $X \oplus I \cong Y \oplus J$ *in $\mathcal{A}$ for some injective objects $I, J \in \mathcal{A}$.*

Proof (1) ⇔ (2): See Lemma 2.1.26.

(2) ⇒ (3): Let $\phi\colon X \to Y$ be a morphism in $\mathcal{A}$ that becomes invertible in $\operatorname{St}\mathcal{A}$. Adding $X \to I_X$ yields a split monomorphism $X \to Y \oplus I_X$, so $X \oplus I \cong Y \oplus I_X$ for some object I. We have $I = 0$ in $\operatorname{St}\mathcal{A}$, so I is injective.

(3) ⇒ (2): Clear. □

2.2 Localisation of Additive and Abelian Categories

There are specific constructions for localising additive and abelian categories. In both cases the localisation amounts to annihilating a class of objects. Also, the additional categorical structure is preserved. This means the localisation provides an additive functor $\mathcal{A} \to \mathcal{A}[S^{-1}]$ when $\mathcal{A}$ is additive and an exact functor when $\mathcal{A}$ is abelian.

Additive Categories

Let $\mathcal{A}$ be an additive category. When $F\colon \mathcal{A} \to \mathcal{B}$ is an additive functor, then the class $S = \{\sigma \in \operatorname{Mor}\mathcal{A} \mid F\sigma \text{ is invertible}\}$ contains the identities and is closed under finite direct sums. The following criterion shows that this is sufficient for $\mathcal{A}[S^{-1}]$ to be an additive category.

Lemma 2.2.1. *Let $\mathcal{A}$ be an additive category and $S \subseteq \operatorname{Mor}\mathcal{A}$ a class of morphisms. Suppose that S contains the identity morphism of each object and that $\sigma, \tau \in S$ implies $\sigma \oplus \tau \in S$. Then $\mathcal{A}[S^{-1}]$ is an additive category and the canonical functor $\mathcal{A} \to \mathcal{A}[S^{-1}]$ is additive.*

Proof We use the characterisation of an additive category from Remark 2.1.3. Also, we make a number of additional observations.

(1) Finite coproducts in a category $\mathcal{C}$ are given by a left adjoint of the diagonal $\Delta\colon \mathcal{C} \to \mathcal{C}^n$ for any $n \geq 0$. Dually, finite products are given by a right adjoint.

(2) If $\mathcal{C}_i$ and $S_i \subseteq \operatorname{Mor}\mathcal{C}_i$ are categories with classes of morphisms, then

$$\Big(\prod_i \mathcal{C}_i\Big)\Big[\Big(\prod_i S_i\Big)^{-1}\Big] \xrightarrow{\sim} \prod_i \mathcal{C}_i[S_i^{-1}].$$

(3) Let (F, G) be an adjoint pair of functors $\mathcal{C} \rightleftarrows \mathcal{D}$. If $S \subseteq \operatorname{Mor}\mathcal{C}$ and $T \subseteq \operatorname{Mor}\mathcal{D}$ are classes of morphisms such that $F(S) \subseteq T$ and $G(T) \subseteq S$, then (F, G) induces an adjoint pair of functors $\mathcal{C}[S^{-1}] \rightleftarrows \mathcal{D}[T^{-1}]$ (Lemma 1.1.6).

Now it follows that $\mathcal{A}[S^{-1}]$ is a category with finite products and coproducts, and the canonical functor $\mathcal{A} \to \mathcal{A}[S^{-1}]$ preserves these (co)products. Moreover, in $\mathcal{A}[S^{-1}]$ the monoid structure on $\operatorname{Hom}(X, Y)$ given by (2.1.2) yields a group structure for all objects X, Y. □

Let $\mathcal{A}$ be an additive category and let $\mathcal{C} \subseteq \mathcal{A}$ be a full additive subcategory. The *additive quotient category* $\mathcal{A}/\mathcal{C}$ of $\mathcal{A}$ with respect to $\mathcal{C}$ has the same objects as $\mathcal{A}$ while the morphisms for objects X, Y are defined by the quotient

$$\operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(X, Y) = \operatorname{Hom}_{\mathcal{A}}(X, Y)/\{\phi \mid \phi \text{ factors through an object in } \mathcal{C}\}.$$

For a morphism ϕ in $\mathcal{A}$ we write $\bar{\phi}$ for the corresponding morphism in $\mathcal{A}/\mathcal{C}$.

Lemma 2.2.2. *Let $\mathcal{A}$ be an additive category and let $\mathcal{C} \subseteq \mathcal{A}$ be a full additive subcategory. Set*

$$S = S(\mathcal{C}) = \{\sigma \in \operatorname{Mor}\mathcal{A} \mid \bar{\sigma} \textit{ is invertible in } \mathcal{A}/\mathcal{C}\}.$$

Then the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ induces an isomorphism $\mathcal{A}[S^{-1}] \xrightarrow{\sim} \mathcal{A}/\mathcal{C}$.

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Proof Consider the canonical functors $P\colon \mathcal{A} \to \mathcal{A}/\mathcal{C}$ and $Q\colon \mathcal{A} \to \mathcal{A}[S^{-1}]$. Clearly, P factors through Q via a functor $\bar{P}$. Now observe for morphisms α, β in $\mathcal{A}$ that $\bar{\alpha} = \bar{\beta}$ implies $Q\alpha = Q\beta$, since Q is additive by Lemma 2.2.1. Thus Q factors through P via a functor $\bar{Q}$. It follows that $\bar{P}\bar{Q} = \mathrm{id}$ and $\bar{Q}\bar{P} = \mathrm{id}$, since P and Q provide solutions of some universal problems. □

Abelian Categories

Let $\mathcal{A}$ be an abelian category. A full additive subcategory $\mathcal{C} \subseteq \mathcal{A}$ is a *Serre subcategory* provided that $\mathcal{C}$ is closed under taking subobjects, quotients and extensions. This means that for every exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$, the object X is in $\mathcal{C}$ if and only if X' and X'' are in $\mathcal{C}$.

Example 2.2.3. The kernel of an exact functor $\mathcal{A} \to \mathcal{B}$ between abelian categories is a Serre subcategory of $\mathcal{A}$.

Fix a Serre subcategory $\mathcal{C}$ of $\mathcal{A}$. We set

$$S(\mathcal{C}) = \{\sigma \in \operatorname{Mor}\mathcal{A} \mid \operatorname{Ker}\sigma, \operatorname{Coker}\sigma \in \mathcal{C}\}$$

and

$$\mathcal{C}^{\perp} = \{Y \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(X,Y) = 0 = \operatorname{Ext}^1_{\mathcal{A}}(X,Y) \text{ for all } X \in \mathcal{C}\}.$$

The *abelian quotient category* $\mathcal{A}/\mathcal{C}$ of $\mathcal{A}$ with respect to $\mathcal{C}$ has the same objects while the morphisms for objects X, Y are defined as follows. There is for each pair of subobjects $X' \subseteq X$ and $Y' \subseteq Y$ an induced map $\operatorname{Hom}_{\mathcal{A}}(X,Y) \to \operatorname{Hom}_{\mathcal{A}}(X', Y/Y')$. The pairs (X', Y') such that both X/X' and Y' lie in $\mathcal{C}$ form a directed set, and one obtains a directed system of abelian groups $\operatorname{Hom}_{\mathcal{A}}(X', Y/Y')$. Then one defines

$$\operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(X,Y) = \operatorname*{colim}_{(X',Y')} \operatorname{Hom}_{\mathcal{A}}(X', Y/Y').$$

The composition of morphisms in $\mathcal{A}$ induces the composition in $\mathcal{A}/\mathcal{C}$.

Lemma 2.2.4. *For a Serre subcategory $\mathcal{C} \subseteq \mathcal{A}$ the following holds.*

(1) *$S(\mathcal{C})$ admits a calculus of left and right fractions.*
(2) *An object in $\mathcal{A}$ is $S(\mathcal{C})$-local if and only if it is in $\mathcal{C}^{\perp}$.*
(3) *The canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ induces an isomorphism $\mathcal{A}[S(\mathcal{C})^{-1}] \xrightarrow{\sim} \mathcal{A}/\mathcal{C}$.*

Proof (1) and (2) are straightforward. For (3) we apply Lemma 1.2.2. Given objects X, Y in $\mathcal{A}$ we have

$$\begin{aligned}\operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(X,Y) &= \operatorname*{colim}_{(X',Y')} \operatorname{Hom}_{\mathcal{A}}(X', Y/Y') \\ &\cong \operatorname*{colim}_{(\sigma,\tau)} \operatorname{Hom}_{\mathcal{A}}(\bar{X}, \bar{Y}) \\ &\cong \operatorname{Hom}_{\mathcal{A}[S^{-1}]}(X,Y)\end{aligned}$$

where $\sigma\colon \bar{X} \to X$ and $\tau\colon Y \to \bar{Y}$ run through all morphisms in $S(\mathcal{C})$. □

A consequence is the following useful observation describing the morphisms in $\mathcal{A}/\mathcal{C}$. For each morphism $\phi\colon X \to Y$ in $\mathcal{A}/\mathcal{C}$ we have a commutative square

$$\begin{array}{ccc} X' & \longrightarrow & Y/Y' \\ \big\downarrow & & \big\uparrow \\ X & \xrightarrow{\ \phi\ } & Y \end{array}$$

such that the other three morphisms are in the image of $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ and the vertical morphisms are isomorphisms in $\mathcal{A}/\mathcal{C}$, since X/X' and Y' lie in $\mathcal{C}$. There is an analogue for exact sequences in $\mathcal{A}/\mathcal{C}$; see Lemma 14.1.9.

The following provides another useful fact about the morphisms in $\mathcal{A}/\mathcal{C}$.

Lemma 2.2.5. *Let $\mathcal{C} \subseteq \mathcal{A}$ be a Serre subcategory and $Y \in \mathcal{A}$. Then the canonical map*

$$\operatorname{Hom}_{\mathcal{A}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(X,Y)$$

is a bijection for all $X \in \mathcal{A}$ if and only if $Y \in \mathcal{C}^{\perp}$.

Proof This follows from Lemma 1.1.2 and Lemma 2.2.4. □

Proposition 2.2.6. *Let $\mathcal{A}$ be an abelian category and $\mathcal{C} \subseteq \mathcal{A}$ a Serre subcategory. Then the following holds.*

(1) *The category $\mathcal{A}/\mathcal{C}$ is abelian and the canonical functor $Q\colon \mathcal{A} \to \mathcal{A}/\mathcal{C}$ is an exact functor that annihilates $\mathcal{C}$.*
(2) *If $\mathcal{B}$ is an abelian category and $F\colon \mathcal{A} \to \mathcal{B}$ is an exact functor that annihilates $\mathcal{C}$, then there exists a unique exact functor $\bar{F}\colon \mathcal{A}/\mathcal{C} \to \mathcal{B}$ such that $F = \bar{F} \circ Q$.*

Proof (1) We apply Lemma 2.2.4. Thus $\mathcal{A}/\mathcal{C} = \mathcal{A}[S^{-1}]$ for $S = S(\mathcal{C})$, and S admits a calculus of left and right fractions. The category $\mathcal{A}/\mathcal{C}$ is additive by Lemma 2.2.1. A morphism $X \to Y$ in $\mathcal{A}/\mathcal{C}$ is up to an isomorphism of the form

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$Q\phi$ for some $\phi\colon X \to Y$ in $\mathcal{A}$. Choosing a cokernel $\psi\colon Y \to Z$ yields for each $A \in \mathcal{A}$ an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(Z, A) \to \operatorname{Hom}_{\mathcal{A}}(Y, A) \to \operatorname{Hom}_{\mathcal{A}}(X, A)$$

and therefore an exact sequence

$$0 \to \operatorname*{colim}_{A \to A'} \operatorname{Hom}_{\mathcal{A}}(Z, A') \to \operatorname*{colim}_{A \to A'} \operatorname{Hom}_{\mathcal{A}}(Y, A') \to \operatorname*{colim}_{A \to A'} \operatorname{Hom}_{\mathcal{A}}(X, A')$$

where $A \to A'$ runs through all morphisms in S starting at A. Thus the sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(Z, A) \to \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(Y, A) \to \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(X, A)$$

is exact by Lemma 1.2.2, and it follows that $Q\psi$ is a cokernel of $Q\phi$. The dual argument shows that each morphism in $\mathcal{A}/\mathcal{C}$ admits a kernel. Clearly, Q preserves kernels and cokernels; so the property of $\mathcal{A}$ to be abelian carries over to $\mathcal{A}/\mathcal{C}$.

(2) If $F\colon \mathcal{A} \to \mathcal{B}$ is an exact functor and $F|_{\mathcal{C}} = 0$, then F inverts all morphisms in S. Thus F factors through $Q\colon \mathcal{A} \to \mathcal{A}/\mathcal{C}$ via a unique functor $\bar{F}\colon \mathcal{A}/\mathcal{C} \to \mathcal{B}$. The functor $\bar{F}$ is exact, because any exact sequence in $\mathcal{A}/\mathcal{C}$ is up to isomorphism the image of an exact sequence in $\mathcal{A}$. □

Remark 2.2.7. (1) The properties (1)–(2) in Proposition 2.2.6 provide a universal property that determines the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ up to a unique isomorphism.

(2) The canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ preserves all coproducts in $\mathcal{A}$ if and only if $\mathcal{C}$ is closed under coproducts; see Lemma 1.1.8.

Next we describe all Serre subcategories of a quotient $\mathcal{A}/\mathcal{C}$.

Proposition 2.2.8. *Let $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ be Serre subcategories of an abelian category $\mathcal{A}$. Then $\mathcal{B}/\mathcal{C}$ identifies with a Serre subcategory of $\mathcal{A}/\mathcal{C}$, and every Serre subcategory of $\mathcal{A}/\mathcal{C}$ is of this form. Moreover, the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ induces an isomorphism $\mathcal{A}/\mathcal{B} \xrightarrow{\sim} (\mathcal{A}/\mathcal{C})/(\mathcal{B}/\mathcal{C})$.*

We capture the situation in the following commutative diagram.

$$\begin{array}{ccccc}
\mathcal{C} & \rightarrowtail & \mathcal{B} & \twoheadrightarrow & \mathcal{B}/\mathcal{C} \\
\| & & \downarrow & & \downarrow \\
\mathcal{C} & \rightarrowtail & \mathcal{A} & \twoheadrightarrow & \mathcal{A}/\mathcal{C} \\
 & & \downarrow & & \downarrow \\
 & & \mathcal{A}/\mathcal{B} & \xrightarrow{\sim} & (\mathcal{A}/\mathcal{C})/(\mathcal{B}/\mathcal{C})
\end{array}$$

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Proof The inclusion $\mathcal{B} \to \mathcal{A}$ induces a fully faithful functor $\mathcal{B}/\mathcal{C} \to \mathcal{A}/\mathcal{C}$ since $\mathcal{B}$ is left and right cofinal with respect to $S(\mathcal{C})$; see Lemma 1.2.5. It is easily checked that $\mathcal{B}/\mathcal{C}$ yields a Serre subcategory of $\mathcal{A}/\mathcal{C}$. If $\mathcal{D} \subseteq \mathcal{A}/\mathcal{C}$ is a Serre subcategory, set $\mathcal{B} := Q^{-1}(\mathcal{D})$. Then $\mathcal{B}/\mathcal{C} \xrightarrow{\sim} \mathcal{D}$. The final assertion is clear, since the kernel of the composite $\mathcal{A} \to \mathcal{A}/\mathcal{C} \to (\mathcal{A}/\mathcal{C})/(\mathcal{B}/\mathcal{C})$ equals $\mathcal{B}$. □

Remark 2.2.9. The above correspondence $\mathcal{B} \mapsto \mathcal{B}/\mathcal{C}$ between Serre subcategories is inclusion preserving, and $\mathcal{B}/\mathcal{B}' \xrightarrow{\sim} (\mathcal{B}/\mathcal{C})/(\mathcal{B}'/\mathcal{C})$ for $\mathcal{B}' \subseteq \mathcal{B}$.

Localisation and Adjoints

Let $\mathcal{A}$ be an abelian category. We consider the situation that the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ given by a Serre subcategory $\mathcal{C}$ admits a right adjoint.

Lemma 2.2.10. *Let $\mathcal{A}$ be an abelian category and $\mathcal{C} \subseteq \mathcal{A}$ a Serre subcategory. Suppose the canonical functor $Q\colon \mathcal{A} \to \mathcal{A}/\mathcal{C}$ admits a right adjoint $Q_\rho\colon \mathcal{A}/\mathcal{C} \to \mathcal{A}$. Then the following holds.*

(1) *The functor Q_ρ is fully faithful and induces an equivalence*

$$\mathcal{A}/\mathcal{C} \xrightarrow{\sim} \mathcal{C}^\perp \qquad \text{with quasi-inverse} \qquad \mathcal{C}^\perp \hookrightarrow \mathcal{A} \xrightarrow{Q} \mathcal{A}/\mathcal{C}.$$

(2) *The adjunction yields for X in $\mathcal{A}$ a natural exact sequence*

$$0 \longrightarrow X' \longrightarrow X \xrightarrow{\eta} Q_\rho Q(X) \longrightarrow X'' \longrightarrow 0$$

with X' and X'' in $\mathcal{C}$.

(3) *The assignment $X \mapsto X'$ gives a right adjoint of the inclusion $\mathcal{C} \to \mathcal{A}$.*

Proof (1) This follows from Proposition 1.1.3 and Lemma 2.2.4.

(2) This follows from the fact that $Q(\eta)$ is invertible.

(3) The map $\operatorname{Hom}_{\mathcal{A}}(C, X') \to \operatorname{Hom}_{\mathcal{A}}(C, X)$ is bijective for $C \in \mathcal{C}$ since $Q_\rho Q(X)$ is in $\mathcal{C}^\perp$. □

We capture the situation in the following diagram

$$\mathcal{C} \underset{I_\rho}{\overset{I}{\rightleftarrows}} \mathcal{A} \underset{Q_\rho}{\overset{Q}{\rightleftarrows}} \mathcal{A}/\mathcal{C}$$

which is a localisation sequence. Each object $X \in \mathcal{A}$ fits into a functorial exact sequence

$$0 \longrightarrow II_\rho(X) \longrightarrow X \longrightarrow Q_\rho Q(X).$$

A Serre subcategory $\mathcal{C} \subseteq \mathcal{A}$ is called *localising* if the canonical functor

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$Q\colon \mathcal{A} \to \mathcal{A}/\mathcal{C}$ admits a right adjoint. Note that in this case $\mathcal{C}$ is closed under all coproducts which exist in $\mathcal{A}$, since Q preserves coproducts.

