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The Choquet–Deny Equation in a Banach Space

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Abstract. Let G be a locally compact group and π a representation of G by weakly* continuous isometries acting in a dual Banach space E. Given a probability measure μ on G, we study the Choquet–Deny equation $\pi(\mu)x = x$, $x \in E$. We prove that the solutions of this equation form the range of a projection of norm 1 and can be represented by means of a "Poisson formula" on the same boundary space that is used to represent the bounded harmonic functions of the random walk of law μ. The relation between the space of solutions of the Choquet–Deny equation in E and the space of bounded harmonic functions can be understood in terms of a construction resembling the W^*-crossed product and coinciding precisely with the crossed product in the special case of the Choquet–Deny equation in the space $E = B(L^2(G))$ of bounded linear operators on $L^2(G)$. Other general properties of the Choquet–Deny equation in a Banach space are also discussed.

1 Introduction

The classical Choquet–Deny theorem asserts that when μ is a regular probability measure on a locally compact abelian group G, then every bounded continuous function $h\colon G \to \mathbb{C}$ which satisfies

$$h(g) = \int_G h(gg')\mu(dg') \tag{1.1}$$

for every $g \in G$, is necessarily constant on the cosets of the smallest closed subgroup G_μ containing the support of μ. The theorem can readily be seen to be equivalent to the statement that whenever π is a representation of G by weakly continuous isomorphisms of a locally convex space E where for every $x \in E$ and $x^* \in E^*$ the function $G \ni g \to \langle \pi(g)x, x^* \rangle \in \mathbb{C}$ is bounded and continuous, then every vector $x \in E$ which satisfies

$$x = \int_G \pi(g)x\mu(dg) \tag{1.2}$$

is necessarily fixed by every $g \in G_\mu$, *i.e.*, $\pi(g)x = x$ for every $g \in G_\mu$.

The Choquet–Deny theorem is known to remain true for some nonabelian groups, too; however, it is not true for all groups. When the theorem fails, then equations (1.1) and (1.2) will have other solutions in addition to those described above. The goal of this article is to understand the connection between the solutions of the classical *Choquet-Deny equation* (1.1) and its functional analytic counterpart (1.2), and

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to uncover some general properties of the spaces of such solutions. We will be concerned mainly with representations that are adjoints of strongly continuous representations by isometries in separable Banach spaces and our group G will, for the most part, be locally compact and second countable.

In general, a bounded Borel function $h\colon G \to \mathbb{C}$ satisfying (1.1) is called a *μ-harmonic* function. Such functions have their origin in probability theory. They are a special case of the harmonic functions of a Markov chain and, as such, have been extensively studied. When G is locally compact, one usually considers the bounded μ-harmonic functions as elements of $L^\infty(G)$. They are then solutions of equation (1.2) where π is now the right regular representation in $L^\infty(G)$. In fact, the right-hand sides of equations (1.1) and (1.2) define a positive weakly* continuous contraction $\pi(\mu)\colon L^\infty(G) \to L^\infty(G)$ which commutes with the left regular representation π_l of G in $L^\infty(G)$ and can be viewed as the average $\int_G \pi(g)\mu(dg)$ of the right regular representation. It is then immediate that the space $\mathcal{H}_\mu \subseteq L^\infty(G)$ of bounded μ-harmonic functions is a weakly* closed self-adjoint subspace of $L^\infty(G)$ containing constants and invariant under π_l. It is a much deeper result, involving the theory of martingales, that bounded μ-harmonic functions can be represented, by means of a "Poisson formula", as bounded Borel functions on a certain "boundary space". As a result, $\mathcal{H}_\mu$ turns out to be isomorphic to an L^∞-space and so has canonically the structure of an abelian W^*-algebra. More precisely, for each $h_1, h_2 \in \mathcal{H}_\mu$ the sequence $\pi(\mu^n)(h_1h_2)$ converges almost everywhere (hence, also weakly*), and the formula

$$h_1 \diamond h_2 = \mathrm{w}^*\text{-}\lim_{n\to\infty} \pi(\mu^n)(h_1h_2) \tag{1.3}$$

defines a product $\diamond$ on $\mathcal{H}_\mu$ under which $\mathcal{H}_\mu$ is an abelian W^*-algebra. The product $\diamond$ coincides with the ordinary product in $L^\infty(G)$ if and only if the Choquet–Deny theorem holds for the particular measure μ. It is also remarkable that $\mathcal{H}_\mu$ is the range of a projection $K\colon L^\infty(G) \to \mathcal{H}_\mu$ of norm 1, which commutes with the left regular representation.

We will show that the solutions of the Choquet–Deny equation (1.2) in a dual Banach space also form the range of a projection of norm 1 and can be represented by means of a Poisson formula on the same boundary space that is used to represent the classical bounded μ-harmonic functions. It will follow that the relation between the space of such solutions and the space $\mathcal{H}_\mu$ of the classical μ-harmonic functions can be understood in terms of a construction closely resembling the well-known construction of the W^*-crossed product, and coinciding precisely with the latter in the special case of the Choquet–Deny equation in the space $B(L^2(G))$ of bounded linear operators on $L^2(G)$. We will also see that the space of solutions of the Choquet–Deny equation in a W^*-algebra has itself canonically the structure of a W^*-algebra.

2 Some Special Cases

Let E be a Banach space which is the dual of a Banach space E_*. The duality between E_* and E will be written $\langle x_*, x\rangle$, $x_* \in E_*$, $x \in E$. Let π_* be a strongly continuous representation of a locally compact group G by isometries in E_*, and let π be the adjoint of π_*, *i.e.*, the representation in E by weakly* continuous isometries given by $\pi(g) = [\pi_*(g^{-1})]^*$. As is well known, the representations π_* and π can be extended to representations of the measure algebra $M(G)$ in E_* and E, resp. Given $\sigma \in M(G)$, $\pi_*(\sigma)$ and $\pi(\sigma)$ are the bounded linear operators which satisfy

$$\langle \pi_*(\sigma)x_*\,, x\rangle = \int_G \langle \pi_*(g)x_*\,, x\rangle\sigma(dg), \tag{2.1}$$

$$\langle x_*\,, \pi(\sigma)x\rangle = \int_G \langle x_*\,, \pi(g)x\rangle\sigma(dg),$$

for all $x_* \in E_*$ and $x \in E$. Clearly, $\|\pi_*(\sigma)\|, \|\pi(\sigma)\| \le \|\sigma\|$ and $\pi(\sigma) = [\pi_*(\widetilde{\sigma})]^*$, where $\widetilde{\sigma}$ denotes the measure $\widetilde{\sigma}(A) = \sigma(A^{-1})$. The Choquet–Deny equations in E_* and E become

$$\pi_*(\mu)x_* = x_* \quad \text{and} \quad \pi(\mu)x = x, \tag{2.2}$$

where $\mu \in M_1(G)$, the subset of probability measures in $M(G)$. Solutions of equations (2.2) will be referred to as μ-*harmonic vectors*, and we will denote by $\mathcal{H}_{\mu\pi_*} \subseteq E_*$ and $\mathcal{H}_{\mu\pi} \subseteq E$ the spaces of such vectors. We note that $\mathcal{H}_{\mu\pi_*}$ and $\mathcal{H}_{\mu\pi}$ always include all those vectors that are fixed by every element of the subgroup G_μ. These are the *trivial* solutions of the Choquet–Deny equation.

