ON THE KERNELS OF REPRESENTATIONS OF FINITE GROUPS

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Let G be a finite group and p a prime number. About five years ago I. M. Isaacs and S. D. Smith [5] gave several character-theoretic characterizations of finite p-solvable groups with p-length 1. Indeed, they proved that if P is a Sylow p-subgroup of G then the next four conditions (1)–(4) are equivalent:

(1) G is p-solvable of p-length 1.

(2) Every irreducible complex representation in the principal p-block of G restricts irreducibly to $N_G(P)$.

(3) Every irreducible complex representation of degree prime to p in the principal p-block of G restricts irreducibly to $N_G(P)$.

(4) Every irreducible modular representation in the principal p-block of G restricts irreducibly to $N_G(P)$.

The purpose of the present paper is to generalize the above result. It can be stated as follows: if B is an arbitrary p-block of G with defect group D, then the following four conditions (1)–(4) are equivalent:

(1) D is contained in the intersection of the kernels of all irreducible modular representations in B.

(2) Every irreducible complex representation in B restricts irreducibly to $N_G(D)$.

(3) Every irreducible complex representation in B whose character has height zero restricts irreducibly to $N_G(D)$.

(4) Every irreducible modular representation in B restricts irreducibly to $N_G(D)$.

Since $O_{p',p}(G)$ is the intersection of the kernels of all irreducible modular representations in the principal p-block of G, our result is a generalization of the result of Isaacs and Smith.

Throughout this paper we use the following notation. For an integer n we write $\nu_p(n)=r$ if $p^r \mid n$ and $p^{r+1} \nmid n$. We write $\mathrm{Irr}(G)$ (respectively $\mathrm{IBr}(G)$) for the set of all irreducible complex (respectively Brauer) characters of G. For a p-block B of G let us denote by $\mathrm{Irr}(B)$ (respectively $\mathrm{IBr}(B)$) the set of all elements of $\mathrm{Irr}(G)$ (respectively $\mathrm{IBr}(G)$) which belong to B, by $k(B)$ the number of elements of $\mathrm{Irr}(B)$, and by $k_0(B)$ the number of elements of $\mathrm{Irr}(B)$ with height zero. When $\chi \in \mathrm{Irr}(G)$ (respectively $\phi \in \mathrm{IBr}(G)$), let $\mathrm{Ker}\,\chi$ (respectively $\mathrm{Ker}\,\phi$) be the kernel of the irreducible complex (respectively modular) representation which corresponds to χ (respectively ϕ). Following [1, pp. 494–495] let $N_B = \bigcap \{\mathrm{Ker}\,\chi \mid \chi \in \mathrm{Irr}(B)\}$ and $N_B^* = \bigcap \{\mathrm{Ker}\,\phi \mid \phi \in \mathrm{IBr}(B)\}$ for a p-block B of G. We write $B_0(G)$ for the principal p-block of G. When H is a subgroup of G and b is a p-block of H, we use the notation b^G in the sense of [3, §57] for the case where b^G is defined. When H is a subgroup of G, for a character ψ of G and a character $\tilde{\psi}$ of H, $\psi|_H$ and $\tilde{\psi}^G$ denote the restriction of ψ to H and the induced character of $\tilde{\psi}$ to G, respectively.

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We write G' for the commutator subgroup of G. If S is a subset of G, $N_G(S)$ denotes the normalizer of S in G. We use the notation $O_{p'}(G)$, $O_p(G)$ and $O_{p',p}(G)$ following custom (cf. [**3**, p. 397]).