Proposition 2.2.11. *Let (F, G) be an adjoint pair of functors*

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

between abelian categories such that F is exact and set $\mathcal{C} = \operatorname{Ker} F$. Then G is fully faithful if and only if F induces an equivalence $\mathcal{A}/\mathcal{C} \xrightarrow{\sim} \mathcal{B}$.

Proof Let $S = \{\sigma \in \operatorname{Mor}\mathcal{A} \mid F\sigma \text{ is invertible}\}$. Then G is fully faithful if and only if F induces an equivalence $\mathcal{A}[S^{-1}] \xrightarrow{\sim} \mathcal{B}$, by Proposition 1.1.3. It remains to observe that $\mathcal{A}[S^{-1}] \xrightarrow{\sim} \mathcal{A}/\mathcal{C}$, by Lemma 2.2.4.

Let us give a more direct proof for one implication. So suppose that G is fully faithful. Then it is easily checked that the counit $\varepsilon_X\colon FG(X) \to X$ is an isomorphism for all $X \in \mathcal{B}$; see Proposition 1.1.3. We show that F satisfies, up to an isomorphism, the universal property of the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$; see Remark 2.2.7. Clearly, F is exact and annihilates $\mathcal{C}$. Now let $H\colon \mathcal{A} \to \mathcal{A}'$ be an exact functor between abelian categories that annihilates $\mathcal{C}$. Set $\bar{H} = H \circ G$. We claim that $\bar{H}$ is exact, that $H \cong \bar{H} \circ F$, and that $\bar{H}$ is unique with these properties. For the exactness, choose an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{B}$ which yields an exact sequence

$$0 \to GX \to GY \to GZ \to X' \to 0$$

in $\mathcal{A}$ since G is left exact. We have $FX' = 0$ since $F \circ G \cong \mathrm{id}$, so $X' \in \mathcal{C}$, and therefore $HX' = 0$. Thus $\bar{H}$ is exact. Let $X \in \mathcal{A}$. Then F maps the unit $\eta_X\colon X \to GF(X)$ to an isomorphism, since the counit ε_{FX} is an inverse. Thus $\operatorname{Ker}\eta_X$ and $\operatorname{Coker}\eta_X$ are in $\mathcal{C}$. It follows that $H\eta$ yields an isomorphism $H \xrightarrow{\sim} \bar{H} \circ F$. If $\tilde{H}\colon \mathcal{B} \to \mathcal{A}'$ is another functor such that $H \cong \tilde{H} \circ F$, then one composes this isomorphism with G. Thus $\bar{H} = H \circ G \cong \tilde{H} \circ F \circ G \cong \tilde{H}$. □

Remark 2.2.12. There are dual versions of Lemma 2.2.10 and Proposition 2.2.11 for abelian categories where the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ admits a left adjoint.

Example 2.2.13. Let $\mathcal{A}$ be an abelian category and $i_*\colon \mathcal{A}' \to \mathcal{A}$ the inclusion of a Serre subcategory. Set $\mathcal{A}'' = \mathcal{A}/\mathcal{A}'$ and suppose that the canonical functor $j^*\colon \mathcal{A} \to \mathcal{A}''$ admits both adjoints. Then one obtains a *recollement* of abelian categories.

$$\mathcal{A}' \;\begin{matrix} \xleftarrow{\;i^*\;} \\ \xrightarrow{\;i_*=i_!\;} \\ \xleftarrow[\;i^!\;]{} \end{matrix}\; \mathcal{A} \;\begin{matrix} \xleftarrow{\;j_!\;} \\ \xrightarrow{\;j^!=j^*\;} \\ \xleftarrow[\;j_*\;]{} \end{matrix}\; \mathcal{A}''$$

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For an object X in $\mathcal{A}$, there are natural exact sequences relating the left and the right halves of the diagram.

$$j_! j^!(X) \longrightarrow X \longrightarrow i_* i^*(X) \longrightarrow 0 \qquad 0 \longrightarrow i_! i^!(X) \longrightarrow X \longrightarrow j_* j^*(X)$$

Each recollement of abelian categories is, up to equivalence, of the above form. A prototypical example arises from the category $\operatorname{Sh}(X)$ of sheaves on a topological space X and the inclusion $i\colon V \to X$ of a closed subset plus the inclusion $j\colon U \to X$ for $U = X \setminus V$.

$$\operatorname{Sh}(V) \;\underset{\overset{\twoheadleftarrow}{i^!}}{\overset{\overset{i^*}{\twoheadleftarrow}}{\rightarrowtail\, i_*=i_! \,\longrightarrow}}\; \operatorname{Sh}(X) \;\underset{\overset{\leftarrowtail}{j_*}}{\overset{\overset{j_!}{\leftarrowtail}}{\longrightarrow\, j^!=j^* \,\twoheadrightarrow}}\; \operatorname{Sh}(U)$$

which involves the following functors:

$$\begin{aligned} i^*, j^* &= \text{restriction} & \qquad i^! &= \text{sections with support} \\ i_*, j_* &= \text{direct image} & \qquad j_! &= \text{extension by zero.} \end{aligned}$$

This example explains the notation.

Categories with Injective Envelopes

Recall that an abelian category has *injective envelopes* if every object admits an injective envelope.

Proposition 2.2.14. *Let $\mathcal{A}$ be an abelian category with injective envelopes and let $\mathcal{C} \subseteq \mathcal{A}$ be a Serre subcategory. Then the inclusion $\mathcal{C} \to \mathcal{A}$ admits a right adjoint if and only if the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ admits a right adjoint. In that case $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are categories with injective envelopes. Moreover, both right adjoints induce a sequence of functors*

$$\operatorname{Inj}(\mathcal{A}/\mathcal{C}) \rightarrowtail \operatorname{Inj}\mathcal{A} \twoheadrightarrow \operatorname{Inj}\mathcal{C}$$

that induces an equivalence

$$(\operatorname{Inj}\mathcal{A})/\operatorname{Inj}(\mathcal{A}/\mathcal{C}) \xrightarrow{\sim} \operatorname{Inj}\mathcal{C}.$$

Proof If the functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ admits a right adjoint, then the inclusion $\mathcal{C} \to \mathcal{A}$ admits a right adjoint, by Lemma 2.2.10. For the other implication, suppose that $\mathcal{C} \to \mathcal{A}$ admits a right adjoint, sending $X \in \mathcal{A}$ to the maximal subobject $tX \subseteq X$ that belongs to $\mathcal{C}$. Choose an injective envelope $X/tX \to I$. Then I belongs to $\mathcal{C}^\perp$ because there are no non-zero subobjects in $\mathcal{C}$. We form

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the following pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X/tX & \longrightarrow & X' & \longrightarrow & tC & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X/tX & \longrightarrow & I & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

and also X' belongs to $\mathcal{C}^{\perp}$. Then $X \mapsto X'$ yields a right adjoint of the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{C}$, since the kernel and cokernel of the morphism $X \to X'$ belong to $\mathcal{C}$ by construction.

Now suppose that both adjoints exist. It is convenient to identify $\mathcal{C}^{\perp} = \mathcal{A}/\mathcal{C}$. If $X \to I$ is an injective envelope in $\mathcal{A}$, then it is easily checked that $tX \to tI$ is an injective envelope in $\mathcal{C}$. In particular, t induces a functor $\operatorname{Inj}\mathcal{A} \to \operatorname{Inj}\mathcal{C}$ that is surjective on isoclasses of objects and full. In fact, for $X \in \operatorname{Inj}\mathcal{C}$ we have $X \cong tE(X)$. Also, any morphism $\phi\colon tX \to tY$ can be extended to a morphism $\tilde{\phi}\colon X \to Y$ since Y is injective, and $t\tilde{\phi} = \phi$. We claim that

$$\operatorname{Ker} t \cap \operatorname{Inj}\mathcal{A} = \mathcal{C}^{\perp} \cap \operatorname{Inj}\mathcal{A} = \operatorname{Inj}(\mathcal{C}^{\perp}).$$

The first equality is clear. Also, an object in $\mathcal{C}^{\perp} \cap \operatorname{Inj}\mathcal{A}$ is injective in $\mathcal{C}^{\perp}$, since the inclusion $\mathcal{C}^{\perp} \to \mathcal{A}$ is left exact. Given an object $X \in \mathcal{C}^{\perp}$, then its injective envelope $E(X)$ is also in $\mathcal{C}^{\perp}$, since $tE(X) = 0$. This yields the second equality and shows that $\mathcal{A}/\mathcal{C}$ has injective envelopes. Morover, it follows that t induces an equivalence between the additive quotient $(\operatorname{Inj}\mathcal{A})/\operatorname{Inj}(\mathcal{C}^{\perp})$ and $\operatorname{Inj}\mathcal{C}$. □

Corollary 2.2.15. *Let $\mathcal{A}$ be an abelian category with injective envelopes and let $\mathcal{C} \subseteq \mathcal{A}$ be a localising subcategory. Then we have $\mathcal{C}^{\perp} \cap \operatorname{Inj}\mathcal{A} = \operatorname{Inj}(\mathcal{C}^{\perp})$ and $\mathcal{C}^{\perp} \subseteq \mathcal{A}$ is closed under injective envelopes.* □

Grothendieck categories form an important class of abelian categories with injective envelopes. Thus we can apply the above proposition.

Proposition 2.2.16. *Let $\mathcal{A}$ be a Grothendieck category and $\mathcal{C} \subseteq \mathcal{A}$ a Serre subcategory that is closed under coproducts. Then $\mathcal{C}$ and the quotient $\mathcal{A}/\mathcal{C}$ are Grothendieck categories. Moreover, the canonical functors $\mathcal{C} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ admit right adjoints.*

Proof Let $G \in \mathcal{A}$ be a generator of $\mathcal{A}$. The right adjoints are constructed as follows. Fix an object $X \in \mathcal{A}$. Observe that the subobjects of X form a set which has its cardinality bounded by 2^{α}, where $\alpha = \operatorname{card}\operatorname{Hom}(G, X)$. The subobjects $C \subseteq X$ with $C \in \mathcal{C}$ form a directed subset and we set $tX := \operatorname{colim}_{C \subseteq X} C$; this is the largest subobject of X belonging to $\mathcal{C}$. Then $X \mapsto tX$ yields a right adjoint of the inclusion $\mathcal{C} \to \mathcal{A}$. The right adjoint of $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ then exists by Proposition 2.2.14.

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The object G is also a generator of $\mathcal{A}/\mathcal{C}$, and the coproduct of all quotients of G that belong to $\mathcal{C}$ is a generator for $\mathcal{C}$. It is straightforward to check that the condition (AB5) holds for $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$. □

Corollary 2.2.17. *A Serre subcategory of a Grothendieck category is localising if and only if it is closed under coproducts.* □

Let $\mathcal{A}$ be a Grothendieck category. We denote by $\operatorname{Sp}\mathcal{A}$ a representative set of the isomorphism classes of indecomposable injective objects in $\mathcal{A}$ (the *spectrum* of $\mathcal{A}$). Note that $\operatorname{Sp}\mathcal{A}$ is a set, because $\mathcal{A}$ has a generator G and each object in $\operatorname{Sp}\mathcal{A}$ is the injective envelope of G/U for some subobject $U \subseteq G$.

Corollary 2.2.18. *Let $\mathcal{A}$ be a Grothendieck category and $\mathcal{C} \subseteq \mathcal{A}$ a localising subcategory. Every injective object $X \in \mathcal{A}$ admits a canonical decomposition $X = X' \oplus X''$ satisfying $tX' = tX$ and $X'' \in \mathcal{C}^{\perp}$. In particular, there is a canonical bijection*

$$\operatorname{Sp}\mathcal{C} \sqcup \operatorname{Sp}\mathcal{A}/\mathcal{C} \xrightarrow{\sim} \operatorname{Sp}\mathcal{A}.$$

Proof Let $X \in \mathcal{A}$ be injective. Then the injective envelope $X' = E(tX)$ is a direct summand of X and $X'' = X/X'$ belongs to $\mathcal{C}^{\perp}$. The map $\operatorname{Sp}\mathcal{C} \sqcup \operatorname{Sp}\mathcal{A}/\mathcal{C} \to \operatorname{Sp}\mathcal{A}$ sends $X \in \operatorname{Sp}\mathcal{C}$ to $E(X)$ and $X \in \operatorname{Sp}\mathcal{A}/\mathcal{C}$ to its image under $\mathcal{A}/\mathcal{C} \xrightarrow{\sim} \mathcal{C}^{\perp} \hookrightarrow \mathcal{A}$. □

Example 2.2.19. (1) Let $\mathcal{A}$ be a length category and denote by $S(\mathcal{A})$ a representative set of the isomorphism classes of simple objects. Then the maps

$$\mathcal{A} \supseteq \mathcal{C} \longmapsto \mathcal{C} \cap S(\mathcal{A}) \qquad \text{and} \qquad S(\mathcal{A}) \supseteq \mathcal{S} \longmapsto \operatorname{Filt}(\mathcal{S}) \subseteq \mathcal{A}$$

give mutually inverse and inclusion preserving bijections between the Serre subcategories of $\mathcal{A}$ and the subsets of $S(\mathcal{A})$.

(2) Let Λ be a *semiprimary ring*. Thus the Jacobson radical $J(\Lambda)$ is nilpotent and $\Lambda/J(\Lambda)$ is semisimple. Denote by $S(\Lambda)$ a representative set of the isomorphism classes of simple Λ-modules. Every Λ-module has a finite filtration with semisimple factors. It follows that the map

$$\operatorname{Mod}\Lambda \supseteq \mathcal{C} \longmapsto \mathcal{C} \cap S(\Lambda)$$

gives an inclusion preserving bijection between the localising subcategories of $\operatorname{Mod}\Lambda$ and the subsets of $S(\Lambda)$.

(3) Let $\mathcal{A}$ be a Grothendieck category that is *locally noetherian*. This means that every object is the directed union of its noetherian subobjects. Let $\operatorname{noeth}\mathcal{A}$ denote the full subcategory of noetherian objects in $\mathcal{A}$. Then the map

$$\mathcal{A} \supseteq \mathcal{C} \longmapsto \mathcal{C} \cap \operatorname{noeth}\mathcal{A}$$

gives an inclusion preserving bijection between the localising subcategories of $\mathcal{A}$ and the Serre subcategories of noeth $\mathcal{A}$.

Categories with Enough Projectives or Injectives

Recall that an abelian category $\mathcal{A}$ has enough projective objects if and only if the inclusion $\mathcal{C} := \operatorname{Proj}\mathcal{A} \hookrightarrow \mathcal{A}$ induces an equivalence $\operatorname{mod}\mathcal{C} \xrightarrow{\sim} \mathcal{A}$; see Proposition 2.1.15. In this case the localisation theory for $\mathcal{A}$ is determined by certain subcategories of $\mathcal{C}$.

Let $\mathcal{C}$ be a category and $\mathcal{X} \subseteq \mathcal{C}$ a full subcategory. Given an object $C \in \mathcal{C}$, a morphism $X \to C$ with $X \in \mathcal{X}$ is called a *right $\mathcal{X}$-approximation* of C if the induced map $\operatorname{Hom}_{\mathcal{C}}(X', X) \to \operatorname{Hom}_{\mathcal{C}}(X', C)$ is surjective for every object $X' \in \mathcal{X}$. The subcategory $\mathcal{X}$ is *contravariantly finite* if every object $C \in \mathcal{C}$ admits a right $\mathcal{X}$-approximation.

Let $\mathcal{C}$ be an additive category. We denote by $\operatorname{Mod}\mathcal{C}$ the category of additive functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$. An additive functor $f\colon \mathcal{C} \to \mathcal{D}$ induces an adjoint pair $(f_!, f^*)$

$$\begin{array}{ccccc} \mathcal{C} & \hookrightarrow & \operatorname{mod}\mathcal{C} & \hookrightarrow & \operatorname{Mod}\mathcal{C} \\ \downarrow f & & \downarrow f_! & & f_! \downarrow \;\uparrow f^* \\ \mathcal{D} & \hookrightarrow & \operatorname{mod}\mathcal{D} & \hookrightarrow & \operatorname{Mod}\mathcal{D} \end{array}$$

where f^* is given by $Y \mapsto Y \circ f$ and $f_!$ is given by $X \mapsto f_!(X)$ via presentations

$$\operatorname{Hom}_{\mathcal{C}}(-, C_1) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(-, C_0) \longrightarrow X \longrightarrow 0$$

and

$$\operatorname{Hom}_{\mathcal{D}}(-, f(C_1)) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(-, f(C_0)) \longrightarrow f_!(X) \longrightarrow 0.$$

The following proposition describes the localisation of an abelian category with enough projective objects.