In the classical case we have $E = L^\infty(G)$, $E_* = L^1(G)$, and π and π_* are the usual right regular representations in $L^\infty(G)$ and $L^1(G)$. While the Choquet–Deny theorem is not true in general, it belongs to the folklore of the subject that the Choquet–Deny equation in $L^1(G)$, and, more generally, in $L^p(G)$, $1 \le p < \infty$, as well as the Choquet–Deny equations in $C_0(G)$ and $M(G)$ admit only trivial solutions, no matter what G and μ are. This fact, as well as a more general conclusion in the same direction, can be easily deduced from the following fundamental result due to Mukherjea [31] and Derriennic [9].

Lemma 2.1 *If G_μ is not compact, then the convolution powers μ^n converge to zero in the weak* topology of $M(G)$.*

Proposition 2.2 *Let π be a representation of G by weakly continuous isomorphisms of a locally convex space E and suppose that π vanishes at infinity,* i.e., *for every $x \in E$ and $x^* \in E^*$ the function $G \ni g \to \langle \pi(g)x\,, x^*\rangle$ belongs to $C_0(G)$. Let v be a solution of the Choquet–Deny equation in E,* i.e.,

$$\langle v\,, x^*\rangle = \int_G \langle \pi(g)v\,, x^*\rangle\mu\,(dg) \tag{2.3}$$

for every $x^* \in E^*$. Then $\pi(g)v = v$ for every $g \in G_\mu$. If G_μ is not compact then $v = 0$.

Proof If G_μ is compact, consider the restriction of π to G_μ and consider μ as a measure on G_μ. Since the Choquet–Deny theorem holds for compact groups, the result follows without difficulty because the function $g \to \langle \pi(g)v\,, x^*\rangle$ is μ-harmonic. If G_μ is not compact, observe that equation (2.3) holds with μ replaced by μ^n; then use Lemma 2.1. ∎

Corollary 2.3 *Let π_* and π denote the right regular representations of G in $C_0(G)$ and $M(G)$, resp., and let $\mu \in M_1(G)$. Then $h \in \mathcal{H}_{\mu\pi_*}$ if and only if h is constant on the left cosets of G_μ. Also, $\sigma \in \mathcal{H}_{\mu\pi}$ if and only if $\sigma * \delta_g = \sigma$ for every $g \in G_\mu$. If G_μ is not compact, then $\mathcal{H}_{\mu\pi_*} = \{0\}$ and $\mathcal{H}_{\mu\pi} = \{0\}$.*

Corollary 2.4 *Let π_* be the right regular representation of G in $L^p(G)$, where $1 \leq p < \infty$. Then $f \in \mathcal{H}_{\mu\pi_*}$ if and only if for every $g \in G_\mu$, $f(xg) = f(x)$ for almost every $x \in G$. If G_μ is not compact, then $\mathcal{H}_{\mu\pi_*} = \{0\}$.*

Proof When $p > 1$, Proposition 2.2 applies. When $p = 1$, embed $L^1(G)$ in $M(G)$ and use Corollary 2.3. ■

Another condition that ensures that the Choquet–Deny equation has only trivial solutions is that of strict convexity. The proof of the following proposition is a routine exercise.

Proposition 2.5 *If the Banach space E (resp., E_*) is strictly convex then the Choquet–Deny equation in E (resp., E_*) admits only trivial solutions.*

Thus, in particular, the Choquet–Deny equation in a Hilbert space or in an L^p-space with $1 < p < \infty$ is not interesting.

We will now introduce two examples which provided the original stimulus for our investigations.

Example 1 Let $\mathcal{X}$ be a locally compact G-space where G is a locally compact group and the mapping $G \times \mathcal{X} \ni (g,x) \to gx \in \mathcal{X}$ is continuous. The action of G on $\mathcal{X}$ gives rise to a strongly continuous representation π_* of G in $C_0(\mathcal{X})$, defined by $(\pi_*(g)f)(x) = f(g^{-1}x)$. The adjoint of π_* is the representation π in $M(\mathcal{X}) = C_0(\mathcal{X})^*$ given by $(\pi(g)\sigma)(A) = \sigma(g^{-1}A)$.

Suppose $\mu \in M_1(G)$ and $\sigma \in M_1(\mathcal{X})$. When both $\mathcal{X}$ and G are second countable, $\pi(\mu)\sigma$ has the following simple probabilistic interpretation. Let X and Y be independent random variables taking values in $\mathcal{X}$ and G, resp. If σ is the distribution of X and μ is the distribution of Y, then $\pi(\mu)\sigma$ is the distribution of the $\mathcal{X}$-valued random variable YX. The solutions of the Choquet–Deny equation $\pi(\mu)\sigma = \sigma$ in $M_1(\mathcal{X})$ are called *μ-stationary* measures. From the probabilistic point of view they are, precisely, the possible limits in $M_1(\mathcal{X})$, in the weak* topology, of probability distributions of sequences of $\mathcal{X}$-valued random variables of the form $Y_nY_{n-1}\cdots Y_1X$, where $\{Y_n\}_{n=1}^{\infty}$ is a sequence of independent identically distributed G-valued random variables whose common distribution is μ and X is an $\mathcal{X}$-valued random variable, independent of $Y_1, Y_2, \ldots$. The sequence $Y_nY_{n-1}\cdots Y_1X$ forms a Markov chain in $\mathcal{X}$ [38, Proposition 4.4, p. 30], [10, §II]. The solutions of the Choquet–Deny equation $\pi_*(\widetilde{\mu})f = f$ in $C_0(\mathcal{X})$ are harmonic functions of this Markov chain. When the Choquet–Deny theorem holds, the μ-stationary measures are just the G_μ-invariant probability measures on $\mathcal{X}$, and the elements of $\mathcal{H}_{\widetilde{\mu}\pi_*}$ are functions $f \in C_0(\mathcal{X})$ with $f(gx) = f(x)$ for every $g \in G_\mu$ and $x \in \mathcal{X}$. ♢

Example 2 Let ρ be a continuous unitary representation of G in a Hilbert space $\mathfrak{H}$. Then the formula $\pi(g)A = \rho(g)A\rho(g^{-1})$ defines a representation π of G in $B(\mathfrak{H})$, the algebra of bounded linear operators on $\mathfrak{H}$. The subrepresentation π_* of π in the ideal $\mathcal{T}(\mathfrak{H})$ of trace class operators is strongly continuous (with respect to the trace norm), and π is the adjoint of π_*. When the Choquet–Deny theorem holds, then $\mathcal{H}_{\mu\pi} = \rho(G_\mu)'$, the commutant of $\rho(G_\mu)$ in $B(\mathfrak{H})$, and $\mathcal{H}_{\mu\pi_*} = \rho(G_\mu)' \cap \mathcal{T}(\mathfrak{H})$.