THEOREM. *Let B be an arbitrary p-block of G with defect group D, and let $N = N_G(D)$. Then the following are equivalent.*

(1) *$D \subseteq N_B^*$.*

(2) *When b is a p-block of N with $b^G = B$, for each $\psi \in \mathrm{Irr}(b)$ there is some $\chi \in \mathrm{Irr}(B)$ such that $\chi|_N = \psi$.*

(3) *When b is a p-block of N with $b^G = B$, for each $\psi \in \mathrm{Irr}(b)$ with height zero there is some $\chi \in \mathrm{Irr}(B)$ such that $\chi|_N = \psi$.*

(4) *When b is a p-block of N with $b^G = B$, for each $\psi \in \mathrm{Irr}(b)$ with height zero there is some $\chi \in \mathrm{Irr}(B)$ such that χ has height zero and $\chi|_N = \psi$.*

(5) *If $\chi \in \mathrm{Irr}(B)$ then $\chi|_N \in \mathrm{Irr}(N)$.*

(6) *If $\chi \in \mathrm{Irr}(B)$ and χ has height zero then $\chi|_N \in \mathrm{Irr}(N)$.*

(7) *If $\phi \in \mathrm{IBr}(B)$ then $\phi|_N \in \mathrm{IBr}(N)$.*

Proof. (7)$\Rightarrow$(1). Since D is normal in N, by [**3**, Theorem 53.9(*ii*)], $D \subseteq \mathrm{Ker}\,\tilde{\phi}$ for all $\tilde{\phi} \in \mathrm{IBr}(N)$. Thus, (7) implies (1).

(1)$\Rightarrow$(5), (7). Let $H = N_B^*$ and $V = N_B$. By [**1**, Propositions (3A) and (3D)], H is p-nilpotent and $V = O_{p'}(H)$. Let $\phi \in \mathrm{IBr}(B)$. Since $H \subseteq \mathrm{Ker}\phi$, we can consider $\phi \in \mathrm{IBr}(G/H)$, so that $\phi \in \mathrm{IBr}(\bar{B})$ for some p-block $\bar{B}$ of G/H. By [**3**, Lemma 64.3(1)], $\bar{B} \subseteq B$. Hence, by [**3**, Lemma 64.3(2)], D contains a Sylow p-subgroup of H. Thus, (1) implies that D is a Sylow p-subgroup of H, so that $H = VD$. Hence, by the Frattini argument [**4**, I 7.8 Satz], $G = HN = VN$. This proves (5) since $V \subseteq \mathrm{Ker}\,\chi$ for all $\chi \in \mathrm{Irr}(B)$. Similarly, we get (7) since $H \subseteq \mathrm{Ker}\,\phi$ for all $\phi \in \mathrm{IBr}(B)$.

(5)$\Rightarrow$(6). Trivial.

(6)$\Rightarrow$(4). Let b be a p-block of N with $b^G = B$. By [**3**, Corollary 54.11 and Lemma 57.4], D is a defect group of b. Let $\psi \in \mathrm{Irr}(b)$ such that ψ has height zero. We can write $\psi^G = \sum_i u_i\chi_i$ where $\chi_i \in \mathrm{Irr}(G)$ and u_i is a non-negative integer for each i. Let $|D| = p^d$. If $\chi_i \in \mathrm{Irr}(B)$ and $u_i \neq 0$, then we have

$$\nu_p(u_i\chi_i(1)) \geq \nu_p(\chi_i(1)) \geq \nu_p(|G|) - d$$
$$= \nu_p(|G:N|) + \nu_p(|N|) - d = \nu_p(|G:N|) + \nu_p(\psi(1)) = \nu_p(\psi^G(1))$$

since ψ has height zero. Hence it follows from [**2**, (3A)] that there is some $\chi_i \in \mathrm{Irr}(B)$ such that $u_i \neq 0$ and $\nu_p(u_i\chi_i(1)) = \nu_p(\psi^G(1))$. This implies that $\chi_i \in \mathrm{Irr}(B)$ has height zero by the above inequality. By Frobenius reciprocity, ψ is a component of $\chi_i|_N$. Thus, (6) implies $\chi_i|_N = \psi$.

(4)$\Rightarrow$(3). Clear.

(5)$\Rightarrow$(2). Similar to the proof of (6)$\Rightarrow$(4).