Proposition 2.2.20. *Let $\mathcal{C}$ be an additive category such that* $\operatorname{mod}\mathcal{C}$ *is abelian. If $\mathcal{D} \subseteq \mathcal{C}$ is a contravariantly finite subcategory, then the sequence of additive functors*

$$\mathcal{D} \xrightarrow{\;i\;} \mathcal{C} \xrightarrow{\;p\;} \mathcal{C}/\mathcal{D}$$

induces a diagram of functors between abelian categories

$$\operatorname{mod}(\mathcal{C}/\mathcal{D}) \underset{p^*}{\overset{p_!}{\leftrightarrows}} \operatorname{mod}\mathcal{C} \underset{i^*}{\overset{i_!}{\leftrightarrows}} \operatorname{mod}\mathcal{D}$$

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which is a colocalisation sequence. The functors i^* and p^* are exact and induce equivalences

$$\operatorname{mod}(\mathcal{C}/\mathcal{D}) \xrightarrow{\sim} \operatorname{Ker} i^* \quad \textit{and} \quad (\operatorname{mod}\mathcal{C})/(\operatorname{Ker} i^*) \xrightarrow{\sim} \operatorname{mod}\mathcal{D}.$$

Proof For any additive functor f the assignment $F \mapsto f^*(F)$ is exact, but we need to show that it maps finitely presented functors to finitely presented functors when f is one of i or p. It suffices to show this when F is representable. In the first case, let $F = \operatorname{Hom}_{\mathcal{C}}(-, C)$ and choose a presentation $D_1 \to D_0 \to C$ with $D_i \in \mathcal{D}$, using that $\mathcal{D}$ is contravariantly finite and that $\mathcal{C}$ has weak kernels. Thus $D_0 \to C$ is a right $\mathcal{D}$-approximation of C, and $D_1 \to D_0$ is given by a right $\mathcal{D}$-approximation of a weak kernel of $D_0 \to C$. This yields a presentation

$$\operatorname{Hom}_{\mathcal{D}}(-, D_1) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(-, D_0) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(-, C)|_{\mathcal{D}} \longrightarrow 0$$

in $\operatorname{mod}\mathcal{D}$. Now let $F = \operatorname{Hom}_{\mathcal{C}/\mathcal{D}}(-, C)$. This yields in $\operatorname{mod}\mathcal{C}$ a presentation

$$\operatorname{Hom}_{\mathcal{C}}(-, D_0) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(-, C) \longrightarrow \operatorname{Hom}_{\mathcal{C}/\mathcal{D}}(-, C) \longrightarrow 0.$$

The equivalence $\operatorname{mod}(\mathcal{C}/\mathcal{D}) \xrightarrow{\sim} \operatorname{Ker} i^*$ is clear, since additive functors $\mathcal{C} \to \operatorname{Ab}$ vanishing on $\mathcal{D}$ identify with additive functors $\mathcal{C}/\mathcal{D} \to \operatorname{Ab}$. The second equivalence follows from the fact that $i^* i_! \cong \operatorname{id}$; see Proposition 2.2.11. □

Example 2.2.21. Let $\mathcal{C}$ be an exact category and $(\mathcal{T}, \mathcal{F})$ a torsion pair for $\mathcal{C}$. Then the subcategory $\mathcal{T} \subseteq \mathcal{C}$ is contravariantly finite. If the torsion pair is split, so $\mathcal{C} = \mathcal{T} \vee \mathcal{F}$, then we have an equivalence $\mathcal{F} \xrightarrow{\sim} \mathcal{C}/\mathcal{T}$.

We have the following converse of Proposition 2.2.20, showing that any colocalisation sequence of abelian categories

$$\mathcal{A}' \;\underset{\rightarrowtail}{\overset{\twoheadleftarrow}{\rightleftarrows}}\; \mathcal{A} \;\underset{\twoheadrightarrow}{\overset{\leftarrowtail}{\rightleftarrows}}\; \mathcal{A}''$$

is of the above form, provided that every object in $\mathcal{A}$ admits a projective cover.

Proposition 2.2.22. *Let $\mathcal{A}$ be an abelian category with projective covers and let $\mathcal{A}' \subseteq \mathcal{A}$ be a Serre subcategory. Suppose that the canonical functors $\mathcal{A}' \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{A}'' := \mathcal{A}/\mathcal{A}'$ admit left adjoints. Set $\mathcal{C} := \operatorname{Proj}\mathcal{A}$, $\mathcal{C}' := \operatorname{Proj}\mathcal{A}'$, and $\mathcal{C}'' := \operatorname{Proj}\mathcal{A}''$. Then the left adjoints restrict to functors*

$$\mathcal{C}'' \overset{i}{\rightarrowtail} \mathcal{C} \overset{p}{\twoheadrightarrow} \mathcal{C}'$$

which induce the following commutative diagram.

$$\begin{array}{ccccc}
\mathcal{A}' & \rightarrowtail & \mathcal{A} & \twoheadrightarrow & \mathcal{A}'' \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\operatorname{mod}\mathcal{C}' & \overset{p^*}{\rightarrowtail} & \operatorname{mod}\mathcal{C} & \overset{i^*}{\twoheadrightarrow} & \operatorname{mod}\mathcal{C}''
\end{array}$$

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Proof We have an equivalence $\mathcal{A} \xrightarrow{\sim} \operatorname{mod} \mathcal{C}$ by Proposition 2.1.15 since $\mathcal{A}$ has enough projectives. Now apply the dual of Proposition 2.2.14 which shows that $\mathcal{A}'$ and $\mathcal{A}''$ have enough projectives. □

Example 2.2.23. Let A be a ring and $e = e^2$ an idempotent in A. Multiplication of A-modules by e identifies with $\operatorname{Hom}_A(eA, -)$ and yields an exact functor $\operatorname{Mod} A \to \operatorname{Mod} eAe$ which has a fully faithful left adjoint given by $- \otimes_{eAe} eA$.

$$\operatorname{Mod} A \underset{\operatorname{Hom}_A(eA,-)}{\overset{-\otimes_{eAe} eA}{\leftrightarrows}} \operatorname{Mod} eAe$$

The kernel of $\operatorname{Hom}_A(eA, -)$ identifies with $\operatorname{Mod} A/AeA$. On the other hand, multiplication by e identifies with $- \otimes_A Ae$ and the corresponding functor $\operatorname{Mod} A \to \operatorname{Mod} eAe$ has a fully faithful right adjoint given by $\operatorname{Hom}_{eAe}(Ae, -)$.

$$\operatorname{Mod} A \underset{\operatorname{Hom}_{eAe}(Ae,-)}{\overset{-\otimes_A Ae}{\rightleftarrows}} \operatorname{Mod} eAe$$

Multiplication by an idempotent can be viewed as evaluation or restriction. Thus the following example generalises the previous one.

Example 2.2.24. Let $\mathcal{C}$ be an essentially small additive category and fix an object $X \in \mathcal{C}$. Set $\mathcal{D} = \operatorname{add} X$ and let $i : \mathcal{D} \to \mathcal{C}$ denote the inclusion. Then the evaluation $F \mapsto F(X)$ induces a functor

$$i^* : \operatorname{Mod} \mathcal{C} \longrightarrow \operatorname{Mod} \mathcal{D} = \operatorname{Mod} \operatorname{End}(X)$$

which gives rise to the following recollement

$$\operatorname{Mod} \mathcal{C}/\mathcal{D} \;\begin{matrix} \xleftarrow{p_!} \\ \xrightarrow{p^*} \\ \xleftarrow[p_*]{} \end{matrix}\; \operatorname{Mod} \mathcal{C} \;\begin{matrix} \xleftarrow{i_!} \\ \xrightarrow{i^*} \\ \xleftarrow[i_*]{} \end{matrix}\; \operatorname{Mod} \operatorname{End}(X)$$

where $p : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ denotes the canonical functor.

Remark 2.2.25. There are dual versions of Proposition 2.2.20 and Proposition 2.2.22 for abelian categories with enough injective objects. For instance, let $\mathcal{A}$ be an abelian category with enough injective objects and let $\mathcal{A}' \subseteq \mathcal{A}$ be a Serre subcategory that is localising. Set $\mathcal{C} = \operatorname{Inj} \mathcal{A}$ and $\mathcal{C}'' = \operatorname{Inj}(\mathcal{A}/\mathcal{A}')$. Then $\mathcal{A} \xrightarrow{\sim} (\operatorname{mod} \mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ and $\mathcal{C}/\mathcal{C}'' \xrightarrow{\sim} \operatorname{Inj} \mathcal{A}'$.

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Pullbacks of Abelian Categories

Each diagram of abelian categories and exact functors

$$\begin{array}{ccc} & & \mathcal{A}_2 \\ & & \big\downarrow{\scriptstyle F_2} \\ \mathcal{A}_1 & \xrightarrow{F_1} & \mathcal{A} \end{array}$$

can be completed to a commutative diagram

$$\begin{array}{ccc} \mathcal{A}_1 \times_{\mathcal{A}} \mathcal{A}_2 & \xrightarrow{P_2} & \mathcal{A}_2 \\ {\scriptstyle P_1}\big\downarrow & & \big\downarrow{\scriptstyle F_2} \\ \mathcal{A}_1 & \xrightarrow{F_1} & \mathcal{A} \end{array}$$

as follows. The objects of $\mathcal{A}_1 \times_{\mathcal{A}} \mathcal{A}_2$ are given by triples (X_1, X_2, μ), where $X_i \in \mathcal{A}_i$ are objects, and $\mu\colon F_1(X_1) \xrightarrow{\sim} F_2(X_2)$ is an isomorphism. A morphism from (X_1, X_2, μ) to (Y_1, Y_2, ν) is a pair (ϕ_1, ϕ_2) of morphisms $\phi_i\colon X_i \to Y_i$ such that $\nu F_1(\phi_1) = F_2(\phi_2)\mu$. The composition of morphisms is given by the formula

$$(\psi_1, \psi_2) \circ (\phi_1, \phi_2) = (\psi_1 \circ \phi_1, \psi_2 \circ \phi_2).$$

It is straightforward to check that $\mathcal{A}_1 \times_{\mathcal{A}} \mathcal{A}_2$ is an abelian category and that the canonical functors $P_i\colon \mathcal{A}_1 \times_{\mathcal{A}} \mathcal{A}_2 \to \mathcal{A}_i$ given by $P_i(X_1, X_2, \mu) = X_i$ are exact.

Proposition 2.2.26. *Let $\mathcal{C}$ be a category and $E_i\colon \mathcal{C} \to \mathcal{A}_i$ functors such that $F_1E_1 \cong F_2E_2$. Then there exists, up to isomorphism, a unique functor $E\colon \mathcal{C} \to \mathcal{A}_1 \times_{\mathcal{A}} \mathcal{A}_2$ such that $P_iE \cong E_i$ for $i = 1, 2$.*

Proof Let $\tau\colon F_1E_1 \xrightarrow{\sim} F_2E_2$ be a natural isomorphism. Then one defines $E\colon \mathcal{C} \to \mathcal{A}_1 \times_{\mathcal{A}} \mathcal{A}_2$ by $E(X) = (E_1(X), E_2(X), \tau_X)$. □

The proposition justifies the notation $\mathcal{A}_1 \times_{\mathcal{A}} \mathcal{A}_2$ and we call the category a *pullback* (strictly speaking, a 2-*pullback*); it is unique, up to equivalence.

The following lemma describes a property of pullbacks *of* abelian categories which is the analogue of a property of a pullback *in* an abelian category.

Lemma 2.2.27. *Let $F_i\colon \mathcal{A}_i \to \mathcal{A}$ be exact functors and suppose that F_1 induces an equivalence $\mathcal{A}_1/\operatorname{Ker} F_1 \xrightarrow{\sim} \mathcal{A}$. Then P_1 restricts to an equivalence $\operatorname{Ker} P_2 \xrightarrow{\sim} \operatorname{Ker} F_1$ and P_2 induces an equivalence $(\mathcal{A}_1 \times_{\mathcal{A}} \mathcal{A}_2)/\operatorname{Ker} P_2 \xrightarrow{\sim} \mathcal{A}_2$.*

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The following diagram illustrates the assertion of the lemma.

$$\begin{array}{ccccc} \operatorname{Ker} P_2 & \rightarrowtail & \mathcal{A}_1 \times_{\mathcal{A}} \mathcal{A}_2 & \xrightarrow{P_2} & \mathcal{A}_2 \\ \downarrow \wr & & \downarrow P_1 & & \downarrow F_2 \\ \operatorname{Ker} F_1 & \rightarrowtail & \mathcal{A}_1 & \xrightarrow{F_1} & \mathcal{A} \end{array}$$

Proof We provide for both functors a quasi-inverse. For $\operatorname{Ker} P_2 \to \operatorname{Ker} F_1$ the quasi-inverse $\operatorname{Ker} F_1 \to \operatorname{Ker} P_2$ is given by $X \mapsto (X, 0, 0)$. Now choose a quasi-inverse $G_1 \colon \mathcal{A} \to \mathcal{A}_1/\operatorname{Ker} F_1$ for $\bar{F}_1 \colon \mathcal{A}_1/\operatorname{Ker} F_1 \xrightarrow{\sim} \mathcal{A}$ together with an isomorphism $\tau \colon \bar{F}_1 G_1 \xrightarrow{\sim} \mathrm{id}$. Then the quasi-inverse $\mathcal{A}_2 \to (\mathcal{A}_1 \times_{\mathcal{A}} \mathcal{A}_2)/\operatorname{Ker} P_2$ is given by $X \mapsto (G_1 F_2(X), X, \tau_{F_2(X)})$. □

2.3 Module Categories and Their Localisations

For several classes of abelian categories we describe specific Serre subcategories and the corresponding localisations. We begin with categories of functors and the interplay between effaceable and left exact functors. Then we consider module categories and see the connection with the localisation of a ring.

Effaceable and Left Exact Functors

Let $\mathcal{A}$ be an abelian category. Fix $F \in \operatorname{mod} \mathcal{A}$ given by a presentation

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{A}}(-, X_2) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(-, X_1) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(-, X_0) \longrightarrow F \longrightarrow 0 \tag{2.3.1}$$

coming from an exact sequence $0 \to X_2 \to X_1 \to X_0$ in $\mathcal{A}$.

Lemma 2.3.2. *For $G \in \operatorname{mod} \mathcal{A}$ we have $\operatorname{Ext}^i(F, G) \cong H^i G(X)$ where $G(X)$ is the complex*

$$\cdots \longrightarrow 0 \longrightarrow G(X_0) \longrightarrow G(X_1) \longrightarrow G(X_2) \longrightarrow 0 \longrightarrow \cdots$$

Proof This is clear since (2.3.1) provides a projective resolution of F. □

The functor F is called *effaceable* if $X_1 \to X_0$ is an epimorphism. This definition does not depend on the presentation of F, since an equivalent condition is that $\operatorname{Hom}(F, G) = 0$ for each representable functor $G = \operatorname{Hom}_{\mathcal{A}}(-, X)$. Let $\operatorname{eff} \mathcal{A}$ denote the full subcategory of effaceable functors.

Proposition 2.3.3. *Let $\mathcal{A}$ be an abelian category. The functor* $\operatorname{mod} \mathcal{A} \to \mathcal{A}$ *that*

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sends Coker $\mathrm{Hom}_{\mathcal{A}}(-, \phi)$ *(given by a morphism* ϕ *in* $\mathcal{A}$*) to* Coker ϕ *provides an exact left adjoint of the Yoneda functor* $\mathcal{A} \to \mathrm{mod}\,\mathcal{A}$ *and induces an equivalence*

$$(\mathrm{mod}\,\mathcal{A})/(\mathrm{eff}\,\mathcal{A}) \xrightarrow{\sim} \mathcal{A}.$$

Proof For the adjointness, see Example 1.1.4. The exactness of the left adjoint follows from Lemma 2.1.8. Now the equivalence is a consequence of Proposition 2.2.11. □

Remark 2.3.4. (1) The inclusion $\mathrm{eff}\,\mathcal{A} \hookrightarrow \mathrm{mod}\,\mathcal{A}$ admits a right adjoint that sends F with presentation (2.3.1) to F' with presentation

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(-, X_2) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(-, X_1) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(-, X) \longrightarrow F' \longrightarrow 0$$

where $X = \mathrm{Coker}(X_2 \to X_1)$.

(2) There is an equivalence $(\mathrm{eff}\,\mathcal{A})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{eff}(\mathcal{A}^{\mathrm{op}})$ given by

$$F \longmapsto F^{\vee} \qquad \text{with} \qquad F^{\vee}(X) = \mathrm{Ext}^2(F, \mathrm{Hom}_{\mathcal{A}}(-, X)).$$

When F is given by (2.3.1), then $F^{\vee}$ has a presentation

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X_0, -) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X_1, -) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X_2, -) \longrightarrow F^{\vee} \longrightarrow 0$$

and we have $F^{\vee\vee} \cong F$.

We give an alternative description of the equivalence in Proposition 2.3.3 when $\mathcal{A} = \mathrm{mod}\,\Lambda$ is the module category of a ring. Let $\underline{\mathrm{mod}}\,\Lambda$ denote the *projectively stable category* which is obtained from $\mathrm{mod}\,\Lambda$ by setting for Λ-modules X and Y

$$\underline{\mathrm{Hom}}_{\Lambda}(X, Y) = \mathrm{Hom}_{\Lambda}(X, Y)/\{\phi \mid \phi \text{ factors through a projective module}\}.$$

Proposition 2.3.5. *Let* Λ *be a right coherent ring so that* $\mathrm{mod}\,\Lambda$ *is abelian. Then the sequence of additive functors* $\mathrm{proj}\,\Lambda \rightarrowtail \mathrm{mod}\,\Lambda \twoheadrightarrow \underline{\mathrm{mod}}\,\Lambda$ *induces a sequence of exact functors*

$$\mathrm{mod}(\underline{\mathrm{mod}}\,\Lambda) \rightarrowtail \mathrm{mod}(\mathrm{mod}\,\Lambda) \xrightarrow{\pi} \mathrm{mod}(\mathrm{proj}\,\Lambda) = \mathrm{mod}\,\Lambda$$

and an equivalence

$$\mathrm{mod}(\underline{\mathrm{mod}}\,\Lambda) \xrightarrow{\sim} \mathrm{Ker}\,\pi = \mathrm{eff}(\mathrm{mod}\,\Lambda).$$

Proof The subcategory $\mathrm{proj}\,\Lambda \subseteq \mathrm{mod}\,\Lambda$ is contravariantly finite. Now apply Proposition 2.2.20. □

Now let $\mathcal{A}$ be an exact category and let $\mathrm{Mod}\,\mathcal{A}$ denote the category of additive functors $\mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$. A functor $F \in \mathrm{Mod}\,\mathcal{A}$ is *locally effaceable* if for each object C in $\mathcal{A}$ and $x \in F(C)$ there exists an admissible epimorphism

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$\phi\colon B \to C$ such that $F(\phi)(x) = 0$. We write $\operatorname{Eff}\mathcal{A}$ for the full subcategory of locally effaceable functors.

Lemma 2.3.6. *When $\mathcal{A}$ is abelian we have* $\operatorname{eff}\mathcal{A} = \operatorname{Eff}\mathcal{A} \cap \operatorname{mod}\mathcal{A}$.

Proof Let $F \in \operatorname{mod}\mathcal{A}$ be given by a presentation (2.3.1). Suppose first that $F \in \operatorname{eff}\mathcal{A}$. An element $x \in F(C)$ is given by a morphism $C \to X_0$, and forming the pullback with $X_1 \to X_0$ yields an epimorphism $\phi\colon B \to C$ such that $F(\phi)(x) = 0$. Thus $F \in \operatorname{Eff}\mathcal{A}$.