When $\mathfrak{H} = L^2(G)$ and ρ is the right regular representation, the Choquet–Deny equation in $B(L^2(G))$ can be viewed as a "non-commutative" or "quantized" version of the classical Choquet–Deny equation. We will refer to π as the right regular representation of G in $B(L^2(G))$. Of course, π commutes with a similarly defined left regular representation π_l and therefore $\mathcal{H}_{\mu\pi}$ is always invariant under π_l. Moreover, since $B(L^2(G))$ contains a copy of $L^\infty(G)$, $\mathcal{H}_{\mu\pi}$ contains a copy of $\mathcal{H}_\mu$. When $G_\mu = G$, the subspace of trivial solutions of the Choquet–Deny equation in $B(L^2(G))$ is just $VN(G)$, the von Neumann algebra generated by ρ_l, the left regular representation of G in $L^2(G)$; $\mathcal{H}_{\mu\pi} = VN(G)$ if and only if the conclusion of the Choquet–Deny theorem is true for μ. The equality $\mathcal{H}_{\mu\pi} = VN(G)$ is possible only when G is an amenable group, *cf.* [27, 39].

We note that, analogously to the classical case, the Choquet–Deny equation in $\mathcal{T}(L^2(G))$ has always only trivial solutions, and when G_μ is not compact, $\mathcal{H}_{\mu\pi_*} = \{0\}$. This follows from Proposition 2.2 when $\mathcal{T}(L^2(G))$ is considered as a locally convex space under the $\sigma\big(\mathcal{T}(L^2(G)), \mathcal{K}(L^2(G))\big)$-topology, where $\mathcal{K}(L^2(G))$ denotes the ideal of compact operators. $\Diamond$

In connection with Example 2 we would like to mention that a very different route to establish a "non-commutative" Choquet–Deny equation has been recently taken by Chu and Lau [5]. There, the duality between $L^\infty(G)$ and $L^1(G)$ is replaced by that between the group von Neumann algebra $VN(G)$ and the Fourier algebra $A(G)$. Moreover, the measure algebra $M(G)$ is replaced by the Fourier–Stieltjes algebra $B(G)$, and probability measures correspond to the elements of the set $P^1(G)$ consisting of bounded continuous positive definite functions σ on G with $\sigma(e) = 1$. The role of the right regular representation of $M(G)$ in $L^\infty(G)$ is played by the canonical action of $B(G)$ on $VN(G)$. For a fixed $\sigma \in B(G)$, the authors study the space $H_\sigma = \{T \in VN(G) \,;\, \sigma T = T\}$ of *σ-harmonic functionals* on $A(G)$. Their investigations show analogies with, but mainly reveal crucial differences from, the classical situation. For instance, for $\sigma \in P^1(G)$, H_σ is always a subalgebra of $VN(G)$ — which is in contrast to the fact that for a probability measure μ, the space of bounded μ-harmonic functions is a subalgebra of $L^\infty(G)$ only if it is trivial. This completely different behaviour is not surprising, since the classical theory of harmonic functions can be recovered from the setting of [5] only for abelian groups.

3 The Preannihilator of $\mathcal{H}_{\mu\pi}$

Given a probability measure μ on a locally compact group G, let J_μ denote the set

$$J_\mu = \overline{\{\varphi - \varphi * \mu \,;\, \varphi \in L^1(G)\}}, \tag{3.1}$$

where the bar means closure with respect to the L^1-norm. Evidently, J_μ is a left ideal in the group algebra $L^1(G)$, whose annihilator in $L^\infty(G)$ is precisely the space $\mathcal{H}_\mu$ of the bounded μ-harmonic functions. As pointed out by Willis [43, 44], ideals of this form appear naturally, not only in the theory of μ-harmonic functions, but also in the study of amenability and certain factorization questions in group algebras. The quotient $L^1(G)/J_\mu$ turns out to be an abstract L^1-space whose pointwise realization is the boundary needed to represent the μ-harmonic functions by means of a Poisson formula. The space $\mathcal{J}$ of all ideals of the form J_μ, where μ ranges over $M_1(G)$, has an interesting order structure when ordered by inclusion [18, 43]. We note that

$$J_\mu \subseteq L^1_0(G, G_\mu) \subseteq L^1_0(G),$$

where $L^1_0(G, G_\mu)$ denotes the kernel of the canonical mapping from $L^1(G)$ to $L^1(G/G_\mu)$, and $L^1_0(G) = L^1_0(G, G)$ is the augmentation ideal

$$L^1_0(G) = \{f \in L^1(G)\,;\, \textstyle\int f = 0\}.$$

Moreover, $L^1_0(G, G_\mu)$ coincides with the preannihilator of the subspace of trivial solutions of the Choquet–Deny equation and the equality $J_\mu = L^1_0(G, G_\mu)$ holds if and only if the conclusion of the Choquet–Deny theorem is true for μ.

Let π_* be a strongly continuous representation of G by isometries in a Banach space E_* and π the adjoint of π_* acting in the dual E of E_*. The analog of J_μ is the closed subspace $J_{\mu\pi}$ of E_* given by

$$J_{\mu\pi} = \overline{\{x_* - \pi_*(\widetilde{\mu})x_*\,;\, x_* \in E_*\}}, \tag{3.2}$$

where the bar now denotes the norm closure in E_*. It is evident that, as in the classical case, $\mathcal{H}_{\mu\pi} = J_{\mu\pi}^\perp$.

Remark 3.1 The subspace of trivial solutions of the Choquet–Deny equation in E is the annihilator of the closed subspace $E_{*0}(G_\mu)$ of E_* spanned by the set $\{x_* - \pi_*(g)x_*\,;\, g \in G_\mu, x_* \in E_*\}$. Thus $E_{*0}(G_\mu)$ plays the role of $L^1_0(G, G_\mu)$; $J_{\mu\pi} \subseteq E_{*0}(G_\mu)$, with equality if and only if the Choquet–Deny theorem in E has only trivial solutions.

Throughout the sequel, it will be convenient to identify $L^1(G)$ with the subspace of absolutely continuous complex measures in $M(G)$. With this convention in force, we obtain the following simple relation between J_μ and $J_{\mu\pi}$. Recall that for every $\sigma \in M(G)$, $\widetilde{\sigma}$ denotes the measure $\widetilde{\sigma}(A) = \sigma(A^{-1})$.