(2)$\Rightarrow$(3). Clear.

(3)$\Rightarrow$(1). By Brauer's first main theorem [**3**, Theorem 58.3], there is a p-block b of N with defect group D such that $b^G = B$. Let $R = N_b$, so that $R \subseteq O_{p'}(N)$ from [**1**, Proposition (3A)]. Let $\phi \in \mathrm{IBr}(b)$. By [**1**, Proposition (3D)] and [**3**, Theorem 53.9(*ii*)],

$RD \subseteq \operatorname{Ker}\phi$. Hence we can consider $\phi \in \operatorname{IBr}(b^*)$ for some p-block b^* of $N/(RD)$. By [**3,** Lemma 64.3(1)], $b^* \subseteq b$. Let $\psi \in \operatorname{Irr}(b^*)$. We can consider $\psi \in \operatorname{Irr}(b)$ with $RD \subseteq \operatorname{Ker}\psi$. Clearly, we may consider $\psi \in \operatorname{Irr}(N/(RD'))$. Since $(RD)/(RD')$ is an abelian normal subgroup of $N/(RD')$, we get from [**4,** V 17.10 Satz] that $\psi(1) \mid |N/(RD)|$. This shows that $\psi \in \operatorname{Irr}(b)$ has height zero since D is a defect group of b. Hence, by (3) there is some $\chi \in \operatorname{Irr}(B)$ with $\chi|_N = \psi$. This shows $RD \subseteq \operatorname{Ker}\chi$. Hence $\{\chi \in \operatorname{Irr}(B) \mid RD \subseteq \operatorname{Ker}\chi\} \neq \emptyset$. Let $L = \bigcap\{\operatorname{Ker}\chi \mid \chi \in \operatorname{Irr}(B),\ RD' \subseteq \operatorname{Ker}\chi\}$, so that $RD' \subseteq L \cap N$. Let $\psi_i \in \operatorname{Irr}(b)$ with $RD' \subseteq \operatorname{Ker}\psi_i$. As above, ψ_i has height zero. Thus, by (3) there is some $\chi_i \in \operatorname{Irr}(B)$ with $\chi_i|_N = \psi_i$, so that $RD' \subseteq \operatorname{Ker}\chi_i$. This implies $L \subseteq \bigcap_i \operatorname{Ker}\chi_i$, so that

$$L \cap N \subseteq \bigcap\{\operatorname{Ker}\psi \mid \psi \in \operatorname{Irr}(b),\ RD' \subseteq \operatorname{Ker}\psi\}.$$

Next, we want to claim

$$\bigcap\{\operatorname{Ker}\psi \mid \psi \in \operatorname{Irr}(b),\ RD' \subseteq \operatorname{Ker}\psi\} = RD'. \tag{*}$$

Let $I = \{\psi \in \operatorname{Irr}(b) \mid RD' \subseteq \operatorname{Ker}\psi\} = \{\psi_1, \ldots, \psi_m\}$. We have already shown $I \neq \emptyset$. Take any ψ_i with $1 \leq i \leq m$. Since RD$' \subseteq \operatorname{Ker}\psi_i$, we can consider $\psi_i \in \operatorname{Irr}(\bar{b}_i)$ for some p-block $\bar{b}_i$ of $N/(RD')$. By [**3,** Lemma 64.3(1)], $\bar{b}_i \subseteq b$ for all $i = 1, \ldots, m$. Then, $I = \bigcup_{i=1}^m \operatorname{Irr}(\bar{b}_i)$. Take any $\bar{b}_i$ with $1 \leq i \leq m$. By [**3,** Lemma 64.3(1)], there is a p-block $\tilde{b}_i$ of N/R with $\bar{b}_i \subseteq \tilde{b}_i \subseteq b$. Let $\tilde{F} = \bigcup_{i=1}^m \operatorname{IBr}(\tilde{b}_i)$ and $\bar{F} = \bigcup_{i=1}^m \operatorname{IBr}(\bar{b}_i)$. Take any $\tilde{b}_i$. Let $\phi \in \operatorname{IBr}(\tilde{b}_i)$. Since $(RD')/R$ is a normal p-subgroup of N/R, $(RD')/R \subseteq \operatorname{Ker}\phi$ from [**3,** Theorem 53.9(*ii*)]. Thus, $RD' \subseteq \operatorname{Ker}\phi$ if we consider $\phi \in \operatorname{IBr}(b)$. Then, $\phi \in \operatorname{IBr}(\bar{b})$ for some p-block $\bar{b}$ of $N/(RD')$. By [**3,** Lemma 64.3(1)], $\bar{b} \subseteq b$. Take any $\psi \in \operatorname{Irr}(\bar{b})$, so that $\psi \in \operatorname{Irr}(b)$ with $RD' \subseteq \operatorname{Ker}\psi$. This shows $\psi \in I$. Hence $\bar{b} = \bar{b}_j$ for some j. So that $\phi \in \operatorname{IBr}(\bar{b}_j)$. Thus, we have $\tilde{F} = \bar{F}$. Then,