Now let $F \in \operatorname{Eff}\mathcal{A}$. Choose $C = X_0$ and take for $x \in F(C)$ the element given by $\mathrm{id}\colon X_0 \to X_0$. This yields an epimorphism $\phi\colon B \to C$ that factors through $X_1 \to X_0$. Thus the morphism $X_1 \to X_0$ is an epimorphism and therefore $F \in \operatorname{eff}\mathcal{A}$. □

We denote by $\operatorname{Lex}\mathcal{A}$ the category of additive functors $F\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ that are *left exact*, that is, each exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ induces an exact sequence $0 \to FZ \to FY \to FX$ of abelian groups.

Proposition 2.3.7. *Let $\mathcal{A}$ be an essentially small exact category.*

(1) *The inclusion* $\operatorname{Lex}\mathcal{A} \to \operatorname{Mod}\mathcal{A}$ *admits an exact left adjoint* $\operatorname{Mod}\mathcal{A} \to \operatorname{Lex}\mathcal{A}$ *that induces an equivalence*

$$(\operatorname{Mod}\mathcal{A})/(\operatorname{Eff}\mathcal{A}) \xrightarrow{\sim} \operatorname{Lex}\mathcal{A}.$$

(2) *The category* $\operatorname{Lex}\mathcal{A}$ *is a Grothendieck category.*

(3) *The Yoneda functor* $\mathcal{A} \to \operatorname{Lex}\mathcal{A}$ *that takes X to* $\operatorname{Hom}_{\mathcal{A}}(-, X)$ *is exact and identifies $\mathcal{A}$ with a full extension closed subcategory of* $\operatorname{Lex}\mathcal{A}$.

Proof Using (Ex1) and (Ex2) one shows that $\operatorname{Eff}\mathcal{A}$ is a Serre subcategory of $\operatorname{Mod}\mathcal{A}$ and closed under coproducts. From (Ex3) it follows that $(\operatorname{Eff}\mathcal{A})^{\perp} = \operatorname{Lex}\mathcal{A}$. Thus the canonical functor $\operatorname{Mod}\mathcal{A} \to \frac{\operatorname{Mod}\mathcal{A}}{\operatorname{Eff}\mathcal{A}}$ admits a fully faithful right adjoint, which identifies $\frac{\operatorname{Mod}\mathcal{A}}{\operatorname{Eff}\mathcal{A}}$ with $\operatorname{Lex}\mathcal{A}$; see Lemma 2.2.10 and Proposition 2.2.16. In particular, $\operatorname{Lex}\mathcal{A}$ is a Grothendieck category.

Now let $\xi\colon 0 \to \operatorname{Hom}_{\mathcal{A}}(-, X) \xrightarrow{\alpha} E \xrightarrow{\beta} \operatorname{Hom}_{\mathcal{A}}(-, Z) \to 0$ be an exact sequence in $\operatorname{Lex}\mathcal{A}$. Then $\operatorname{Coker}\beta$ is locally effaceable, and there exists an admissible epimorphism $V \to Z$ inducing the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(-, U) & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(-, V) & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(-, Z) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(-, X) & \longrightarrow & E & \longrightarrow & \operatorname{Hom}_{\mathcal{A}}(-, Z) & \longrightarrow & 0
\end{array}$$

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Apply condition (Ex3) by forming the following pushout.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

Then the bottom row identifies with ξ, and therefore the image of the Yoneda functor $\mathcal{A} \to \operatorname{Lex}\mathcal{A}$ is extension closed. □

The injective objects in $\operatorname{Lex}\mathcal{A}$ admit the following explicit description. The functors $I_X = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Hom}_{\mathcal{A}}(X, -), \mathbb{Q}/\mathbb{Z})$ (with $X \in \mathcal{A}$) form a set of injective cogenerators for the abelian category $\operatorname{Mod}\mathcal{A}$. Thus the direct summands of products $\prod_\alpha I_{X_\alpha}$ are precisely the injective objects in $\operatorname{Mod}\mathcal{A}$. Moreover,

$$\operatorname{Inj}(\operatorname{Lex}\mathcal{A}) = \{F \in \operatorname{Inj}(\operatorname{Mod}\mathcal{A}) \mid F \text{ is exact}\}.$$

Epimorphisms of Rings

A ring homomorphism $\phi\colon A \to B$ is by definition an *epimorphism of rings* if for any pair of homomorphisms $\psi, \psi'\colon B \to C$ we have that $\psi\phi = \psi'\phi$ implies $\psi = \psi'$. An equivalent condition is that restriction of scalars $\phi^*\colon \operatorname{Mod} B \to \operatorname{Mod} A$ is fully faithful [197, Proposition XI.1.2]. In fact, we have an adjoint pair $(\phi_!, \phi^*)$ with counit $X \otimes_A B \to X$ given by scalar multiplication for any B-module X. Then ϕ^* is fully faithful if and only if the counit is an isomorphism for all X if and only if $B \otimes_A B \xrightarrow{\sim} B$. It follows that the adjoint pair $(\phi_!, \phi^*)$ gives rise to a localisation functor $\phi^* \circ \phi_!\colon \operatorname{Mod} A \to \operatorname{Mod} A$ when ϕ is an epimorphism, cf. Proposition 1.1.5.

Proposition 2.3.8. *Let* $L\colon \operatorname{Mod} A \to \operatorname{Mod} A$ *be a localisation functor. Then the following are equivalent.*

(1) *The functor* L *is, up to an equivalence, of the form* $\phi^* \circ \phi_!$ *for some ring epimorphism* $\phi\colon A \to B$.
(2) *The subcategory* $\operatorname{Im} L$ *is closed under all coproducts and cokernels.*

Proof (1) ⇒ (2): An epimorphism $\phi\colon A \to B$ yields an adjoint pair $(\phi_!, \phi^*)$, and we have $\operatorname{Im}\phi^* = \operatorname{Im} L$ for $L = \phi^* \circ \phi_!$. Clearly, ϕ^* is right exact and preserves coproducts.

(2) ⇒ (1): Recall from Proposition 1.1.5 that a localisation functor L can be written as the composite $L = G \circ F$ given by an adjoint pair (F, G) such that F is a quotient functor and G is fully faithful. Let $\mathcal{B} = \{X \in \operatorname{Mod} A \mid X \xrightarrow{\sim} L(X)\}$ be the localised category. It is of the form $S^\perp$ for a class S of morphisms in $\operatorname{Mod} A$, so closed under all limits in $\operatorname{Mod} A$; see Proposition 1.1.3. Also,

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$\mathcal{B} = \operatorname{Im} L$ is closed under colimits and it follows that $\mathcal{B}$ is abelian. The inclusion $G\colon \mathcal{B} \to \operatorname{Mod} A$ is exact, and therefore F takes projectives to projectives. It follows that FA is a projective generator of $\mathcal{B}$. Also, $\operatorname{Hom}_{\mathcal{B}}(FA, -)$ preserves coproducts since G preserves coproducts. Set $B = \operatorname{End}_{\mathcal{B}}(FA)$. It follows that $\operatorname{Hom}_{\mathcal{B}}(FA, -)\colon \mathcal{B} \to \operatorname{Mod} B$ is an equivalence. Let $\phi\colon A \to B$ denote the homomorphism that is induced by F. Then the composite

$$\operatorname{Mod} A \xrightarrow{\ F\ } \mathcal{B} \xrightarrow{\operatorname{Hom}(FA,-)} \operatorname{Mod} B$$

is isomorphic to $\phi_! = - \otimes_B B$, and therefore $L \cong \phi^* \circ \phi_!$. □

Examples of ring epimorphisms arise from localising a ring by universally inverting a set of fixed elements.

Universal Localisation

Let A be a ring and Σ a set of morphisms between finitely generated projective A-modules. The *universal localisation* of A with respect to Σ is a ring A_Σ together with a ring homomorphism $q\colon A \to A_\Sigma$ satisfying the following:

(UL1) For every $\sigma \in \Sigma$, the morphism $\sigma \otimes_A A_\Sigma$ is invertible.

(UL2) For every ring homomorphism $f\colon A \to B$ such that $\sigma \otimes_A B$ is invertible for all $\sigma \in \Sigma$, there exists a unique ring homomorphisms $\bar{f}\colon A_\Sigma \to B$ such that $f = \bar{f} q$.

The universal localisation solves a universal problem and is therefore unique. In particular, a universal localisation is an epimorphism of rings.

Any element $x \in A$ can be viewed as a morphism $\lambda_x\colon A \to A$ (left multiplication by x). Thus the universal localisation generalises the localisation of A with respect to a subset $S \subseteq A$, because we have $A[S^{-1}] = A_\Sigma$ for $\Sigma = \{\lambda_x \mid x \in S\}$.

We sketch the construction of A_Σ. Set $\mathcal{C} = \operatorname{proj} A$ so that $\Sigma \subseteq \operatorname{Mor} \mathcal{C}$. We may assume that Σ contains the identity morphism of each object and that $\sigma, \tau \in \Sigma$ implies $\sigma \oplus \tau \in \Sigma$. Then $\mathcal{C}[\Sigma^{-1}]$ is an additive category and the canonical functor $\mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ is additive, by Lemma 2.2.1. Set $A_\Sigma = \operatorname{End}_{\mathcal{C}[\Sigma^{-1}]}(A)$. The functor $\operatorname{Hom}_{\mathcal{C}[\Sigma^{-1}]}(A, -)$ makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{-\otimes_A A_\Sigma} & \operatorname{proj} A_\Sigma \\ & {\scriptstyle\text{can}}\searrow \quad \nearrow{\scriptstyle \operatorname{Hom}_{\mathcal{C}[\Sigma^{-1}]}(A,-)} & \\ & \mathcal{C}[\Sigma^{-1}] & \end{array}$$

and identifies the idempotent completion of $\mathcal{C}[\Sigma^{-1}]$ with $\operatorname{proj} A_\Sigma$.

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There is an alternative construction of A_Σ. Set $\mathcal{A} = \operatorname{Mod} A$ and consider the full subcategory $\mathcal{A}' \subseteq \mathcal{A}$ of A-modules X such that $\operatorname{Hom}_A(\sigma, X)$ is invertible for all $\sigma \in \Sigma$. It is easily checked that $\mathcal{A}'$ is closed under taking (co)kernels, (co)products, and extensions. Moreover, the inclusion $\mathcal{A}' \to \mathcal{A}$ admits a left adjoint $F\colon \mathcal{A} \to \mathcal{A}'$ (for instance by [84, Satz 8.5] or [1, Theorem 1.39]) which takes A to a projective generator of $\mathcal{A}'$. Set $A_\Sigma = \operatorname{End}_A(FA)$. Then we obtain an equivalence

$$\operatorname{Hom}_A(FA, -)\colon \mathcal{A}' \xrightarrow{\sim} \operatorname{Mod} A_\Sigma.$$

The inverse is given by the canonical functor $\operatorname{Mod} A_\Sigma \to \operatorname{Mod} A$, via restriction of scalars along the morphism $A \to A_\Sigma$ induced by F. Now set $\bar\Sigma = \{\sigma \in \operatorname{Mor} \mathcal{A} \mid \sigma \otimes_A A_\Sigma \text{ is invertible}\}$. Then it follows from Proposition 1.1.3 that the following diagram commutes

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T=-\otimes_A A_\Sigma} & \operatorname{Mod} A_\Sigma \\ & {\scriptstyle\text{can}}\searrow \qquad \nearrow{\scriptstyle\operatorname{Hom}_{\mathcal{A}[\bar\Sigma^{-1}]}(A,-)} & \\ & \mathcal{A}[\bar\Sigma^{-1}] & \end{array}$$

which equals the 'completion' of the above diagram for $\operatorname{proj} A$. Note that $\bar T = \operatorname{Hom}_{\mathcal{A}[\bar\Sigma^{-1}]}(A, -)$ is an equivalence.

In general, the universal localisation A_Σ is not a flat A-module.

Example 2.3.9. Let Σ be a set of morphisms between finitely generated projective A-modules such that $\operatorname{proj.dim} \operatorname{Coker} \sigma \le 1$ for all $\sigma \in \Sigma$. Then $\operatorname{Mod} A_\Sigma$ identifies with $\mathcal{C}^\perp$ where $\mathcal{C} = \{\operatorname{Ker} \sigma, \operatorname{Coker} \sigma \mid \sigma \in \Sigma\}$ and

$$\mathcal{C}^\perp = \{X \in \operatorname{Mod} A \mid \operatorname{Hom}_A(C, X) = 0 = \operatorname{Ext}^1_A(C, X) \text{ for all } C \in \mathcal{C}\}.$$

2.4 Commutative Noetherian Rings

We consider modules over commutative rings. There is a notion of support for modules which yields a classification of Serre subcategories for the category of noetherian modules. This extends to a classification of localising subcategories for the category of all modules provided the ring is noetherian. Also, we discuss injective and artinian modules.

Let A be a commutative ring. For the main results of this section we need to assume that A is noetherian.

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Support of Modules

Let A be a commutative ring. The *spectrum* $\operatorname{Spec} A$ of A is the set of *prime ideals* $\mathfrak{p} \subseteq A$. A subset of $\operatorname{Spec} A$ is *Zariski closed* if it is of the form

$$\mathcal{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{a} \subseteq \mathfrak{p}\}$$

for some ideal $\mathfrak{a}$ of A. A subset $\mathcal{V}$ of $\operatorname{Spec} A$ is *specialisation closed* if for any pair $\mathfrak{p} \subseteq \mathfrak{q}$ of prime ideals, $\mathfrak{p} \in \mathcal{V}$ implies $\mathfrak{q} \in \mathcal{V}$. For $\mathfrak{p} \in \operatorname{Spec} A$ set $S = A \setminus \mathfrak{p}$ and denote by $A_\mathfrak{p} = A[S^{-1}]$ the localisation. Note that $X \mapsto X_\mathfrak{p} := X \otimes_A A_\mathfrak{p}$ yields an exact functor $\operatorname{Mod} A \to \operatorname{Mod} A_\mathfrak{p}$. The *support* of an A-module X is the subset

$$\operatorname{Supp} X = \{\mathfrak{p} \in \operatorname{Spec} A \mid X_\mathfrak{p} \neq 0\}.$$

Observe that this is a specialisation closed subset of $\operatorname{Spec} A$.

Lemma 2.4.1. *We have* $\operatorname{Supp} A/\mathfrak{a} = \mathcal{V}(\mathfrak{a})$ *for each ideal* $\mathfrak{a}$ *of* A.

Proof Fix $\mathfrak{p} \in \operatorname{Spec} A$ and let $S = A \setminus \mathfrak{p}$. Recall that for any A-module X, an element x/s in $S^{-1}X = X_\mathfrak{p}$ is zero if and only if there exists $t \in S$ such that $tx = 0$. Thus we have $(A/\mathfrak{a})_\mathfrak{p} = 0$ if and only if there exists $t \in S$ with $t(1 + \mathfrak{a}) = t + \mathfrak{a} = 0$ if and only if $\mathfrak{a} \not\subseteq \mathfrak{p}$. □

Lemma 2.4.2. *Let* $0 \to X' \to X \to X'' \to 0$ *be an exact sequence of* A*-modules. Then* $\operatorname{Supp} X = \operatorname{Supp} X' \cup \operatorname{Supp} X''$.

Proof The sequence $0 \to X'_\mathfrak{p} \to X_\mathfrak{p} \to X''_\mathfrak{p} \to 0$ is exact for each $\mathfrak{p}$ in $\operatorname{Spec} A$. □

Lemma 2.4.3. *Let* $X = \sum_i X_i$ *be an* A*-module, written as a sum of submodules* X_i*. Then* $\operatorname{Supp} X = \bigcup_i \operatorname{Supp} X_i$.

Proof The assertion is clear if the sum $\sum_i X_i$ is direct, since

$$\bigoplus_i (X_i)_\mathfrak{p} = \Big(\bigoplus_i X_i\Big)_\mathfrak{p}.$$

As $X_i \subseteq X$ for all i one gets $\bigcup_i \operatorname{Supp} X_i \subseteq \operatorname{Supp} X$, from Lemma 2.4.2. On the other hand, $X = \sum_i X_i$ is a factor of $\bigoplus_i X_i$, so $\operatorname{Supp} X \subseteq \bigcup_i \operatorname{Supp} X_i$. □

We write $\operatorname{Ann} X$ for the ideal of elements in A that annihilate X; it is the kernel of the natural homomorphism $A \to \operatorname{End}_A(X)$.

Lemma 2.4.4. *We have* $\operatorname{Supp} X \subseteq \mathcal{V}(\operatorname{Ann} X)$*, with equality when* X *is in* $\operatorname{mod} A$.

Proof Write $X = \sum_i X_i$ as a sum of cyclic modules $X_i \cong A/\mathfrak{a}_i$. Then

$$\operatorname{Supp} X = \bigcup_i \operatorname{Supp} X_i = \bigcup_i \mathcal{V}(\mathfrak{a}_i) \subseteq \mathcal{V}\Big(\bigcap_i \mathfrak{a}_i\Big) = \mathcal{V}(\operatorname{Ann} X),$$

and equality holds if the sum is finite. □

Lemma 2.4.5. *Let $X \neq 0$ be an A-module. If $\mathfrak{p}$ is maximal in the set of ideals which annihilate a non-zero element of X, then $\mathfrak{p}$ is prime.*

Proof Suppose $0 \neq x \in X$ and $\mathfrak{p}x = 0$. Let $a, b \in A$ with $ab \in \mathfrak{p}$ and $a \notin \mathfrak{p}$. Then $(\mathfrak{p}, b)$ annihilates $ax \neq 0$, so the maximality of $\mathfrak{p}$ implies $b \in \mathfrak{p}$. Thus $\mathfrak{p}$ is prime. □

Lemma 2.4.6. *Let $X \neq 0$ be a noetherian A-module. There exists a submodule of X which is isomorphic to $A/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$.*

Proof The ring $\bar{A} = A/(\operatorname{Ann} X)$ is noetherian. Thus the set of ideals of $\bar{A}$ annihilating a non-zero element has a maximal element. Now apply Lemma 2.4.5. □

Lemma 2.4.7. *For each noetherian A-module X there exists a finite filtration*

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$$

such that each factor X_i/X_{i-1} is isomorphic to $A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. In that case we have $\operatorname{Supp} X = \bigcup_i \mathcal{V}(\mathfrak{p}_i)$.