Theorem 3.2 *$J_{\mu\pi} = \pi_*(\widetilde{J_\mu})E_* = \{\pi_*(\widetilde{\varphi})x_*\,;\, \varphi \in J_\mu,\ x_* \in E_*\}$ and $J_{\mu\pi} = \pi_*(\widetilde{J_\mu})J_{\mu\pi}$.*

Proof Given $\varphi \in J_\mu$, $x_* \in E_*$, and $x \in \mathcal{H}_{\mu\pi}$, we obtain

$$\langle \pi_*(\widetilde{\varphi})x_*\,, x\rangle = \langle x_*\,, \pi(\varphi)x\rangle = \int_G \langle x_*\,, \pi(g)x\rangle \varphi(dg) = 0,$$

because the function $g \to \langle x_*, \pi(g)x\rangle$ is μ-harmonic. Hence, $\pi_*(\widetilde{J}_\mu)E_* \subseteq J_{\mu\pi}$. In particular, this means that $\pi_*(\widetilde{J}_\mu)J_{\mu\pi} \subseteq J_{\mu\pi}$, and so $J_{\mu\pi}$ is a left Banach $\widetilde{J}_\mu$-module. But as pointed out in [43, p. 203], if $\{\varepsilon_\alpha\}_{\alpha\in A}$ is a bounded approximate identity in $L^1(G)$ then

$$\eta_{\alpha n} = \varepsilon_\alpha * (\delta_e - \tfrac{1}{n}\sum_{i=1}^{n}\mu^i), \quad \alpha \in A,\ n \in \mathbb{N},$$

is a bounded right approximate identity for J_μ. Since

$$\lim_{n\to\infty} \|\pi_*(\tfrac{1}{n}\sum_{i=1}^{n}\mu^i)x_*\| = 0$$

for every $x_* \in J_{\mu\pi}$, it easily follows that the $\widetilde{J}_\mu$-module $J_{\mu\pi}$ has a bounded left approximate identity. Hence, Cohen's factorization theorem yields $\pi_*(\widetilde{J}_\mu)J_{\mu\pi} = J_{\mu\pi}$. Thus $J_{\mu\pi} = \pi_*(\widetilde{J}_\mu)E_*$. ■

Example 2 (continued) Suppose that E_* is a Banach algebra with multiplication denoted by $\star$. It is immediate that if the identity

$$\pi_*(\sigma)(x_* \star y_*) = x_* \star (\pi_*(\sigma)y_*)$$

holds for all $x_*, y_* \in E_*$ and $\sigma \in M(G)$, then $J_{\mu\pi}$ will be a left ideal in E_*. This is, of course, the case when π_* is the right regular representation in $L^1(G)$ and $\star$ is the convolution in $L^1(G)$. Continuing our discussion of the "non-commutative" Choquet–Deny equation in $B(L^2(G))$, we wish to point out here that the convolution in $L^1(G)$ has, in fact, an analog in $\mathcal{T}(L^2(G))$, and that with respect to this "non-commutative" convolution, $J_{\mu\pi}$ is a left ideal in $\mathcal{T}(L^2(G))$, exactly as in the classical case. Such "non-commutative" convolution was recently introduced and studied [32, 33], and we believe that ideals of the form $J_{\mu\pi}$ may be useful in further investigation of the resulting "non-commutative" version of the group algebra $L^1(G)$ [37].

The convolution in $\mathcal{T}(L^2(G))$ can be defined as follows. Let π_l denote the left regular representation of G in $B(L^2(G))$ and π_{l*} its restriction to $\mathcal{T}(L^2(G))$. Recall that there is a canonical mapping $\kappa\colon \mathcal{T}(L^2(G)) \to L^1(G) \subseteq M(G)$, which commutes with the left and right regular representations and is given by $\kappa(S)(A) = \mathrm{tr}[SF(A)]$ for every Borel set $A \subseteq G$, where $F(A) \in B(L^2(G))$ is the operator of multiplication by the characteristic function of A; κ is the preadjoint of the canonical embedding of $L^\infty(G)$ in $B(L^2(G))$. One then defines convolution of two trace class operators S and T by

$$S * T = \pi_{l*}(\kappa(S))T. \tag{3.3}$$

Since $\kappa(S * T) = \kappa(S) * \kappa(T)$, it easily follows that $*$ is an associative algebra product with the further property that

$$\mathrm{tr}(S * T) = \mathrm{tr}(S)\,\mathrm{tr}(T)$$

for all $S, T \in \mathcal{T}(L^2(G))$. Then $\big(\mathcal{T}(L^2(G)), *\big)$ is a Banach algebra with respect to the trace norm. Since the left and right regular representations commute, it is also evident that

$$\pi_*(\sigma)(S * T) = S * (\pi_*(\sigma)T)$$

for all $\sigma \in M(G)$ and $S, T \in \mathcal{T}(L^2(G))$. Hence, we obtain the following.

Proposition 3.3 *$J_{\mu\pi}$ is a left ideal in $(\mathcal{T}(L^2(G)), *)$.*[1]

In connection with Remark 3.1 we note that for $E_* = \mathcal{T}(L^2(G))$, $E_{*0}(G)$ is the preannihilator $VN(G)_\perp$ of $VN(G)$ in $\mathcal{T}(L^2(G))$ and is properly contained in the augmentation ideal of $(\mathcal{T}(L^2(G)), *)$, unless $G = \{e\}$.

The "non-commutative" convolution can also be defined in essentially the same way in the space $\mathcal{N}(L^p(G))$ of nuclear operators on $L^p(G)$, when $1 < p < \infty$. Since $\mathcal{N}(L^p(G))$ is the predual of $B(L^p(G))$, it will remain true that the preannihilator of the space of solutions of the Choquet–Deny equation in $B(L^p(G))$ is a left ideal in $\mathcal{N}(L^p(G))$. $\Diamond$

As we already mentioned, in the classical case the quotient $L^1(G)/J_\mu$ is an abstract L^1-space whose pointwise realization is the boundary needed to represent the μ-harmonic functions by means of a Poisson formula. This fact can be established by a purely functional analytic argument, see [43]. It is therefore conceivable that a purely functional analytic argument could also be used to obtain a Poisson formula for the μ-harmonic vectors and relate it to the formula for the classical bounded μ-harmonic functions. However, we will not attempt to pursue this approach here, choosing instead a more basic, probabilistic approach based on martingale theory. The price to be paid for this is the assumptions of separability that we will need to impose on the predual E_* of our Banach space and, for our main results, also on the group G. On the other hand, the probabilistic approach yields certain powerful convergence results which do not seem possible to obtain by other means.

4 Random Walks and Their Harmonic Functions

In this section we review elements of the classical theory of the bounded μ-harmonic functions and those elements of the theory of random walks that are needed to develop our generalization. With the exception of some more specialized results pertaining to the theory of μ-boundaries, most of the material presented here is a special case of the basic theory of Markov chains [34, 38].