$$\begin{aligned}\bigcap_{i=1}^m N^*_{\tilde{b}_i} &= \bigcap_{\phi\in\tilde{F}} \operatorname{Ker}\phi \cong \bigcap_{\phi\in\bar{F}} (\operatorname{Ker}\phi/((RD')/R))\\ &= \Big(\bigcap_{\phi\in\tilde{F}} \operatorname{Ker}\phi\Big)\Big/((RD')/R) = \Big(\bigcap_{i=1}^m N^*_{\tilde{b}_i}\Big)\Big/((RD'/R).\end{aligned} \tag{**}$$

Take any $\tilde{b}_i$. Let $\tilde{\psi} \in \operatorname{Irr}(\tilde{b}_i)$. By [**1,** Proposition (3B*)], $N_{\tilde{b}_i} = O_{p'}(\operatorname{Ker}\tilde{\psi})$. When we consider $\tilde{\psi} \in \operatorname{Irr}(b)$, we write ψ for $\tilde{\psi}$. Then, similarly $R = N_b = O_{p'}(\operatorname{Ker}\psi)$. Since $\operatorname{Ker}\tilde{\psi} = (\operatorname{Ker}\psi)/R$, $N_{\tilde{b}_i} = 1$. Hence, by [**1,** Proposition (3D)], $N^*_{\tilde{b}_i}$ is a p-group for all i. Thus, $\bigcap_{i=1}^m N^*_{\bar{b}_i}$ is also a p-group from (**). Hence, by [**1,** Propositions (3A) and (3D)], $\bigcap_{i=1}^m N_{\bar{b}_i} = 1$. Since $\bigcap_{i=1}^m N_{\bar{b}_i} = (\bigcap_{\psi\in I} \operatorname{Ker}\psi)/(RD')$, we get (*).

Hence, $L \cap N = RD'$. Let $K = \bigcap\{\operatorname{Ker}\chi \mid \chi \in \operatorname{Irr}(B),\ RD \subseteq \operatorname{Ker}\chi\}$. We get $K \cap N = RD$ as for $L \cap N$. Since $D' \subseteq L \cap D$, $(LD)/L$ is isomorphic to a factor group of D/D', so that $(LD)/L$ is abelian. We have shown that there is some $\chi \in \operatorname{Irr}(B)$ with $RD \subseteq \operatorname{Ker}\chi$. Hence $K \subseteq \operatorname{Ker}\chi$. We can consider $\chi \in \operatorname{Irr}(\bar{B})$ for some p-block $\bar{B}$ of G/K, so that $\bar{B} \subseteq B$ by [**3,** Lemma 64.3(1)]. Then, by [**3,** Lemma 64.3(2)], D is a Sylow p-subgroup of K since $D \subseteq K$. Hence $(LD)/L$ is an abelian Sylow p-subgroup of K/L. Let $M/L = N_{K/L}((LD)/L)$. By Sylow's theorem, $M = L(M \cap N)$. Hence $M \subseteq L(K \cap N) = L(RD) = LD$, so that $(LD)/L = M/L$. Thus, by Burnside's theorem [**3,** Theorem 18.7], K/L is p-nilpotent. Let $C/L = O_{p'}(K/L)$, so that $C \cap LD = L$. Hence, $C \cap D \subseteq L \cap D = L \cap N \cap D = RD' \cap D = D'$. Since D is a p-group, $D' \subseteq \Phi(D)$ where $\Phi(D)$ is the Frattini subgroup of D. Then it