Proof Repeated application of Lemma 2.4.6 yields a chain of submodules $0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ of X such that each X_i/X_{i-1} is isomorphic to $A/\mathfrak{p}_i$ for some $\mathfrak{p}_i$. This chain stabilises since X is noetherian, and therefore $\bigcup_i X_i = X$.

The last assertion follows from Lemma 2.4.2 and Lemma 2.4.1. □

For a class $\mathcal{C} \subseteq \operatorname{Mod} A$ we set

$$\operatorname{Supp} \mathcal{C} = \bigcup_{X \in \mathcal{C}} \operatorname{Supp} X.$$

Proposition 2.4.8. *Let A be a commutative noetherian ring. Then the assignment $\mathcal{C} \mapsto \operatorname{Supp} \mathcal{C}$ induces a bijection between*

- *the set of Serre subcategories of* $\operatorname{mod} A$*, and*
- *the set of specialisation closed subsets of* $\operatorname{Spec} A$.

Its inverse takes $\mathcal{V} \subseteq \operatorname{Spec} A$ to $\{X \in \operatorname{mod} A \mid \operatorname{Supp} X \subseteq \mathcal{V}\}$.

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Proof Both maps are well defined by Lemma 2.4.2 and Lemma 2.4.4. If $\mathcal{V} \subseteq \operatorname{Spec} A$ is a specialisation closed subset, let $\mathcal{C}_{\mathcal{V}}$ denote the smallest Serre subcategory containing $\{A/\mathfrak{p} \mid \mathfrak{p} \in \mathcal{V}\}$. Then we have $\operatorname{Supp} \mathcal{C}_{\mathcal{V}} = \mathcal{V}$, by Lemma 2.4.1 and Lemma 2.4.2. Now let $\mathcal{C}$ be a Serre subcategory of $\operatorname{mod} A$. Then

$$\operatorname{Supp} \mathcal{C} = \{\mathfrak{p} \in \operatorname{Spec} A \mid A/\mathfrak{p} \in \mathcal{C}\}$$

by Lemma 2.4.7. It follows that $\mathcal{C} = \mathcal{C}_{\mathcal{V}}$ for each Serre subcategory $\mathcal{C}$, where $\mathcal{V} = \operatorname{Supp} \mathcal{C}$. Thus $\operatorname{Supp} \mathcal{C}_1 = \operatorname{Supp} \mathcal{C}_2$ implies $\mathcal{C}_1 = \mathcal{C}_2$ for each pair $\mathcal{C}_1, \mathcal{C}_2$ of Serre subcategories. □

Corollary 2.4.9. *Let X and Y be in* $\operatorname{mod} A$. *Then* $\operatorname{Supp} Y \subseteq \operatorname{Supp} X$ *if and only if Y belongs to the smallest Serre subcategory containing X.*

Proof With $\mathcal{C}$ denoting the smallest Serre subcategory containing X, there is an equality $\operatorname{Supp} \mathcal{C} = \operatorname{Supp} X$ by Lemma 2.4.2. Now apply Proposition 2.4.8. □

Corollary 2.4.10. *The assignment* $\mathcal{C} \mapsto \operatorname{Supp} \mathcal{C}$ *induces a bijection between*

– *the set of localising subcategories of* $\operatorname{Mod} A$*, and*
– *the set of specialisation closed subsets of* $\operatorname{Spec} A$.

Proof The proof is essentially the same as that of Proposition 2.4.8 if we observe that any A-module X is the sum $X = \sum_i X_i$ of its finitely generated submodules; see also Example 2.2.19. Note that X belongs to a localising subcategory $\mathcal{C}$ if and only if all X_i belong to $\mathcal{C}$. In addition, we use that $\operatorname{Supp} X = \bigcup_i \operatorname{Supp} X_i$; see Lemma 2.4.3. □

Injective Modules

Let A be a commutative noetherian ring. For an A-module X we say that $\mathfrak{p} \in \operatorname{Spec} A$ is *associated* to X if $A/\mathfrak{p}$ is isomorphic to a submodule of X. The set of associated primes is denoted by $\operatorname{Ass} X$.

Lemma 2.4.11. *We have* $\operatorname{Supp} X = \bigcup_{\mathfrak{p} \in \operatorname{Ass} X} \mathcal{V}(\mathfrak{p})$ *for each A-module X.*

Proof We have $\mathcal{V}(\mathfrak{p}) \subseteq \operatorname{Supp} X$ when $A/\mathfrak{p} \subseteq X$, by Lemma 2.4.1 and Lemma 2.4.2. For the other direction, let $\mathfrak{p} \in \operatorname{Supp} X$, and we need to show that $\mathfrak{p} \in \operatorname{Ass} X$ when $\mathfrak{p}$ is minimal in $\operatorname{Supp} X$. We may assume that X is finitely generated, and as in Lemma 2.4.7 we have submodules

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$$

such that each factor X_i/X_{i-1} is isomorphic to $A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$.

Choose $\mathfrak{p} = \mathfrak{p}_i$ to be minimal in $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$, and let i be minimal such that $\mathfrak{p} = \mathfrak{p}_i$. Pick $x \in X_i \setminus X_{i-1}$. Then $\mathfrak{p}_1 \cdots \mathfrak{p}_i \subseteq \operatorname{Ann} Ax \subseteq \mathfrak{p}$. Also $\mathfrak{p}_j \nsubseteq \mathfrak{p}$ for $j < i$ and therefore $\mathfrak{p}_1 \cdots \mathfrak{p}_{i-1} \nsubseteq \mathfrak{p}$. Pick $a \in \mathfrak{p}_1 \cdots \mathfrak{p}_{i-1} \setminus \mathfrak{p}$. Then $\operatorname{Ann} Aax = \mathfrak{p}$, and therefore $\mathfrak{p} \in \operatorname{Ass} X$. □

Lemma 2.4.12. *Let* $\mathfrak{p} \in \operatorname{Spec} A$. *Then* $\operatorname{Ass} A/\mathfrak{p} = \{\mathfrak{p}\}$.

Proof We have $A/\mathfrak{q} \cong X \subseteq A/\mathfrak{p}$ if and only if $\mathfrak{q}$ equals the ideal annihilating $a + \mathfrak{p}$ for some $a \in A$. Then $b \in \mathfrak{q}$ if and only if $ab \in \mathfrak{p}$ if and only if $b \in \mathfrak{p}$, since $\mathfrak{p}$ is prime. □

Recall that for an A-module X, $E(X)$ denotes an injective envelope.

Lemma 2.4.13. *We have* $\operatorname{Ass} E(X) = \operatorname{Ass} X$ *for every* A*-module* X.

Proof Clearly, $\operatorname{Ass} X \subseteq \operatorname{Ass} E(X)$. If $A/\mathfrak{p} \cong X' \subseteq E(X)$ for some $\mathfrak{p} \in \operatorname{Spec} A$, then $X' \cap X \neq 0$, and we have $A/\mathfrak{q} \cong X'' \subseteq X' \cap X$ for some $\mathfrak{q} \in \operatorname{Spec} A$, by Lemma 2.4.6. This implies $\mathfrak{p} = \mathfrak{q}$, by Lemma 2.4.12. □

Corollary 2.4.14. *Let* X *be an* A*-module. Then* $\operatorname{Supp} X = \operatorname{Supp} E(X)$. *Therefore localising subcategories of* $\operatorname{Mod} A$ *are closed under injective envelopes.*

Proof We have $\operatorname{Ass} E(X) = \operatorname{Ass} X$ by Lemma 2.4.13, and then Lemma 2.4.11 implies that $\operatorname{Supp} E(X) = \operatorname{Supp} X$. If $\mathcal{C} \subseteq \operatorname{Mod} \mathcal{A}$ is localising and $X \in \mathcal{C}$, then $E(X) \in \mathcal{C}$ by Corollary 2.4.10. □

Corollary 2.4.15. *The assignments* $\mathfrak{p} \mapsto E(A/\mathfrak{p})$ *and* $X \mapsto \operatorname{Ass} X$ *yield mutually inverse bijections between* $\operatorname{Spec} A$ *and* $\operatorname{Sp}(\operatorname{Mod} A)$.

Proof We have $\operatorname{Ass}(E(A/\mathfrak{p})) = \{\mathfrak{p}\}$ by Lemma 2.4.12 and Lemma 2.4.13. On the other hand, if X is indecomposable injective, then $\operatorname{Ass} X \neq \varnothing$ by Lemma 2.4.6. Clearly, $X \cong E(A/\mathfrak{p})$ when $\mathfrak{p} \in \operatorname{Ass} X$. □

For a subset $\mathcal{U} \subseteq \operatorname{Spec} A$ we set

$$\operatorname{Inj}_{\mathcal{U}} A = \{X \in \operatorname{Inj} A \mid \operatorname{Ass} X \subseteq \mathcal{U}\}.$$

Corollary 2.4.16. *Let* $\mathcal{V} \subseteq \operatorname{Spec} A$ *be specialisation closed and set* $\mathcal{W} = \operatorname{Spec} A \setminus \mathcal{V}$. *Then we have for* $\operatorname{Inj} A$ *a split torsion pair* $(\operatorname{Inj}_{\mathcal{V}} A, \operatorname{Inj}_{\mathcal{W}} A)$.

Proof Consider the localising subcategory

$$\mathcal{C} = \{X \in \operatorname{Mod} A \mid \operatorname{Supp} X \subseteq \mathcal{V}\};$$

see Corollary 2.4.10. Because $\mathcal{C}$ is closed under injective envelopes by Corollary 2.4.14, we have

$$\operatorname{Inj}_{\mathcal{V}} A = \mathcal{C} \cap \operatorname{Inj} \mathcal{A} \qquad \text{and} \qquad \operatorname{Inj}_{\mathcal{W}} A = \mathcal{C}^{\perp} \cap \operatorname{Inj} \mathcal{A}. \qquad □$$

Localising subcategories of module categories over non-commutative rings are usually not closed under injective envelopes.

Example 2.4.17. Let k be a field and $\Lambda = \left[\begin{smallmatrix} k & 0 \\ k & k \end{smallmatrix}\right]$. Consider the simple Λ-module $S = e\Lambda$ where $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. The localising subcategory generated by S consists of all direct sums of copies of S since $\operatorname{Ext}^1_\Lambda(S, S) = 0$; so it does not contain $E(S) = \operatorname{Hom}_k(\Lambda e, k)$.

Artinian Modules

Let A be a commutative ring and let $\mathfrak{a}$ be an ideal. We set $\operatorname{gr}(A)_n = \mathfrak{a}^n/\mathfrak{a}^{n+1}$ for $n \in \mathbb{Z}$, where $\mathfrak{a}^n = A$ for all $n \le 0$. The *associated graded ring*

$$\operatorname{gr}(A) = \bigoplus_{n\in\mathbb{Z}} \operatorname{gr}(A)_n$$

is $\mathbb{Z}$-graded with multiplication induced by that in A.

Lemma 2.4.18. *If the ideal* $\mathfrak{a}$ *is finitely generated over* A*, then* $\operatorname{gr}(A)$ *is a finitely generated* $A/\mathfrak{a}$*-algebra.*

Proof Let $x_1, \dots, x_n$ generate $\mathfrak{a}$. Then $\operatorname{gr}(A) = (A/\mathfrak{a})[\bar{x}_i, \dots \bar{x}_n]$, where $\bar{x}_i = x_i + \mathfrak{a}^2$, and $\operatorname{gr}(A)$ is a quotient of the polynomial ring $(A/\mathfrak{a})[X_i, \dots X_n]$ as a graded ring. □

For an A-module X and $m \in \mathbb{Z}$ let X_m denote the submodule of elements annihilated by $\mathfrak{a}^m$. We set $\operatorname{gr}^{\mathfrak{a}}(X)_m = X_{-m+1}/X_{-m}$ and obtain a graded $\operatorname{gr}(A)$-module

$$\operatorname{gr}^{\mathfrak{a}}(X) = \bigoplus_{n\in\mathbb{Z}} \operatorname{gr}^{\mathfrak{a}}(X)_n.$$

The assignment $(x, a) \mapsto xa$ yields an $A/\mathfrak{a}$-bilinear map

$$\operatorname{gr}^{\mathfrak{a}}(X)_m \times \operatorname{gr}(A)_n \longrightarrow \operatorname{gr}^{\mathfrak{a}}(X)_{m+n},$$

which induces a homomorphism

$$\mu_X\colon\ \operatorname{gr}^{\mathfrak{a}}(X) \longrightarrow \operatorname{Hom}_{A/\mathfrak{a}}(\operatorname{gr}(A), X_1)$$

of graded $\operatorname{gr}(A)$-modules since $\operatorname{gr}^{\mathfrak{a}}(X)_0 = X_1$.

For each submodule $U \subseteq X$ let $\mathfrak{a}_U$ denote the graded ideal of $\operatorname{gr}(A)$ consisting in degree n of elements $a \in \operatorname{gr}(A)_n$ such that $xa = 0$ for all $x \in ((U \cap X_{n+1}) + X_n)/X_n$.

Lemma 2.4.19. *Let A be a commutative noetherian ring and let* $\mathfrak{m}$ *be a maximal ideal. Then the injective envelope* $E(A/\mathfrak{m})$ *is artinian over* A.

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Proof Set $X = E(A/\mathfrak{m})$. We consider $\mathrm{gr}(A)$ for $\mathfrak{a} = \mathfrak{m}$ and $\mathrm{gr}^{\mathfrak{m}}(X)$. First observe that $X = \bigcup_{n\geq 0} X_n$. To see this, let $U \subseteq X$ be a finitely generated submodule. Then we have $\operatorname{Supp} U \subseteq \operatorname{Supp} X = \{\mathfrak{m}\}$ by Corollary 2.4.14. Thus U admits a finite filtration with factors isomorphic to $A/\mathfrak{m}$ by Lemma 2.4.7. This means U is annihilated by $\mathfrak{m}^n$ for some $n \geq 0$, so $U \subseteq X_n$.

Our first observation implies that μ_X is an isomorphism. For submodules U, V of X, it follows that $\mathfrak{m}_U = \mathfrak{m}_V$ implies

$$(U \cap X_{n+1}) + X_n = (V \cap X_{n+1}) + X_n$$

for all n. Thus $U \cap X_{n+1} = V \cap X_{n+1}$ for all n by induction, and therefore $U = V$. Clearly, $U \subseteq V$ implies $\mathfrak{m}_V \subseteq \mathfrak{m}_U$. Thus X is artinian, because $\mathrm{gr}(A)$ is noetherian by Lemma 2.4.18. □

Proposition 2.4.20. *For a module X over a commutative noetherian ring the following are equivalent.*

(1) *The module X is artinian.*
(2) *The module X is a union of finite length submodules and the socle of X has finite length.*
(3) *The socle of X has finite length and all prime ideals in* Supp *X are maximal.*

Proof (1) ⇒ (2): The module X is a union of its finitely generated submodules, which are both artinian and noetherian, and therefore of finite length. A semisimple artinian module has finite length. Thus $\operatorname{soc} X$ has finite length.

(2) ⇒ (3): This follows from Lemma 2.4.3, since the support of a finite length module consists of prime ideals which are maximal.

(3) ⇒ (1): We have $\operatorname{Supp} E(X) = \operatorname{Supp} X$ by Corollary 2.4.14. Then Lemma 2.4.19 implies that $E(X)$ is artinian. Thus X is artinian. □

Graded Rings and Modules

The preceding results about modules over commutative noetherian rings generalise to graded modules over graded rings. We sketch the appropriate setting.

Fix an abelian *grading group G* and let A be a *G-graded ring*. Thus A is a ring together with a decomposition of the underlying abelian group

$$A = \bigoplus_{g\in G} A_g$$

such that the multiplication satisfies $A_g A_h \subseteq A_{g+h}$ for all $g, h \in G$. An element in A is called *homogeneous* of *degree g* if it belongs to A_g for some $g \in G$.

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We consider graded A-modules and homogeneous ideals of A. An A-module M is *G-graded* if the underlying abelian group admits a decomposition

$$M = \bigoplus_{g \in G} M_g$$

such that the multiplication satisfies $M_g A_h \subseteq M_{g+h}$ for all $g, h \in G$. We write $\operatorname{GrMod} A$ for the category of graded A-modules (with degree zero morphisms) and $\operatorname{grmod} A$ for the full subcategory of finitely presented modules. Later on we will consider the full subcategory $\operatorname{grproj} A$ of finitely generated projective modules and the projectively stable category $\underline{\operatorname{grmod}}\, A$.

Now suppose that G is endowed with a symmetric bilinear form

$$(-,-)\colon G \times G \longrightarrow \mathbb{Z}/2.$$

A typical example is $G = \mathbb{Z}$ with $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/2$ the multiplication map modulo two. We say that A is *G-graded commutative* when $xy = (-1)^{(g,h)} yx$ for all homogeneous $x \in A_g$, $y \in A_h$. A homogeneous element in A is *even* if it belongs to A_g for some $g \in G$ satisfying $(g, h) = 0$ for all $h \in G$.

Let us fix such a G-graded commutative ring A. Note that all homogeneous ideals are automatically two-sided. The graded localisation of A at a multiplicative set consisting of even (and therefore central) homogeneous elements is the obvious one and enjoys the usual properties; in particular, it is again a G-graded commutative ring. Similarly, one localises any graded A-module at such a multiplicative set. For instance, when $\mathfrak{p}$ is a homogeneous prime ideal of A and M is a graded A-module, then $M_\mathfrak{p}$ is the localisation of M with respect to the multiplicative set of even homogeneous elements in $A \setminus \mathfrak{p}$.

Suppose now that A is *noetherian* as a G-graded ring, that is, the ascending chain condition holds for homogeneous ideals of A. Then all results of this section carry over to the category of graded A-modules. However, it is necessary to twist. For any graded A-module M and $g \in G$, the *twisted module* $M(g)$ is the A-module M with the new grading defined by $M(g)_h = M_{g+h}$ for each $h \in G$. For instance, in Lemma 2.4.6 one shows that each graded non-zero module has a submodule of the form $(A/\mathfrak{p})(g)$ for some homogeneous prime ideal $\mathfrak{p}$ and some $g \in G$. This affects all subsequent statements. The following is then the analogue of Proposition 2.4.8.