Let G be a locally compact group and μ a probability measure on G. By the (right) random walk of law μ one means the Markov chain with state space G and transition probability $\Pi(g, A) = \mu(g^{-1}A)$. The position of the random walk after its n-th step ($n = 0, 1, \dots$) can be expressed as the product $Y_0Y_1 \cdots Y_n$ where Y_0 is the initial position and $Y_1, Y_2, \dots$ are independent G-valued random variables distributed

[1]The definition of the "non-commutative" convolution given here is the reverse of the definition originally given in [32, 33], *i.e.,* our $S * T$ is $T * S$ in [32, 33]; this ensures that $J_{\mu\pi}$ is, as the classical J_μ, a *left* ideal.

according to the law μ. In general, Y_0 is also a random variable, independent of $Y_1, Y_2, \dots$ and distributed according to a law ν.

Let G^∞ denote the product space $G^\infty = \prod_{n=0}^{\infty} G$ (the space of paths $\omega = \{\omega_n\}_{n=0}^{\infty}$ of the random walk), $X_n \colon G^\infty \to G$, $n = 0, 1, \dots$, the canonical projections, and $\mathcal{B}^\infty$ the product σ-algebra $\prod_{n=0}^{\infty} \mathcal{B} = \sigma(X_0, X_1, \dots)$, where $\mathcal{B}$ stands for the σ-algebra of Borel subsets of G. The law μ of the random walk and the starting measure ν define a measure Q_ν on $(G^\infty, \mathcal{B}^\infty)$, called the *Markov measure*. The canonical projections X_n become random variables on the probability space $(G^\infty, \mathcal{B}^\infty, Q_\nu)$, with distributions $\nu * \mu^n$. When G is second countable, Q_ν is the image of the product measure $\nu \times \mu \times \mu \times \cdots$ on $(G^\infty, \mathcal{B}^\infty)$ under the mapping $G^\infty \ni (\omega_0, \omega_1, \omega_2, \dots) \mapsto (\omega_0, \omega_0\omega_1, \omega_0\omega_1\omega_2, \dots) \in G^\infty$. [2]

In the case that ν is a point measure δ_g, we will write Q_g rather than Q_{δ_g}. The function $G \times \mathcal{B}^\infty \ni (g, A) \mapsto Q_g(A)$ is, in fact, a transition probability from $(G, \mathcal{B})$ to $(G^\infty, \mathcal{B}^\infty)$, and one has

$$Q_\nu(A) = \int_G Q_g(A)\nu(dg), \quad A \in \mathcal{B}^\infty. \tag{4.1}$$

The transition probability $\Pi(g, A) = \mu(g^{-1}A)$ is invariant with respect to the action of G on G by left translations, *i.e.*, $\Pi(gg', gA) = \Pi(g', A)$ for all $g, g' \in G$ and $A \in \mathcal{B}$. There is also a related action of G on the space of paths G^∞, namely, $g\{\omega_n\}_{n=0}^{\infty} = \{g\omega_n\}_{n=0}^{\infty}$. With this action, $(G^\infty, \mathcal{B}^\infty)$ is a Borel G-space. The Markov measures Q_g satisfy

$$Q_{gg'}(gA) = Q_{g'}(A), \quad g, g' \in G,\ A \in \mathcal{B}^\infty, \tag{4.2}$$

i.e., the transition probability $(g, A) \to Q_g(A)$ is G-invariant.

In general, when $f \colon \mathcal{X} \to \mathcal{X}'$ is a Borel function from a Borel space $(\mathcal{X}, \mathcal{A})$ into a Borel space $(\mathcal{X}', \mathcal{A}')$ and σ a measure on $(\mathcal{X}, \mathcal{A})$, we will write $f\sigma$ for the measure $(f\sigma)(A') = \sigma(f^{-1}(A'))$, $A' \in \mathcal{A}'$. When $(\mathcal{X}, \mathcal{A})$ is a Borel G-space and $g \in G$, $g\sigma$ will stand for the measure $(g\sigma)(A) = \sigma(g^{-1}A)$, $A \in \mathcal{A}$. The convolution $\nu * \sigma$ of a measure ν on G with σ is defined by

$$(\nu * \sigma)(A) = \int_G (g\sigma)(A)\nu(dg), \quad A \in \mathcal{A}, \tag{4.3}$$

provided that the function $G \ni g \mapsto (g\sigma)(A)$ is Borel for every $A \in \mathcal{A}$. Using the above notation, we have $Q_g = gQ_e$ (4.2) and $Q_\nu = \nu * Q_e$ (4.1).

Let $\mathbb{B}(G)$ and $\mathbb{B}(G^\infty)$ denote the algebras of bounded complex-valued Borel functions on G and G^∞, resp., equipped with the sup-norms. The natural action of G on $\mathbb{B}(G)$ and $\mathbb{B}(G^\infty)$ is the action $(gf)(x) = f(g^{-1}x)$ associated with the left regular representations. It follows from (4.2) that the formula

$$(Rf)(g) = \int_{G^\infty} f(\omega)Q_g(d\omega) = \int_{G^\infty} f(g\omega)Q_e(d\omega) \tag{4.4}$$

[2]Second countability ensures that the mapping $(\omega_0, \omega_1, \omega_2, \dots) \mapsto (\omega_0, \omega_0\omega_1, \omega_0\omega_1\omega_2, \dots)$ is measurable. To ensure this without second countability requires extending the infinite product measure $\nu \times \mu \times \mu \times \cdots$ to a σ-algebra larger than $\mathcal{B}^\infty$.

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defines an equivariant contraction $R\colon \mathbb{B}(G^\infty) \to \mathbb{B}(G)$.

We will denote by r the right regular representation of G in $\mathbb{B}(G)$, as well as its extension to a representation of $M(G)$.

The *Markov shift* ϑ is the transformation $\vartheta\colon G^\infty \to G^\infty$ given by $\vartheta(\{\omega_n\}_{n=0}^\infty) = \{\omega_{n+1}\}_{n=0}^\infty$. Then ϑ commutes with the natural G-action on G^∞ and transforms the Markov measure Q_ν into the Markov measure $Q_{\nu*\mu}$, *i.e.*,

$$\vartheta Q_\nu = Q_{\nu*\mu}. \tag{4.5}$$

The formula $\theta f = f \circ \vartheta$ defines an injective homomorphism θ of $\mathbb{B}(G^\infty)$ into itself. It is an immediate consequence of (4.1) and (4.5) that

$$R\theta = r(\mu)R. \tag{4.6}$$

The σ-algebra $\mathcal{B}^{(i)} = \{A \in \mathcal{B}^\infty \ ; \ \vartheta^{-1}(A) = A\}$ is called the *invariant σ-algebra*. Elements of $\mathcal{B}^{(i)}$ are called *invariant sets*, and $\mathcal{B}^{(i)}$-measurable functions are called *invariant random variables.* A $\mathcal{B}^\infty$-measurable function $f\colon G^\infty \to \mathbb{C}$ is $\mathcal{B}^{(i)}$-measurable if and only if $f \circ \vartheta = f$. Since ϑ and the G-action on G^∞ commute, $\mathcal{B}^{(i)}$ is preserved by the G-action and therefore $(G^\infty, \mathcal{B}^{(i)})$ is also a Borel G-space.