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follows from [**4,** IV 4.7 Satz] that C is p-nilpotent. Hence, $C = X(C \cap D)$ where $X = O_{p'}(C)$. Since X is normal in K and since $K = XD$, K is also p-nilpotent. We know from [**1,** Proposition (3A)] that $N_B \subseteq O_{p'}(K)$. On the other hand, there exists $\chi \in \mathrm{Irr}(B)$ with $RD \subseteq \mathrm{Ker}\ \chi$. By [**1,** Proposition (3B*)], $N_B = O_{p'}(\mathrm{Ker}\ \chi)$. Since $O_{p'}(K)$ is normal in $\mathrm{Ker}\ \chi$, $O_{p'}(K) \subseteq O_{p'}(\mathrm{Ker}\ \chi)$. Hence $N_B = O_{p'}(K)$. Then, by [**1,** Proposition (3D)], we have $K/N_B \subseteq O_p(G/N_B) = N_B^*/N_B$. Hence $D \subseteq N_B^*$. This completes the proof of the theorem.

COROLLARY 1. *Let G, B, D and N be as above and satisfy $D \subseteq N_B^*$. Let b be a p-block of N with $b^G = B$. Then*

(i) $p \nmid |G:N|$.

(ii) $k(b) \leq k(B)$.

(iii) $k_0(b) \leq k_0(B)$.

Proof. Since there exists $\psi \in \mathrm{Irr}(b)$ with height zero, we get (i) by Theorem (4). We have (ii) and (iii) from Theorem (2) and Theorem (4), respectively.

COROLLARY 2 (cf. [**5,** Theorems 2 and 4]). *Let P be a Sylow p-subgroup of G, and let $N_0 = N_G(P)$, $B_0 = B_0(G)$ and $b_0 = B_0(N_0)$. Then the following are equivalent.*

(1) *G is p-solvable of p-length* 1.

(2) *For each $\psi \in \mathrm{Irr}(b_0)$ there is some $\chi \in \mathrm{Irr}(B_0)$ such that $\chi|_{N_0} = \psi$.*

(3) *For each $\psi \in \mathrm{Irr}(b_0)$ with $p \nmid \psi(1)$ there is some $\chi \in \mathrm{Irr}(B_0)$ such that $\chi|_{N_0} = \psi$.*

(4) *If $\chi \in \mathrm{Irr}(B_0)$ then $\chi|_{N_0} \in \mathrm{Irr}(N_0)$.*

(5) *If $\chi \in \mathrm{Irr}(B_0)$ with $p \nmid \chi(1)$ then $\chi|_{N_0} \in \mathrm{Irr}(N_0)$.*

(6) *If $\phi \in \mathrm{IBr}(B_0)$ then $\phi|_{N_0} \in \mathrm{IBr}(N_0)$.*

Proof. Clearly, G is p-solvable of p-length 1 if and only if $P \subseteq O_{p',p}(G)$. By [**3,** Theorem 65.2(2)], $N_{B_0}^* = O_{p',p}(G)$. Since $p \nmid |G:N_0|$, by Brauer's third main theorem [**3,** Theorem 65.4], the corollary is a special case of the theorem.

REMARK. Concerning kernels of representations of finite groups there is a result of G. O. Michler [**6**].

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