Proposition 2.4.21. *The assignment $\mathcal{C} \mapsto \operatorname{Supp} \mathcal{C}$ induces a bijection between*

- *the set of Serre subcategories of* $\operatorname{grmod} A$ *that are closed under twists, and*
- *the set of specialisation closed sets of homogeneous prime ideals of A.* □

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Example 2.4.22. Let k be a field and $\mathbb{X} = \mathbb{P}^1_k$ the projective line with homogeneous coordinate ring $S = k[x_0, x_1]$. Then a theorem of Serre [188] provides the following localisation sequence

$$\operatorname{GrMod}_0 S \rightleftarrows \operatorname{GrMod} S \underset{\Gamma_*(\mathbb{X},-)}{\rightleftarrows} \operatorname{Qcoh} \mathbb{X}$$

where $\operatorname{GrMod}_0 S$ denotes the category of torsion modules. Note that $\operatorname{GrMod}_0 S$ is the localising subcategory corresponding to the category $\operatorname{grmod}_0 S$ of finite length modules. These are precisely the modules with support only containing the unique maximal homogeneous ideal of positive degree elements. The fact that the subcategory $\operatorname{GrMod}_0 S$ is not closed under products leads to an example showing that products in $\operatorname{Qcoh} \mathbb{X}$ need not be exact.

For each $n \geq 0$, we have a canonical map

$$\pi_n \colon \mathscr{O}(-n) \otimes_k \operatorname{Hom}_{\mathbb{X}}(\mathscr{O}(-n), \mathscr{O}) \longrightarrow \mathscr{O}$$

which is an epimorphism in $\operatorname{Qcoh} \mathbb{X}$. We claim that the product

$$\pi \colon \prod_{n \geq 0} \big(\mathscr{O}(-n) \otimes_k \operatorname{Hom}_{\mathbb{X}}(\mathscr{O}(-n), \mathscr{O})\big) \longrightarrow \prod_{n \geq 0} \mathscr{O}$$

is not an epimorphism. Taking graded global sections gives for each $n \geq 0$ the multiplication map

$$\Gamma_*(\mathbb{X}, \pi_n) \colon S(-n) \otimes_k S_n \longrightarrow S$$

which is a morphism of graded S-modules with cokernel of finite length. However, the cokernel of

$$\Gamma_*(\mathbb{X}, \pi) = \prod_{n \geq 0} \Gamma_*(\mathbb{X}, \pi_n)$$

is not a torsion module. The left adjoint of $\Gamma_*(\mathbb{X}, -)$ is exact and takes $\Gamma_*(\mathbb{X}, \pi)$ to π. It follows that the cokernel of π is non-zero, because the left adjoint of $\Gamma_*(\mathbb{X}, -)$ annihilates exactly those S-modules which are torsion.

2.5 Grothendieck Categories

We study the basic properties of Grothendieck categories. It is shown that an abelian category is a Grothendieck category if and only if it is the localisation of a module category. From this we deduce that objects in a Grothendieck category admit injective envelopes. Also, it follows that any Grothendieck category is a locally presentable category. This means that every object is an α-filtered colimit of α-presentable objects for some regular cardinal α. Finally, we characterise the coherent functors for any locally presentable category.

The Embedding Theorem

Let $\mathcal{A}$ be an abelian category and suppose that $\mathcal{A}$ admits arbitrary coproducts. We fix an object $C \in \mathcal{A}$ and set $\Lambda = \operatorname{End}(C)$. Then the functor

$$H \colon \mathcal{A} \longrightarrow \operatorname{Mod}\Lambda, \quad X \mapsto \operatorname{Hom}(C, X)$$

admits a left adjoint $T\colon \operatorname{Mod}\Lambda \to \mathcal{A}$. We obtain this by first extending the equivalence $\operatorname{add}\Lambda \to \operatorname{add}C$ to a functor $\tilde{T}\colon \operatorname{Add}\Lambda \to \operatorname{Add}C$ preserving coproducts. Then extend $\tilde{T}$ to a right exact functor $\operatorname{Mod}\Lambda \to \mathcal{A}$.

Recall that C is a generator for $\mathcal{A}$ if for every object $X \in \mathcal{A}$ the canonical morphism $\coprod_{\phi \in \operatorname{Hom}(C,X)} C \to X$ is an epimorphism.

Lemma 2.5.1. *Suppose that filtered colimits in $\mathcal{A}$ are exact and that C is a generator. If $\phi\colon X \to H(Y)$ is a monomorphism in* $\operatorname{Mod}\Lambda$*, then the adjoint morphism $\psi\colon T(X) \to Y$ is a monomorphism.*

Proof Suppose $K = \operatorname{Ker}\psi \neq 0$. Choose an epimorphism $\Lambda^{(I)} \to X$ which yields an epimorphism $\pi\colon T(\Lambda^{(I)}) \to T(X)$. Write $\Lambda^{(I)} = \bigcup_{J \subseteq I} \Lambda^J$ as filtered colimit, where $J \subseteq I$ runs through all finite subsets. This implies $T(\Lambda^{(I)}) = \bigcup_{J \subseteq I} T(\Lambda^J)$ and therefore

$$\bigcup_{J \subseteq I} \big(\pi^{-1}(K) \cap T(\Lambda^J)\big) = \pi^{-1}(K) \neq 0.$$

Thus we obtain a non-zero morphism

$$\tau\colon T(\Lambda) \to \pi^{-1}(K) \cap T(\Lambda^J) \hookrightarrow T(\Lambda^J) \to T(X)$$

such that $\psi\tau = 0$, since $C = T(\Lambda)$ is a generator. Note that $\tau = T(\sigma)$ for some $\sigma\colon \Lambda \to X$ which yields the following commutative diagram.

$$\begin{array}{ccccc} \Lambda & \xrightarrow{\ \sigma\ } & X & & \\ \downarrow & & \downarrow & \searrow^{\phi} & \\ HT(\Lambda) & \xrightarrow{H(\tau)} & HT(X) & \xrightarrow{H(\psi)} & H(Y) \end{array}$$

We have $\phi\sigma = 0$ and this implies $\sigma = 0$ since ϕ is a monomorphism. This is a contradiction since $T(\sigma) \neq 0$, and therefore $\operatorname{Ker}\psi = 0$. □

The following is known as the *Gabriel–Popescu theorem.*

Theorem 2.5.2 (Gabriel–Popescu)**.** *Let $\mathcal{A}$ be a category such that filtered colimits are exact. Given C, H, and T as above, the following are equivalent.*

(1) *C is a generator for $\mathcal{A}$.*
(2) *H is fully faithful.*
(3) *T is exact and induces an equivalence* $(\operatorname{Mod}\Lambda)/(\operatorname{Ker}T) \xrightarrow{\sim} \mathcal{A}$.

Proof (1) ⇔ (2): Clearly, C is a generator when H is faithful. For the converse suppose that C is a generator. For $X \in \mathcal{A}$ consider the counit $\varepsilon_X : TH(X) \to X$. Then we need to show that this is invertible for all $X \in \mathcal{A}$; see Proposition 1.1.3. Each morphism $C \to X$ factors through ε_x since e_C is invertible, and therefore ε_X is an epimorphism. On the other hand, ε_x is adjoint to $\mathrm{id}: H(X) \to H(X)$ and therefore a monomorphism by Lemma 2.5.1.

(1) & (2) ⇒ (3): We show that T is exact. Then it follows from Proposition 2.2.11 that T induces an equivalence $(\operatorname{Mod}\Lambda)/(\operatorname{Ker}T) \xrightarrow{\sim} \mathcal{A}$.

For the exactness of T we apply the criterion from Corollary 2.1.16. Thus we need to show that for each exact sequence $X \to Y \to Z$ of projective Λ-modules, the sequence $T(X) \to T(Y) \to T(Z)$ is exact. To show this, it suffices to prove that for each exact sequence $0 \to X \to Y \to Z \to 0$ of Λ-modules, the sequence $0 \to T(X) \to T(Y) \to T(Z) \to 0$ is exact provided that Y is projective. Moreover, it suffices to show that $T(X) \to T(Y)$ is a monomorphism since T is right exact. We may assume that $Y = \Lambda^{(I)}$ is free and write this as the filtered colimit $Y = \operatorname{colim} Y_J$, where $Y_J = \Lambda^J$ and $J \subseteq I$ runs through all finite subsets. Then $X \to Y$ is the filtered colimit of monomorphisms $X_J \to Y_J$, where $X_J = X \cap Y_J$. The morphism $T(X_J) \to T(Y_J) = C^J$ is adjoint to $X_J \to Y_J = H(C^J)$ and therefore a monomorphism by Lemma 2.5.1. It remains to note that T preserves colimits and that filtered colimits in $\mathcal{A}$ are exact. Thus $T(X) \to T(Y)$ is a monomorphism since it identifies with the filtered colimit of monomorphisms $T(X_J) \to T(Y_J)$.

(3) ⇒ (2): See Proposition 2.2.11. □

Corollary 2.5.3. *An abelian category is a Grothendieck category if and only if it is the localisation of a module category, so of the form* $(\operatorname{Mod}\Lambda)/\mathcal{C}$ *for some ring* Λ *and a localising subcategory* $\mathcal{C} \subseteq \operatorname{Mod}\Lambda$.

Proof Combine Theorem 2.5.2 with Proposition 2.2.16. □

Injective Envelopes

We are now able to establish injective envelopes in Grothendieck categories.

Corollary 2.5.4. *A Grothendieck category admits arbitrary products, and every object admits an injective envelope.*

Proof Fix a Grothendieck category $\mathcal{A}$. We apply the above Theorem 2.5.2 and identify $\mathcal{A}$ with $(\operatorname{Ker}T)^{\perp} \subseteq \operatorname{Mod}\Lambda$. The category $\operatorname{Mod}\Lambda$ has arbitrary products, and $(\operatorname{Ker}T)^{\perp}$ is closed under products. From this the first assertion follows. The existence of injective envelopes in $\mathcal{A}$ follows from Corollary 2.2.15, once we have shown that $\operatorname{Mod}\Lambda$ has injective envelopes.

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We proceed in two steps. Set $\mathcal{A} = \operatorname{Mod}\Lambda$ and fix an object $X \in \mathcal{A}$.

(1) *The object X admits an embedding into an injective object.* It suffices to find an injective cogenerator, say E, because then $X \to \prod_{\phi \in \operatorname{Hom}(X,E)} E$ is a monomorphism.

If $\Lambda = \mathbb{Z}$, then

$$\mathbb{Q}/\mathbb{Z} \cong \coprod_{p \text{ prime}} \mathbb{Z}_{p^\infty} \cong E\Big(\coprod_{p \text{ prime}} \mathbb{Z}/(p) \Big)$$

is an injective cogenerator. This can be shown using the notion of a divisible module. For an arbitrary ring Λ, we use restriction of scalars via the canonical homomorphism $\mathbb{Z} \to \Lambda$. So $\operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q}/\mathbb{Z})$ is an injective cogenerator since

$$\operatorname{Hom}_\Lambda(-, \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q}/\mathbb{Z})) \cong \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$$

by adjunction.

(2) *The object X admits an essential embedding into an injective object.* Let $\phi \colon X \to E$ be a monomorphism such that E is injective. Consider the partially ordered set of subobjects $\{E' \subseteq E \mid \operatorname{Im}\phi \hookrightarrow E' \text{ essential}\}$. Using the fact that filtered colimits are exact, it follows that this has a maximal element by Zorn's lemma, say E_0. It is easily checked that $X \to E_0$ is an injective envelope. In fact, choose a maximal subobject $E'' \subseteq E$ such that $E'' \cap E_0 = 0$, using again Zorn's lemma. Then the composite $E_0 \hookrightarrow E \twoheadrightarrow E/E''$ is an essential monomorphism and therefore an isomorphism by the maximality of E_0. Thus the inclusion $E_0 \hookrightarrow E$ is split and E_0 is injective. □

Corollary 2.5.5. *A Grothendieck category admits an injective cogenerator.*

Proof Fix a generator C and choose $E = \prod_{C' \subseteq C} E(C/C')$ where $C' \subseteq C$ runs through all subjects. It follows that any non-zero morphism $C \to X$ can be extended to a non-zero morphism $X \to E$. □

Decompositions into Indecomposables

We provide a brief discussion about decompositions of objects into indecomposable objects. In particular, we include a result about the uniqueness of such decompositions into indecomposable objects with local endomorphism rings.

Recall that an object X is *indecomposable* if $X \neq 0$ and if $X = X_1 \oplus X_2$ implies $X_1 = 0$ or $X_2 = 0$.

A non-zero object X is called *uniform* provided any two non-zero subobjects intersect non-trivially. Clearly, X is uniform if and only if its injective envelope $E(X)$ is indecomposable. An object X is called *super-decomposable* if X has no indecomposable direct summands. Note that $E(X)$ is super-decomposable if and only if X has no uniform subobjects. This is clear since a direct summand E of $E(X)$ is the injective envelope of the intersection $E \cap X$.

Example 2.5.6. Let $\Lambda = k\langle x, y\rangle$ be the free algebra on two generators. Then the Λ-module $E(\Lambda)$ is super-decomposable.

To see this, observe that if $a \in \Lambda$, then $ax\Lambda \cap ay\Lambda = 0$. Thus Λ has no uniform right ideals, and hence $E(\Lambda)$ is super-decomposable.

A ring is called *local* if all non-invertible elements form a proper ideal. Thus an object is indecomposable if its endomorphism ring is local.

Lemma 2.5.7. *If X is an indecomposable injective object in a Grothendieck category, then* $\operatorname{End}(X)$ *is a local ring.*

Proof We need to show that if ϕ and ψ in $\operatorname{End}(X)$ are non-invertible, then $\phi + \psi$ is non-invertible. If ϕ or ψ is a monomorphism, then it splits. Thus we need to show that $\operatorname{Ker}\phi \neq 0$ and $\operatorname{Ker}\psi \neq 0$ implies $\operatorname{Ker}(\phi+\psi) \neq 0$. But this is clear, since X is the injective envelope of any non-zero subobject. Thus

$$0 \neq (\operatorname{Ker}\phi) \cap (\operatorname{Ker}\psi) \subseteq \operatorname{Ker}(\phi + \psi). \qquad \square$$

The following is known as *Krull–Remak–Schmidt–Azumaya theorem.*

Theorem 2.5.8 (Krull–Remak–Schmidt–Azumaya)**.** *Let X be an object in a Grothendieck category with decompositions $X = \coprod_{i\in I} X_i$ and $X = \coprod_{j\in J} Y_j$ such that* $\operatorname{End}(X_i)$ *is a local ring for all i and Y_j is indecomposable for all j. Then there is a bijection $\sigma\colon I \xrightarrow{\sim} J$ such that $X_i \cong Y_{\sigma(i)}$ for all $i \in I$.*

Proof See for example [156, Section 4.8]. □

The appropriate tool for studying decompositions of objects in a Grothendieck category is its spectral category. Let $\mathcal{A}$ be a Grothendieck category and denote by Ess the class of essential monomorphisms in $\mathcal{A}$. This class admits a calculus of right fractions and is closed under coproducts. We obtain the canonical functor

$$P\colon \mathcal{A} \longrightarrow \mathcal{A}[\mathrm{Ess}^{-1}]$$

and call $\mathcal{A}[\mathrm{Ess}^{-1}]$ the *spectral category* of $\mathcal{A}$. It is not difficult to show that this is again a Grothendieck category which is split exact [82, Satz 1.3].

We have the following explicit description of the spectral category.

Proposition 2.5.9. *The canonical functor* $\mathcal{A} \to \mathcal{A}[\mathrm{Ess}^{-1}]$ *restricted to* $\operatorname{Inj}\mathcal{A}$ *induces an equivalence* $(\operatorname{Inj}\mathcal{A})/\operatorname{Rad}(\operatorname{Inj}\mathcal{A}) \xrightarrow{\sim} \mathcal{A}[\mathrm{Ess}^{-1}]$.

The assertion says that P induces for $X, Y \in \operatorname{Inj} \mathcal{A}$ an isomorphism

$$\operatorname{Hom}_{\mathcal{A}}(X,Y)/\operatorname{Rad}_{\mathcal{A}}(X,Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}[\operatorname{Ess}^{-1}]}(X,Y).$$

Proof The functor P identifies each object X with its injective envelope $E(X)$. Thus the restriction $P|_{\operatorname{Inj}\mathcal{A}}$ is essentially surjective. This restriction is also surjective on morphisms, because each morphism in $\mathcal{A}[\operatorname{Ess}^{-1}]$ is given by a right fraction $X \xleftarrow{\sigma} X' \xrightarrow{\alpha} Y$ (Lemma 1.2.1). Indeed, α extends to a morphisms $\bar{\alpha}\colon X \to Y$ when Y is injective, and then the right fraction equals $P(\bar{\alpha})$. Finally, we apply Proposition 2.1.23 and see that P annihilates a morphism ϕ in $\operatorname{Inj}\mathcal{A}$ if and only if ϕ is radical, since P is left exact. □

Locally Presentable Categories

A cardinal α is called *regular* if α is not the sum of fewer than α cardinals, all smaller than α. For example, $\aleph_0$ is regular because the sum of finitely many finite cardinals is finite. Also, the successor κ^+ of every infinite cardinal κ is regular. In particular, there are arbitrarily large regular cardinals.

Let α be a regular cardinal. A category $\mathcal{I}$ is called *α-filtered* if

(Fil1) the category is non-empty,

(Fil2) for each family $(x_i)_{i\in I}$ of fewer than α objects there is an object x with morphisms $x_i \to x$ for all i, and

(Fil3) for each family $(\phi_i : x \to y)_{i\in I}$ of fewer than α morphisms there exists a morphism $\psi\colon y \to z$ such that $\psi\phi_i = \psi\phi_j$ for all i, j.

An *α-filtered colimit* is the colimit of a functor $\mathcal{I} \to \mathcal{C}$ such that the category $\mathcal{I}$ is α-filtered. An *α-small colimit* is the colimit of a functor $\mathcal{I} \to \mathcal{C}$ such that the category $\mathcal{I}$ has fewer than α morphisms.

We record a characteristic property of α-filtered categories; it is well known when $\alpha = \aleph_0$ and says that α-filtered colimits in the category of sets commute with α-small limits.

Lemma 2.5.10. *For a regular cardinal α let $F\colon \mathcal{I}\times\mathcal{J} \to \operatorname{Set}$ be a functor such that $\mathcal{I}$ is α-filtered and $\mathcal{J}$ is α-small. Then the canonical map*

$$\operatorname*{colim}_i \lim_j F(i,j) \longrightarrow \lim_j \operatorname*{colim}_i F(i,j)$$

is bijective.