We will denote by $\mathbb{B}_i$ the algebra of bounded complex-valued invariant random variables. $\mathbb{B}_i$ is precisely the subalgebra of $\mathbb{B}(G^\infty)$ consisting of the fixed points of the homomorphism $\theta\colon \mathbb{B}(G^\infty) \to \mathbb{B}(G^\infty)$. By (4.6), for every $f \in \mathbb{B}_i$, $r(\mu)Rf = Rf$, *i.e.*, Rf is a bounded μ-harmonic function. We note that this property of R is equivalent to having

$$Q_e(A) = (\mu * Q_e)(A) \tag{4.7}$$

for every $A \in \mathcal{B}^{(i)}$.

We will denote by $\mathbb{H}_\mu$ the space of bounded μ-harmonic functions in $\mathbb{B}(G)$, equipped with the sup norm. Then $\mathbb{H}_\mu$ is invariant under the left regular representation; by the action of G on $\mathbb{H}_\mu$ we will always mean the action associated with the left regular representation.

An invariant set $A \in \mathcal{B}^{(i)}$ will be called *universally null* (resp., *universally conull*) if $Q_g(A) = 0$ (resp., $Q_g(A) = 1$) for every $g \in G$. We will say that a property dependent on $\omega \in G^\infty$ holds *universally almost everywhere* (u.a.e.) if it holds for ω in a universally conull set.

Let $\mathcal{N}_u$ denote the collection of universally null sets. An invariant random variable $f\colon G^\infty \to \mathbb{C}$ will be called *universally essentially bounded* if

$$\|f\|_u = \inf_{\Delta\in\mathcal{N}_u} \Big(\sup_{\omega\in G^\infty-\Delta} |f(\omega)| \Big) < \infty. \tag{4.8}$$

$\|\cdot\|_u$ is a C^*-norm on the $*$-algebra $\mathbb{L}_i^\infty(\mu)$ of equivalence classes of the universally essentially bounded invariant random variables, where two such random variables are equivalent when they coincide u.a.e. Since $\mathcal{N}_u$ is invariant under the action of G on G^∞, the natural action of G on $\mathbb{B}_i$ (the left regular representation) induces an action of G on $\mathbb{L}_i^\infty(\mu)$, and the contraction R of equation (4.4) induces an equivariant contraction, which we denote also by R, of $\mathbb{L}_i^\infty(\mu)$ into $\mathbb{H}_\mu$. The following fundamental result [34, Proposition V.2.4] is a well-known consequence of the Martingale Convergence Theorem.

Proposition 4.1 *R is an equivariant isometric isomorphism of $\mathbb{L}_i^\infty(\mu)$ onto $\mathbb{H}_\mu$. Moreover, for every $h \in \mathbb{H}_\mu$, the sequence $\{h \circ X_n\}_{n=0}^\infty$ converges u.a.e. to $R^{-1}h$.*

We will find it useful in our treatment of the μ-harmonic vectors in Section 6 that the sequence $\{h \circ X_n\}_{n=0}^\infty$ converges u.a.e. not only when $h \in \mathbb{H}_\mu$ but also when h is a μ-subharmonic function. A bounded above Borel function $h\colon G \to \mathbb{R}$ is called *μ-subharmonic* if

$$h(g) \leq \int_G h(gg')\mu(dg') \tag{4.9}$$

holds for every $g \in G$. Clearly, if h is bounded harmonic, then $|h|$ is subharmonic. It is also easy to see that if $h_1, \dots, h_n$ are subharmonic functions, then so is $h = \max_{1\leq i\leq n} h_i$. The next proposition is a direct consequence of the theory of submartingales.

Proposition 4.2 *Given a bounded above subharmonic function $h\colon G \to \mathbb{R}$, the sequence $h \circ X_n$ converges u.a.e. to an invariant random variable $f\colon G^\infty \to \mathbb{R}$ such that for every $g \in G$,*

$$h(g) \leq \int_{G^\infty} f(\omega)Q_g(d\omega).$$

Returning to the μ-harmonic functions, observe that since $\mathbb{L}_i^\infty(\mu)$ is an abelian C^*-algebra, $\mathbb{H}_\mu$ itself is an abelian C^*-algebra when equipped with the product

$$h_1 \diamond h_2 = R[(R^{-1}h_1)(R^{-1}h_2)].$$

The following intrinsic description of this product is an immediate consequence of Proposition 4.1 and the Dominated Convergence Theorem.

Corollary 4.3 *Given $h_1, h_2 \in \mathbb{H}_\mu$, the sequence $r(\mu^n)(h_1h_2)$ converges pointwise to $h_1 \diamond h_2$.*

In general, $\diamond$ differs from the usual pointwise product of functions. It is not hard to see that $\diamond$ coincides with the usual product if and only if for every $A \in \mathcal{B}^{(i)}$ and every $g \in G$, $Q_g(A)$ is either 0 or 1 (since $Q_g = gQ_e$, this is equivalent to having $Q_e(A) \in \{0,1\}$ for every $A \in \mathcal{B}^{(i)}$). The Hewitt–Savage 0-1 law [13], [30, p. 190] implies that for second countable abelian groups $\diamond$ is always identical with the usual product. This, in turn, implies the Choquet–Deny theorem, see [30, pp. 192–193].

The measure μ is called *spread out* if for some n the convolution power μ^n is nonsingular (of course, on a discrete group every measure is spread out). Bounded μ-harmonic functions of a spread out measure are right uniformly continuous [4, Proposition I.6, p. 23]. It can be shown that for spread out μ, $\diamond$ is the usual product if and only if every bounded μ-harmonic function is constant on the left cosets of G_μ, *i.e.,* the Choquet–Deny equation in $\mathbb{B}(G)$ has only trivial solutions. When G is nilpotent, or is compactly generated and has polynomial growth, then for every spread out μ the Choquet–Deny equation has only trivial solutions [4, Proposition IV.10,

p. 98], [22]. The largest class known today of countable groups for which this is the case is the class of FC-hypercentral groups [17, Theorem 4.8]. On the other hand, $\diamond$ differs from the usual product for every probability measure (spread out or not) for which G_μ is nonamenable.[3] We also note that every σ-compact amenable locally compact group admits an absolutely continuous probability measure μ such that $G = G_\mu$ and the Choquet–Deny equation has only trivial solutions [27, 39]; but amenability of G alone does not guarantee that every absolutely continuous probability measure has this property [27], [21, Theorem 3.13], [22, Theorem 3.16].

Since every G-invariant function $f: G \to \mathbb{C}$ is constant, Proposition 4.1 also has the following corollary.