Proof Adapt the proof of the case $\alpha = \aleph_0$; see [142, Section IX.2]. □

Now fix an additive category $\mathcal{A}$ and suppose that $\mathcal{A}$ is cocomplete. An object

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$X \in \mathcal{A}$ is called *α-presentable* if $\operatorname{Hom}(X,-)$ preserves α-filtered colimits, that is, for every α-filtered colimit $\operatorname{colim}_{i\in\mathcal{I}} Y_i$ in $\mathcal{A}$ the canonical map

$$\operatorname*{colim}_i \operatorname{Hom}(X, Y_i) \longrightarrow \operatorname{Hom}(X, \operatorname*{colim}_i Y_i)$$

is bijective. Let $\mathcal{A}^\alpha$ denote the full subcategory of α-presentable objects.

Lemma 2.5.11. *The α-presentable objects are closed under taking α-small colimits.*

Proof Let $\operatorname{colim}_{i\in\mathcal{I}} X_i$ be an α-small colimit of α-presentable objects X_i. For an α-filtered colimit $\operatorname{colim}_{j\in\mathcal{J}} Y_j$ we compute

$$\begin{aligned}\operatorname*{colim}_j \operatorname{Hom}(\operatorname*{colim}_i X_i, Y_j) &\cong \operatorname*{colim}_j \lim_i \operatorname{Hom}(X_i, Y_j)\\ &\cong \lim_i \operatorname*{colim}_j \operatorname{Hom}(X_i, Y_j)\\ &\cong \operatorname{Hom}(\operatorname*{colim}_i X_i, \operatorname*{colim}_j Y_j)\end{aligned}$$

where the second isomorphism follows from the fact that α-small limits commute with α-filtered colimits in the category of sets, by Lemma 2.5.10. □

Lemma 2.5.12. *Let $\alpha \le \beta$ be regular cardinals. Then any colimit of α-presentable objects can be written canonically as a β-filtered colimit of β-presentable objects, which are β-small colimits of α-presentable objects.*

Proof Let $X : \mathcal{I} \to \mathcal{A}$ be a functor such that $X(i)$ is α-presentable for each $i \in \mathcal{I}$. Consider the set $\binom{\mathcal{I}}{\beta}$ of all subcategories $\mathcal{J} \subseteq \mathcal{I}$ having fewer than β morphisms. This set is partially ordered by inclusion and can be viewed as a category, which is β-filtered. For each $\mathcal{J} \subseteq \mathcal{I}$ set $X(\mathcal{J}) = \operatorname{colim} X|_{\mathcal{J}}$; this induces a functor $X_\beta : \binom{\mathcal{I}}{\beta} \to \mathcal{A}$. Then it is straightforward to check that the morphisms $X(\mathcal{J}) \to \operatorname{colim} X$ induce an isomorphism $\phi\colon \operatorname{colim} X_\beta \xrightarrow{\sim} \operatorname{colim} X$. In fact, for each $i \in \mathcal{I}$ there is a canonical morphism $X(i) \to \operatorname{colim} X_\beta$. These morphisms are compatible and induce the inverse of ϕ. It remains to observe that each $X(\mathcal{J})$ is β-presentable by Lemma 2.5.11. □

A cocomplete category $\mathcal{A}$ is called *locally α-presentable* if the category $\mathcal{A}^\alpha$ is essentially small and each object is an α-filtered colimit of α-presentable objects. The category is *locally presentable* if it is locally α-presentable for some regular cardinal α.

Lemma 2.5.13. *Let $\mathcal{A}$ be a locally presentable category. Then*

$$\mathcal{A} = \bigcup_\alpha \mathcal{A}^\alpha$$

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where α runs through all regular cardinals.

If $\mathcal{A}$ is locally α-presentable, then $\mathcal{A}$ is locally β-presentable for all $\beta \geq \alpha$. Moreover, $\mathcal{A}^\beta$ equals the closure of $\mathcal{A}^\alpha$ under β-small colimits.

Proof Let $X \in \mathcal{A}$ be the α-filtered colimit of α-presentable objects, given by a functor $\mathcal{I} \to \mathcal{A}$. Choose a regular cardinal $\beta \geq \alpha$ such that $\mathcal{I}$ has fewer than β morphisms. Then X is β-presentable by Lemma 2.5.11.

Let $\mathcal{A}$ be locally α-presentable. Then every object is a β-filtered colimit of β-presentable objects, by Lemma 2.5.12. In fact, we can choose β-presentable objects that are β-small colimits of α-presentable objects. In particular, every β-presentable object is of this form. □

Next we consider more specifically the category $\mathcal{A} = \operatorname{Mod}\Lambda$ for a ring Λ.

Lemma 2.5.14. *Let Λ be a ring, α a regular cardinal, and $n \geq 0$ an integer. If a Λ-module X admits a free presentation*

$$\Lambda^{(\alpha_{n+1})} \longrightarrow \Lambda^{(\alpha_n)} \longrightarrow \cdots \longrightarrow \Lambda^{(\alpha_0)} \longrightarrow X \longrightarrow 0$$

with $\alpha_p < \alpha$ for $0 \leq p \leq n+1$, then $\operatorname{Ext}^n_\Lambda(X,-)$ preserves α-filtered colimits.

Proof We view the presentation of X as a complex and have

$$\operatorname{Ext}^n_\Lambda(X,-) \cong H^n \operatorname{Hom}_\Lambda(\Lambda^{(\alpha_p)},-).$$

For an α-filtered colimit $\operatorname{colim}_{i\in\mathcal{I}} Y_i$ of Λ-modules we compute

$$\begin{aligned}\operatorname*{colim}_i \operatorname{Ext}^n_\Lambda(X,Y_i) &\cong \operatorname*{colim}_i H^n \operatorname{Hom}_\Lambda(\Lambda^{(\alpha_p)},Y_i)\\ &\cong H^n \operatorname*{colim}_i \operatorname{Hom}_\Lambda(\Lambda^{(\alpha_p)},Y_i)\\ &\cong H^n \operatorname{Hom}_\Lambda(\Lambda^{(\alpha_p)},\operatorname*{colim}_i Y_i)\\ &\cong \operatorname{Ext}^n_\Lambda(X,\operatorname*{colim}_i Y_i).\end{aligned}$$

The second isomorphism follows from the fact that taking α-filtered colimits is exact, and the third isomorphism uses that $\Lambda^{(\alpha_p)}$ is α-presentable for each $p \leq n+1$, by Lemma 2.5.11. □

Lemma 2.5.15. *Let Λ be a ring. For every family of Λ-modules $(X_i)_{i\in I}$ and every $n \geq 0$ we have a canonical isomorphism*

$$\operatorname{Ext}^n_\Lambda\Big(\coprod_i X_i,-\Big) \xrightarrow{\sim} \prod_i \operatorname{Ext}^n_\Lambda(X_i,-).$$

Proof Choose a projective resolution $p(X_i) \to X_i$ for each i. Because taking

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(co)products of modules is exact, we obtain for every Λ-module Y

$$\begin{aligned}\operatorname{Ext}^n_\Lambda\Big(\coprod_i X_i, Y\Big) &\cong H^n \operatorname{Hom}\Big(\coprod_i p(X_i), Y\Big)\\ &\cong H^n \prod_i \operatorname{Hom}(p(X_i), Y)\\ &\cong \prod_i \operatorname{Ext}^n_\Lambda(X_i, Y).\end{aligned}$$

□

Proposition 2.5.16. *Any Grothendieck category is locally presentable.*

Proof Fix a Grothendieck category $\mathcal{A}$ with generator C and set $\Lambda = \operatorname{End}(C)$. We deduce the assertion from Theorem 2.5.2. Let $T\colon \operatorname{Mod}\Lambda \to \mathcal{A}$ be the exact left adjoint of the full faithful functor $\operatorname{Hom}(C, -)$. Then $\operatorname{Hom}(C, -)$ identifies $\mathcal{A}$ with $(\operatorname{Ker} T)^\perp$ by Lemma 2.2.10. Now choose a generator K of $\operatorname{Ker} T$. It is not difficult to check that $(\operatorname{Ker} T)^\perp = K^\perp$, since any exact sequence $0 \to X' \to K^{(\alpha)} \to X \to 0$ in $\operatorname{Mod}\Lambda$ (α any cardinal) yields an exact sequence

$$\begin{aligned}0 \longrightarrow \operatorname{Hom}(X,-) \longrightarrow \operatorname{Hom}(K^{(\alpha)},-) \longrightarrow \operatorname{Hom}(X',-) \longrightarrow\\ \longrightarrow \operatorname{Ext}^1(X,-) \longrightarrow \operatorname{Ext}^1(K^{(\alpha)},-) \longrightarrow \operatorname{Ext}^1(X',-) \longrightarrow \cdots\end{aligned}$$

and keeping in mind that

$$\operatorname{Ext}^i(K^{(\alpha)}, -) \cong \operatorname{Ext}^i(K, -)^\alpha \qquad (i \geq 0)$$

by Lemma 2.5.15. Now choose a free presentation

$$\Lambda^{(\alpha_2)} \longrightarrow \Lambda^{(\alpha_1)} \longrightarrow \Lambda^{(\alpha_0)} \longrightarrow K \longrightarrow 0$$

and a regular cardinal α such that $\alpha_i < \alpha$ for all i. Then it follows from Lemma 2.5.14 that $\operatorname{Hom}(K, -)$ and $\operatorname{Ext}^1(K, -)$ preserve α-filtered colimits. Thus the functor $\operatorname{Hom}(C, -)$ preserves α-filtered colimits, because it identifies with the inclusion $\mathcal{A} \hookrightarrow \operatorname{Mod}\Lambda$. Then the lemma below implies that T maps α-presentable objects to α-presentable objects.

Any Λ-module is a filtered colimit of finitely presented modules (Proposition 11.1.9), and therefore an α-filtered colimit of α-presentable modules by Lemma 2.5.12. Applying the functor T it follows that any object in $\mathcal{A}$ is an α-filtered colimit of α-presentable objects. □

Lemma 2.5.17. *Let (F, G) be an adjoint pair of functors and α a regular cardinal. If G preserves α-filtered colimits, then F maps α-presentable objects to α-presentable objects.*

Proof For an α-presentable object X and an α-filtered colimit $\operatorname{colim}_{i\in\mathcal{I}} Y_i$ we have

$$\begin{aligned}\operatorname*{colim}_i \operatorname{Hom}(FX, Y_i) &\cong \operatorname*{colim}_i \operatorname{Hom}(X, GY_i)\\ &\cong \operatorname{Hom}(X, \operatorname*{colim}_i GY_i)\\ &\cong \operatorname{Hom}(X, G(\operatorname*{colim}_i Y_i))\\ &\cong \operatorname{Hom}(FX, \operatorname*{colim}_i Y_i).\end{aligned}$$ □

Remark 2.5.18. Let $\mathcal{C}$ be an essentially small additive category and fix a regular cardinal α. When $\mathcal{C}$ has α-small colimits we write

$$\operatorname{Ind}_\alpha \mathcal{C} := \operatorname{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$$

for the category of left exact functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ preserving α-small products. This category is locally α-presentable with

$$\mathcal{C} \xrightarrow{\sim} (\operatorname{Ind}_\alpha \mathcal{C})^\alpha.$$

Conversely, for any locally α-presentable additive category $\mathcal{A}$ the assignment $X \mapsto \operatorname{Hom}_{\mathcal{A}}(-, X)|_{\mathcal{A}^\alpha}$ induces an equivalence

$$\mathcal{A} \xrightarrow{\sim} \operatorname{Ind}_\alpha(\mathcal{A}^\alpha).$$

This generalises (with similar proofs) a correspondence for locally finitely presented categories, which is the case $\alpha = \aleph_0$ (Theorem 11.1.15). A consequence is the fact that a locally presentable category is complete, because the subcategory $\operatorname{Ind}_\alpha \mathcal{C} \subseteq \operatorname{Mod} \mathcal{C}$ is closed under limits.

Remark 2.5.19. Let $\mathcal{A}^{\mathbf{2}}$ denote the category of morphisms in $\mathcal{A}$. If $\mathcal{A}$ is locally α-presentable, then $\mathcal{A}^{\mathbf{2}}$ is locally α-presentable and $(\mathcal{A}^\alpha)^{\mathbf{2}} \xrightarrow{\sim} (\mathcal{A}^{\mathbf{2}})^\alpha$. This means that each morphism in $\mathcal{A}$ can be written as an α-filtered colimit of morphisms in $\mathcal{A}^\alpha$.

Localisation of Grothendieck Categories

In the following we sketch the localisation theory for Grothendieck categories, using the fact that any Grothendieck category $\mathcal{A}$ admits a filtration $\mathcal{A} = \bigcup_\alpha \mathcal{A}^\alpha$. In fact, we will see that $\mathcal{A}^\alpha$ is abelian when α is sufficiently large.

Lemma 2.5.20. *Let $\mathcal{A}$ be a locally α-presentable Grothendieck category. Then $\mathcal{A}^\alpha$ is abelian if and only if $\mathcal{A}^\alpha$ is closed under kernels. Moreover, in this case the inclusion $\mathcal{A}^\alpha \to \mathcal{A}$ is exact and $\mathcal{A}^\alpha$ is an extension closed subcategory.*

Proof We use the fact that $\mathcal{A}^\alpha$ is closed under cokernels. Thus when $\mathcal{A}^\alpha$ is closed under kernels, then $\mathcal{A}^\alpha$ is abelian and the inclusion $\mathcal{A}^\alpha \to \mathcal{A}$ is exact. Conversely, suppose that $\mathcal{A}^\alpha$ is abelian. Given an exact sequence $0 \to X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$ in $\mathcal{A}^\alpha$, we need to show that it is also exact in $\mathcal{A}$. Let $\operatorname{colim} X_i$ be the kernel of ψ in $\mathcal{A}$, written as α-filtered colimit of objects in $\mathcal{A}^\alpha$. Each $X_i \to Y$ factors through ϕ, so $\operatorname{colim} X_i \to Y$ factors through ϕ. Thus ϕ is a kernel in $\mathcal{A}$.

In order to show that $\mathcal{A}^\alpha$ is extension closed, let $\eta\colon 0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathcal{A}$ with $X, Z \in \mathcal{A}^\alpha$. Write $Y = \operatorname{colim} Y_i$ as α-filtered colimit of objects in $\mathcal{A}^\alpha$. Then η is the colimit of exact sequences $0 \to X_i \to Y_i \to Z$, and for some index i_0 the induced morphisms $\phi\colon X_{i_0} \to X$ and $Y_{i_0} \to Z$ are epimorphisms. It follows that Y is isomorphic to the cokernel of $\operatorname{Ker}\phi \to Y_{i_0}$ and therefore in $\mathcal{A}^\alpha$. □

Proposition 2.5.21. *Let $\mathcal{A}$ be a Grothendieck category and α a regular cardinal. Suppose that $\mathcal{A}$ is locally α-presentable and that $\mathcal{A}^\alpha$ is abelian. For a localising subcategory $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} \cap \mathcal{A}^\alpha$ generates $\mathcal{B}$, the following holds.*

(1) *$\mathcal{B}$ and $\mathcal{A}/\mathcal{B}$ are locally α-presentable Grothendieck categories.*
(2) *$\mathcal{B}^\alpha = \mathcal{B} \cap \mathcal{A}^\alpha$ and the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ induces an equivalence*

$$\mathcal{A}^\alpha/\mathcal{B}^\alpha \xrightarrow{\sim} (\mathcal{A}/\mathcal{B})^\alpha.$$

(3) *The inclusion $\mathcal{B} \to \mathcal{A}$ induces a localisation sequence.*

$$\begin{array}{ccccc} \mathcal{B}^\alpha & \rightarrowtail & \mathcal{A}^\alpha & \twoheadrightarrow & \mathcal{A}^\alpha/\mathcal{B}^\alpha \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B} & \rightleftarrows & \mathcal{A} & \rightleftarrows & \mathcal{A}/\mathcal{B} \end{array}$$

Proof The proof amounts to identifying the sequence $\mathcal{B} \rightarrowtail \mathcal{A} \twoheadrightarrow \mathcal{A}/\mathcal{B}$ with the sequence $\operatorname{Ind}_\alpha(\mathcal{B}^\alpha) \to \operatorname{Ind}_\alpha(\mathcal{A}^\alpha) \to \operatorname{Ind}_\alpha(\mathcal{A}^\alpha/\mathcal{B}^\alpha)$ which is induced by $\mathcal{B}^\alpha \rightarrowtail \mathcal{A}^\alpha \twoheadrightarrow \mathcal{A}^\alpha/\mathcal{B}^\alpha$. Proposition 11.1.31 gives the details when $\alpha = \aleph_0$, and the general case is similar. □

Let $\mathcal{C}$ be an essentially small additive category and fix a regular cardinal α. We write

$$\operatorname{mod}_\alpha \mathcal{C} := (\operatorname{Mod}\mathcal{C})^\alpha \qquad \text{and} \qquad \operatorname{proj}_\alpha \mathcal{C} := \operatorname{Proj}\mathcal{C} \cap \operatorname{mod}_\alpha \mathcal{C},$$

where $\operatorname{Proj}\mathcal{C}$ denotes the full subcategory of projective objects in $\operatorname{Mod}\mathcal{C}$. It is easily checked that $X \in \operatorname{Mod}\mathcal{C}$ belongs to $\operatorname{mod}_\alpha \mathcal{C}$ if and only if there is a presentation

$$\coprod_{i\in I} \operatorname{Hom}_{\mathcal{C}}(-, C_i) \longrightarrow \coprod_{j\in J} \operatorname{Hom}_{\mathcal{C}}(-, D_j) \longrightarrow X \longrightarrow 0$$

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satisfying $\operatorname{card} I, \operatorname{card} J < \alpha$; see Lemma 2.5.12.

The next lemma shows that $\operatorname{mod}_\alpha \mathcal{C}$ is abelian when α is sufficiently large.

Lemma 2.5.22. *The following conditions are equivalent.*

(1) *The kernel of each morphism in* $\operatorname{mod} \mathcal{C}$ *belongs to* $\operatorname{mod}_\alpha \mathcal{C}$.
(2) *The category* $\operatorname{proj}_\alpha \mathcal{C}$ *has weak kernels.*
(3) *The category* $\operatorname{mod}_\alpha \mathcal{C}$ *is abelian.*

Proof (1) ⇒ (2): We apply Lemma 2.5.11. The objects in $\operatorname{proj}_\alpha \mathcal{C}$ are precisely the direct summands of coproducts $X = \coprod_{i \in I} \operatorname{Hom}_{\mathcal{C}}(-, X_i)$ with $\operatorname{card} I < \alpha$. Clearly, X is the filtered colimit of subobjects $\coprod_{i \in J} \operatorname{Hom}_{\mathcal{C}}(-, X_i)$ with $\operatorname{card} J < \aleph_0$. This colimit is α-small, and it follows that any morphism $X \to Y$ in $\operatorname{proj}_\alpha \mathcal{C}$ is an α-small filtered colimit of morphisms $X_\lambda \to Y_\lambda$ in $\operatorname{proj} \mathcal{C} \subseteq \operatorname{mod} \mathcal{C}$. Thus

$$\operatorname{Ker}(X \to Y) = \operatorname*{colim}_\lambda \operatorname{Ker}(X_\lambda \to Y_\lambda)$$

belongs to $\operatorname{mod}_\alpha \mathcal{C}$. It remains to observe that each object in $\operatorname{mod}_\alpha \mathcal{C}$ is the quotient of an object in $\operatorname{proj}_\alpha \mathcal{C}$.

(2) ⇒ (3): That $\operatorname{mod}_\alpha \mathcal{C}$ is abelian follows from Lemma 2.1.6 since each object in $\operatorname{mod}_\alpha \mathcal{C}$ is the cokernel of a morphism in $\operatorname{proj}_\alpha \mathcal{C}$, and therefore

$$\operatorname{mod}_\alpha \mathcal{C} \xrightarrow{\sim} \operatorname{mod}(\operatorname{proj}_\alpha \mathcal{C}).$$

(3) ⇒ (1): This is clear, since $\operatorname{mod} \mathcal{C} \subseteq \operatorname{mod}_\alpha \mathcal{C}$. □

Corollary 2.5.23. *Let $\mathcal{A}$ be a Grothendieck category. There exists a regular cardinal α_0 such that for all regular $\alpha \geq \alpha_0$ the category $\mathcal{A}^\alpha$ is abelian and an extension closed subcategory of $\mathcal{A}$ with exact inclusion $\mathcal{A}^\alpha \to \mathcal{A}$.*

Proof We apply Theorem 2.5.2 and write $\mathcal{A}$ as the quotient $(\operatorname{Mod} \Lambda)/\mathcal{C}$ for some ring Λ and a localising subcategory $\mathcal{C} \subseteq \operatorname{Mod} \Lambda$. Choose α such that $\operatorname{mod}_\alpha \Lambda$ is abelian and $\mathcal{C} \cap \operatorname{mod}_\alpha \Lambda$ generates $\mathcal{C}$. Then the assertion follows from Proposition 2.5.21. More precisely, for $\alpha \geq \alpha_0$ we have $(\operatorname{mod}_\alpha \Lambda)/\mathcal{C}^\alpha \xrightarrow{\sim} \mathcal{A}^\alpha$. Thus $\mathcal{A}^\alpha$ is abelian and the inclusion $\mathcal{A}^\alpha \to \mathcal{A}$ is exact. Also, $\mathcal{A}^\alpha$ is extension closed by Lemma 2.5.20. □

When $\mathcal{C}$ has α-small colimits, then the Yoneda functor $\mathcal{C} \to \operatorname{mod}_\alpha \mathcal{C}$ admits a left adjoint; it is the α-small colimit preserving functor $\operatorname{mod}_\alpha \mathcal{C} \to \mathcal{C}$ taking each representable functor $\operatorname{Hom}_{\mathcal{C}}(-, X)$ to X. The special case $\alpha = \aleph_0$ is Example 1.1.4. Let $\operatorname{eff}_\alpha \mathcal{C}$ denote the full subcategory of $\operatorname{mod}_\alpha \mathcal{C}$ consisting of the objects annihilated by this left adjoint, and set $\operatorname{Eff}_\alpha \mathcal{C} := \operatorname{Ind}_\alpha(\operatorname{eff}_\alpha \mathcal{C})$. Then the following is an analogue of Proposition 2.3.3.

Proposition 2.5.24. *Let $\mathcal{C}$ be an essentially small abelian category with α-small coproducts and suppose that* $\operatorname{Ind}_\alpha \mathcal{C}$ *is a Grothendieck category. Then the inclusion* $\operatorname{Ind}_\alpha \mathcal{C} \to \operatorname{Mod} \mathcal{C}$ *induces a localisation sequence of abelian categories*

$$\operatorname{Eff}_\alpha \mathcal{C} \rightleftarrows \operatorname{Mod} \mathcal{C} \rightleftarrows \operatorname{Ind}_\alpha \mathcal{C}$$

which restricts to the localisation sequence

$$\operatorname{eff}_\alpha \mathcal{C} \rightleftarrows \operatorname{mod}_\alpha \mathcal{C} \rightleftarrows \mathcal{C}.$$

Proof The inclusion $\operatorname{Ind}_\alpha \mathcal{C} \to \operatorname{Mod} \mathcal{C}$ has a left adjoint; it is the colimit preserving functor which is the identity on the representable functors. This left adjoint is exact by an analogue of Theorem 2.5.2, and it sends α-presentable objects to α-presentable objects, since the right adjoint preserves α-filtered colimits; see Lemma 2.5.17. This yields the left adjoint of the Yoneda functor $\mathcal{C} \to \operatorname{mod}_\alpha \mathcal{C}$. The rest then follows from Proposition 2.5.21. □

The following immediate consequence provides a canonical presentation of a Grothendieck category as the quotient of a module category.

Corollary 2.5.25. *Let $\mathcal{A}$ be a locally α-presentable Grothendieck category such that $\mathcal{C} = \mathcal{A}^\alpha$ is abelian. Then*

$$(\operatorname{Mod} \mathcal{C})/(\operatorname{Eff}_\alpha \mathcal{C}) \xrightarrow{\sim} \mathcal{A}. \qquad \square$$

Coherent Functors

Let $\mathcal{A}$ be a cocomplete additive category. We call a functor $F\colon \mathcal{A} \to \mathrm{Ab}$ *coherent* if there is an exact sequence

$$\operatorname{Hom}_{\mathcal{A}}(Y,-) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(X,-) \longrightarrow F \longrightarrow 0.$$

More precisely, we say for a regular cardinal α that F is *α-coherent* if X and Y are α-presentable objects. Note that every coherent functor is α-coherent for some regular cardinal α when $\mathcal{A}$ is locally presentable, thanks to Lemma 2.5.13.

Recall that a locally presentable category is complete and cocomplete; see Remark 2.5.18.

Theorem 2.5.26. *Let $\mathcal{A}$ be a locally α-presentable category. Then a functor $F\colon \mathcal{A} \to \mathrm{Ab}$ is α-coherent if and only if F preserves products and α-filtered colimits.*

The proof requires some preparations. In particular, we need a characterisation of finitely presented functors in terms of a tensor product.

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Let $\mathcal{C}$ be an essentially small additive category. Recall that there exists a *tensor product*

$$\operatorname{Mod}(\mathcal{C}) \times \operatorname{Mod}(\mathcal{C}^{\mathrm{op}}) \longrightarrow \mathrm{Ab}, \qquad (X,Y) \longmapsto X \otimes_{\mathcal{C}} Y,$$

where the tensor functors $X \otimes_{\mathcal{C}} -$ and $- \otimes_{\mathcal{C}} Y$ are determined by the fact that they preserve colimits and that for $C \in \mathcal{C}$ there are natural isomorphisms

$$X \otimes_{\mathcal{C}} \operatorname{Hom}_{\mathcal{C}}(C,-) \cong X(C) \quad \text{and} \quad \operatorname{Hom}_{\mathcal{C}}(-,C) \otimes_{\mathcal{C}} Y \cong Y(C).$$

Recall that $Y \in \operatorname{Mod}(\mathcal{C}^{\mathrm{op}})$ is finitely presented if there is a presentation

$$\operatorname{Hom}_{\mathcal{C}}(D,-) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(C,-) \longrightarrow Y \longrightarrow 0.$$

Proposition 2.5.27. *A functor $Y \in \operatorname{Mod}(\mathcal{C}^{\mathrm{op}})$ is finitely presented if and only if the functor $- \otimes_{\mathcal{C}} Y$ preserves all products.*

Proof Let $Y \in \operatorname{Mod}(\mathcal{C}^{\mathrm{op}})$. We choose a family of objects $(X_i)_{i\in I}$ in $\operatorname{Mod}\mathcal{C}$ and consider the canonical map

$$\alpha_Y \colon \Big(\prod_i X_i\Big) \otimes_{\mathcal{C}} Y \longrightarrow \prod_i (X_i \otimes_{\mathcal{C}} Y).$$

When $Y = \operatorname{Hom}_{\mathcal{C}}(C,-)$ for some $C \in \mathcal{C}$, then α_Y is a bijection. Now choose an exact sequence $Y_1 \to Y_0 \to Y \to 0$ in $\operatorname{Mod}(\mathcal{C}^{\mathrm{op}})$ and consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
(\prod_i X_i) \otimes_{\mathcal{C}} Y_1 & \longrightarrow & (\prod_i X_i) \otimes_{\mathcal{C}} Y_0 & \longrightarrow & (\prod_i X_i) \otimes_{\mathcal{C}} Y & \longrightarrow & 0 \\
\downarrow{\scriptstyle \alpha_{Y_1}} & & \downarrow{\scriptstyle \alpha_{Y_0}} & & \downarrow{\scriptstyle \alpha_Y} & & \\
\prod_i (X_i \otimes_{\mathcal{C}} Y_1) & \longrightarrow & \prod_i (X_i \otimes_{\mathcal{C}} Y_0) & \longrightarrow & \prod_i (X_i \otimes_{\mathcal{C}} Y) & \longrightarrow & 0
\end{array}$$

When $Y_t = \operatorname{Hom}_{\mathcal{C}}(C_t,-)$ for $C_0, C_1 \in \mathcal{C}$, then all vertical maps are bijective. Thus $- \otimes_{\mathcal{C}} Y$ preserves all products.

It is convenient to set

$$h_C = \operatorname{Hom}_{\mathcal{C}}(-,C) \qquad (C \in \mathcal{C})$$

and for any family of objects $(C_i)_{i\in I}$ in $\mathcal{C}$ we consider the canonical map

$$\beta_Y \colon \Big(\prod_i h_{C_i}\Big) \otimes_{\mathcal{C}} Y \longrightarrow \prod_i \big(h_{C_i} \otimes_{\mathcal{C}} Y\big).$$

Suppose that β_Y is surjective. We claim that Y is finitely generated. To this end consider the product of representable functors

$$\prod_{C \in \mathcal{C}} h_C^{Y(C)}$$

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so that the canonical map

$$\beta\colon \Big(\prod_{C\in\mathcal{C}} h_C^{Y(C)}\Big)\otimes_{\mathcal{C}} Y \longrightarrow \prod_{C\in\mathcal{C}} (h_C\otimes_{\mathcal{C}} Y)^{Y(C)} = \prod_{C\in\mathcal{C}} Y(C)^{Y(C)}$$

is surjective. For any finite subset

$$I\subseteq \coprod_{C\in\mathcal{C}} Y(C)$$

there is by Yoneda's lemma an induced morphism $\coprod_{i\in I} h_{C_i}\to Y$ and we denote by Y_I its image. Then $Y=\operatorname{colim} Y_I$ and therefore

$$\operatorname*{colim}_I \Big(\prod_{C\in\mathcal{C}} h_C^{Y(C)}\Big)\otimes_{\mathcal{C}} Y_I \xrightarrow{\sim} \Big(\prod_{C\in\mathcal{C}} h_C^{Y(C)}\Big)\otimes_{\mathcal{C}} Y.$$

It follows that for some finite set I_0 there is an element

$$x\in \Big(\prod_{C\in\mathcal{C}} h_C^{Y(C)}\Big)\otimes_{\mathcal{C}} Y_{I_0}$$

such that $\beta(x)=\mathrm{id}_Y$, and therefore $Y=Y_{I_0}$ is finitely generated.

Now choose an exact sequence $0\to Y_1\to Y_0\to Y\to 0$ in $\operatorname{Mod}(\mathcal{C}^{\mathrm{op}})$ and consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
 & & (\prod_i h_{C_i})\otimes_{\mathcal{C}} Y_1 & \longrightarrow & (\prod_i h_{C_i})\otimes_{\mathcal{C}} Y_0 & \longrightarrow & (\prod_i h_{C_i})\otimes_{\mathcal{C}} Y & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \beta_{Y_1}} & & \downarrow{\scriptstyle \beta_{Y_0}} & & \downarrow{\scriptstyle \beta_{Y}} & & \\
0 & \longrightarrow & \prod_i (h_{C_i}\otimes_{\mathcal{C}} Y_1) & \longrightarrow & \prod_i (h_{C_i}\otimes_{\mathcal{C}} Y_0) & \longrightarrow & \prod_i (h_{C_i}\otimes_{\mathcal{C}} Y) & \longrightarrow & 0
\end{array}$$

Suppose that β_Y is bijective. Then Y is finitely generated and we may choose $Y_0=\operatorname{Hom}_{\mathcal{C}}(C,-)$ for some $C\in\mathcal{C}$. Thus β_{Y_0} is bijective and it follows that β_{Y_1} is surjective. Then Y_1 is finitely generated, and we conclude that Y is finitely presented. □

Proof of Theorem 2.5.26 Suppose first that F is α-coherent. A representable functor $\operatorname{Hom}_{\mathcal{A}}(X,-)$ preserves products and α-filtered colimits provided that X is α-presentable. Clearly, this property is preserved when one passes to the cokernel of a morphism $\operatorname{Hom}_{\mathcal{A}}(Y,-)\to\operatorname{Hom}_{\mathcal{A}}(X,-)$ where X and Y are α-presentable.

Now suppose that F preserves products and α-filtered colimits. Let $\mathcal{C}$ denote the full subcategory of α-presentable objects in $\mathcal{A}$. We set $G=F|_{\mathcal{C}}$ and note that

$$F(X)\cong \operatorname{Hom}_{\mathcal{A}}(-,X)|_{\mathcal{C}}\otimes_{\mathcal{C}} G \qquad (X\in\mathcal{A}) \tag{2.5.28}$$

since F preserves α-filtered colimits and every object in $\mathcal{A}$ is an α-filtered colimit of objects in $\mathcal{C}$.

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The assumption on F to preserve products implies that any family of objects $(C_i)_{i\in I}$ in $\mathcal{C}$ induces an isomorphism

$$\Big(\prod_i \operatorname{Hom}_{\mathcal{C}}(-,X_i)\Big)\otimes_{\mathcal{C}} G \xrightarrow{\sim} \prod_i \big(\operatorname{Hom}_{\mathcal{C}}(-,X_i)\otimes_{\mathcal{C}} G\big).$$

We conclude from Proposition 2.5.27 that G has a presentation

$$\operatorname{Hom}_{\mathcal{C}}(Y,-)\longrightarrow \operatorname{Hom}_{\mathcal{C}}(X,-)\longrightarrow G\longrightarrow 0$$

with $X,Y\in\mathcal{C}$. Combining this presentation with the isomorphism (2.5.28) gives a presentation

$$\operatorname{Hom}_{\mathcal{A}}(Y,-)\longrightarrow \operatorname{Hom}_{\mathcal{A}}(Y,-)\longrightarrow F\longrightarrow 0$$

of F. Thus F is α-coherent. □

Notes

We follow Gabriel [79] and recall that abelian categories were introduced by Buchsbaum and Grothendieck in order to generalise the homological methods of Cartan and Eilenberg [46]. The localisation theory for abelian categories is developed in Gabriel's thesis [79], following Grothendieck's fundamental work [94]. In particular, [79] contains the description of Serre and localising subcategories for commutative noetherian rings. Also, the idea of presenting a Grothendieck category as a category of left exact functors is from [79]. Exact categories were introduced by Heller under the name 'abelian category' [107]; we follow expositions by Keller and Quillen [120, 165].

Projective and injective objects are important ingredients of homological algebra. We focus on injective objects because Grothendieck categories always have enough injectives, but not necessarily enough projectives. The study of injective modules goes back to the work of Baer [21]; the notion of an injective envelope was introduced by Eckmann and Schopf [70]. In [71] Eilenberg proposes an axiomatic description of minimal resolutions. Our treatment of projective covers and injective envelopes in terms of minimal decompositions of morphisms follows closely [131].

Finitely presented (or coherent) functors were studied in a famous article by Auslander [7]. Closely related is the correspondence between additive categories with weak kernels and abelian categories having enough projective objects, which is due to Freyd [75]. The notion of an effaceable functor goes

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back to Grothendieck [94]. The presentation of an abelian category as the quotient of the category of finitely presented functors modulo the subcategory of effaceable functors is also known as 'Auslander's formula' [139].

The notion of a recollement was introduced by Beilinson, Bernšteĭn and Deligne [26] in their study of perverse sheaves; it describes a diagram of six additive functors and makes sense equally for abelian as for triangulated categories. Universal localisations of (not necessarily commutative) rings were introduced by Cohn [54] and Schofield [182]; see also [33].

Grothendieck categories were introduced by Grothendieck in his Tôhoku paper [94] as an appropriate setting for homological algebra. While coproducts and filtered colimits are exact in Grothendieck categories, taking products need not be exact. The example of sheaves on the projective line over a field was suggested by Keller. The embedding theorem for Grothendieck categories is due to Popescu and Gabriel [159]. The Krull–Remak–Schmidt–Azumaya theorem is Azumaya's generalisation of the uniqueness result for decompositions of finite length modules into indecomposables [19]. Gabriel and Oberst introduced the spectral category of a Grothendieck category [82]; it provides a general context for the study of direct sum decompositions.

Locally presentable categories were introduced and studied by Gabriel and Ulmer [84]; for a modern account see [1]. The characterisation of coherent functors on locally presentable categories is taken from [128]; it generalises the characterisation of functors preserving products and filtered colimits for module categories by Crawley-Boevey [59]. The crucial ingredient of its proof is Lenzing's theorem which characterises finitely presented modules via their tensor functors [138].