Corollary 4.4 *G acts ergodically on $\mathbb{L}_i^\infty(\mu)$,* i.e., *if $f \in \mathbb{L}_i^\infty(\mu)$ and $gf = f$ for every $g \in G$, then f is constant u.a.e.*

We will now discuss the Choquet–Deny equation in $L^\infty(G)$. Recall that the space of μ-harmonic functions in $L^\infty(G)$ is denoted by $\mathcal{H}_\mu$. Evidently, every $h \in \mathbb{H}_\mu$ defines an element of $\mathcal{H}_\mu$. When the Haar measure λ is σ-finite, *i.e.,* when G is σ-compact, one can prove using martingales that every element of $\mathcal{H}_\mu$ arises in this way, *i.e.,* $\mathcal{H}_\mu$ is precisely the space of equivalence classes of the elements of $\mathbb{H}_\mu$ modulo λ [9, Proposition 2]. In fact, σ-finiteness of λ is also used in the proofs of Proposition 4.5 and Corollary 4.6 below.

We will therefore assume for the remainder of this section that G is σ-compact.

For spread out μ, every $h \in \mathbb{H}_\mu$ is continuous. Therefore for such μ, $\mathcal{H}_\mu = \mathbb{H}_\mu$. However, in general, $\mathcal{H}_\mu$ can be very different from $\mathbb{H}_\mu$. For example, when G is abelian and μ is any discrete probability measure with $G_\mu = G$, then $\mathcal{H}_\mu = \mathbb{C}1$ by the Choquet–Deny theorem; at the same time every bounded Borel function constant on the cosets of the subgroup generated by the discrete support of μ is μ-harmonic.

By (4.1) and (4.2) the Markov measure Q_λ, where λ is the left Haar measure, is a G-invariant measure on $(G^\infty, \mathcal{B}^\infty)$, and therefore the natural action of G on G^∞ induces an action of G on $L^\infty(G^\infty, \mathcal{B}^\infty, Q_\lambda)$ and on $L^\infty(G^\infty, \mathcal{B}^{(i)}, Q_\lambda)$. We will denote the latter space by $L_i^\infty(\mu)$. Equation (4.4) defines an equivariant contraction of $L_i^\infty(\mu)$ into $L^\infty(G)$ which (abusing notation) we will still denote by R. It is clear that $RL_i^\infty(\mu) \subseteq \mathcal{H}_\mu$. The following analog of Proposition 4.1 is also a consequence of the theory of martingales.

Proposition 4.5 *R is an equivariant isometric isomorphism of $L_i^\infty(\mu)$ onto $\mathcal{H}_\mu$. For every $h \in \mathcal{H}_\mu$ the sequence $\{h \circ X_n\}_{n=0}^\infty$ converges Q_λ-a.e. to $R^{-1}h$.*

We note that in general the measure Q_λ fails to be σ-finite when restricted to $\mathcal{B}^{(i)}$.[4] However, by (4.1) the measure class of the Markov measure Q_ν is completely determined by the measure class of ν. Thus when β is any finite measure equivalent

[3]When G_μ is nonamenable, then by, e.g., [10, p. 213], there exists a nonconstant bounded continuous μ-harmonic function $h: G_\mu \to \mathbb{C}$. A version of the argument in [30, pp. 192–193] shows that $0 < Q_e(A) < 1$ for some $A \in \mathcal{B}^{(i)}$. So $\diamond$ differs from the usual product.

[4]When μ is spread out, σ-finiteness of Q_λ implies that every bounded μ-harmonic function is constant on the cosets of G_μ, see [21, Proposition 2.6] and [22, Lemma 2.3].

to λ, then Q_β is a finite measure equivalent to Q_λ, and one can replace Q_λ by Q_β. Consequently, $L_i^\infty(\mu)$ and $\mathcal{H}_\mu$ are abelian W^*-algebras. Denoting by $\diamond$ the product

$$h_1 \diamond h_2 = R[(R^{-1}h_1)(R^{-1}h_2)]$$

in $\mathcal{H}_\mu$, one can easily prove the following analog of Corollary 4.3. Now r denotes the right regular representation in $L^\infty(G)$.

Corollary 4.6 *Given $h_1, h_2 \in \mathcal{H}_\mu$, the sequence $r(\mu^n)(h_1h_2)$ converges λ-a.e. to $h_1 \diamond h_2$ and (hence) converges in the weak* topology.*

Remark 4.7 Let $L_i^1(\mu)$ denote the space of complex measures on $(G^\infty, \mathcal{B}^{(i)})$, absolutely continuous with respect to Q_λ. Note that if $\varphi \in L^1(G) \subseteq M(G)$, then $\varphi * Q_e \in L_i^1(\mu)$ and $\langle \varphi , Rf \rangle = \langle \varphi * Q_e , f \rangle$ for every $f \in L_i^\infty(\mu)$. Hence, R is weakly* continuous as a mapping into $L^\infty(G)$, and the preadjoint $R_*\colon L^1(G) \to L_i^1(\mu)$ of R is given by $R_*\varphi = \varphi * Q_e$. In fact, by Lemma 4.8 below, the inverse mapping $R^{-1}\colon \mathcal{H}_\mu \to L_i^\infty(\mu)$ is also weakly* continuous. Thus the weak* (ultraweak) topology of $\mathcal{H}_\mu$, endowed with the above W^*-algebra structure, coincides with the restriction of the weak* topology of $L^\infty(G)$ to $\mathcal{H}_\mu$. The kernel of R_* is precisely the preannihilator J_μ of $\mathcal{H}_\mu$ in $L^1(G)$. It follows that $L^1(G)/J_\mu$ is isometrically isomorphic to $L_i^1(\mu)$, *i.e.*, $L^1(G)/J_\mu$ is an abstract L^1-space, a result already mentioned in Section 3.

Lemma 4.8 *Let X and Y be Banach spaces and T a weakly* continuous isometry of a weakly* closed subspace $V \subseteq X^*$ into Y^*. Then $W = TV$ is also weakly* closed and $T^{-1}\colon W \to V$ is weakly* continuous.*

Proof The proof is a routine application of the Krein–Smulian theorem (e.g., [8, Theorem 7, Ch. V.5 and Corollary 11, Ch. V.3]). ∎

Proposition 4.9 *The following conditions are equivalent:*

(i) *The product $\diamond$ in $\mathcal{H}_\mu$ coincides with the usual product in $L^\infty(G)$.*
(ii) *For each $A \in \mathcal{B}^{(i)}$, $Q_g(A) \in \{0, 1\}$ for λ-a.e. $g \in G$.*
(iii) *The Choquet–Deny theorem is true for μ,* i.e., *every bounded continuous μ-harmonic function is constant on the left cosets of G_μ.*
(iv) *The Choquet–Deny equation in $L^\infty(G)$ has only trivial solutions.*

Proof That (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) is straightforward.

(iv) $\Rightarrow$ (i): Let $h_1, h_2 \in \mathcal{H}_\mu$. Then for every $\varphi \in L^1(G)$,

$$\langle \varphi , \pi(\mu^n)(h_1h_2) \rangle = \int_G \langle \varphi , \pi(g)(h_1h_2) \rangle \mu^n\,(dg) = \int_G \langle \varphi , h_1h_2 \rangle \mu^n\,(dg) = \langle \varphi , h_1h_2 \rangle.$$

Hence, $\pi(\mu^n)(h_1h_2) = h_1h_2$ and thus $h_1 \diamond h_2 = h_1h_2$.

(i) $\Rightarrow$ (iii): Let h be a bounded continuous μ-harmonic function. Then $|h|^2$ is also μ-harmonic. So for every $g \in G$,

$$\int_G |h(gt) - h(g)|^2 \mu\,(dt) = \int_G |h(gt)|^2 - h(gt)^* h(g) - h(gt)h(g)^* + |h(g)|^2 \mu\,(dt) = 0.$$

Using continuity of h this implies that $h(gt) = h(g)$ for every $t \in \operatorname{supp}\mu$. Thus $r(t)h = h$ for every $t \in \operatorname{supp}\mu$ and, hence, also for every $t \in G_\mu$. Therefore h must indeed be constant on the left cosets of G_μ. ■

Since every universally null set is Q_λ-null, $L_i^\infty(\mu)$ is canonically the quotient of $\mathbb{L}_i^\infty(\mu)$. Continuity of the bounded μ-harmonic functions of a spread out measure and equation (4.1) imply that for spread out μ each of the Markov measures Q_g, $g \in G$, is absolutely continuous with respect to Q_λ. Consequently, an invariant set $A \in \mathcal{B}^{(i)}$ is universally null if and only if it is Q_λ-null, and so $\mathbb{L}_i^\infty(\mu)$ and $L_i^\infty(\mu)$ coincide for spread out μ. As our discussion of the relation between $\mathbb{H}_\mu$ and $\mathcal{H}_\mu$ indicates, for nonspread out μ, $\mathbb{L}_i^\infty(\mu)$ and $L_i^\infty(\mu)$ can be very different. Moreover, in general, $\mathbb{L}_i^\infty(\mu)$ is not a W^*-algebra.[5]

In contrast to the case of spread out probability measures (or probability measures on discrete groups) the Choquet–Deny theorem for general measures on continuous noncompact nonabelian locally compact groups has been verified only for 2-step nilpotent groups [12], nilpotent [SIN] groups [19], and some more special groups [12].

The formula

$$h(g) = \int_{G^\infty} f(\omega) Q_g\,(d\omega)$$

is the Poisson formula for the bounded μ-harmonic functions, and the Borel space $(G^\infty, \mathcal{B}^{(i)})$ can be regarded as a "boundary". However, this boundary is not a "nice" space, its pathological feature being the fact that the σ-algebra $\mathcal{B}^{(i)}$ does not separate points (unless $G = \{e\}$). Moreover, $(G^\infty, \mathcal{B}^{(i)})$ itself does not depend on μ, this dependence being encoded entirely in the properties of the Poisson kernel Q_g. Apart from the problem of extending the Choquet–Deny theorem to nonabelian groups, a major problem in the theory of bounded μ-harmonic functions has been to find a realization of the boundary as a topological (at least) Hausdorff space whose points can be regarded as limit points of the trajectories of the random walk (see [24–26, 28] and references therein). This theory is outside the scope of this article and we will confine ourselves to mentioning only certain generalities regarding the possibility of realizing the boundary as a "nice" space, related to the fact that $\mathcal{H}_\mu$ is an abelian W^*-algebra.

By a G-space $(\mathcal{X}, \mathcal{A}, \alpha)$ we mean a Borel G-space $(\mathcal{X}, \mathcal{A})$ with a quasiinvariant measure α. When $(\mathcal{X}, \mathcal{A}, \alpha)$ and $(\mathcal{X}', \mathcal{A}', \alpha')$ are two such G-spaces, we say that $L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$ is *G-isomorphic* to $L^\infty(\mathcal{X}', \mathcal{A}', \alpha')$ if there exists an equivariant $*$-isomorphism of the $*$-algebra $L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$ onto $L^\infty(\mathcal{X}', \mathcal{A}', \alpha')$. We define the *$\mu$-boundary* as any G-space $(\mathcal{X}, \mathcal{A}, \alpha)$ such that $L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$ is G-isomorphic to $L_i^\infty(\mu)$. We

[5] For example, when $\mu = \delta_e$, then $\mathbb{L}_i^\infty(\mu) \cong \mathbb{B}(G)$; when $G = \mathbb{R}$ and μ is a discrete probability measure whose discrete support generates $\mathbb{Q}$, then $\mathbb{L}_i^\infty(\mu) \cong \mathbb{B}(\mathbb{R}/\mathbb{Q})$.

remark that since Q_λ is equivalent to a finite measure, the same is true about the quasiinvariant measure α on any μ-boundary. Moreover, a G-space $(\mathcal{X}, \mathcal{A}, \alpha)$ is a μ-boundary if and only if there exists an equivariant identity-preserving isometry of $L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$ onto $\mathcal{H}_\mu$.

Let us assume that G is second countable. Then $L^1(G)$ is separable and therefore so is the predual of $\mathcal{H}_\mu$. Hence, $L^1_i(\mu)$ is also separable. Moreover, given $f \in L^\infty_i(\mu)$ and $\varphi \in L^1_i(\mu)$, the function $G \ni g \to \langle \varphi, gf \rangle \in \mathbb{C}$ is Borel (in fact, continuous). These properties permit a routine application of the classical Mackey's theorem about pointwise realization of group actions [29]. It shows that there always exist μ-boundaries that are standard Borel spaces such that the map $G \times \mathcal{X} \ni (g, x) \to gx \in \mathcal{X}$ is Borel. Moreover, by a theorem of Varadarajan [42, Theorem 3.2] one can even take $\mathcal{X}$ to be a compact metric space with the map $G \times \mathcal{X} \ni (g, x) \to gx \in \mathcal{X}$ continuous. If $(\mathcal{X}, \mathcal{A}, \alpha)$ is a standard μ-boundary with the map $G \times \mathcal{X} \ni (g, x) \to gx \in \mathcal{X}$ Borel and $\Phi \colon L^\infty(\mathcal{X}, \mathcal{A}, \alpha) \to L^\infty_i(\mu)$ is the equivariant isomorphism, then there exists a probability measure ρ on $\mathcal{X}$ (the Poisson kernel) such that

$$(R\Phi f)(g) = \int_{\mathcal{X}} f(gx)\rho(dx) \pmod{\lambda}$$

for every $f \in L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$.

Another realization of the μ-boundary, often called the Poisson space [4], is the spectrum of the C*-subalgebra of $\mathcal{H}_\mu$ consisting of left uniformly continuous bounded μ-harmonic functions; the disadvantage of the Poisson space is that it usually is not metrizable.

Finally, we note that the action of G on a μ-boundary has certain very distinctive properties. It is always approximately transitive and amenable [7, 15, 16, 45], and when μ is spread out it is strongly approximately transitive [22]. There are also further properties of a more probabilistic nature related to the convergence properties of the underlying random walk [10, 11, 20]. Amenability of the G-action on the μ-boundary is intimately related to the existence of a norm 1 equivariant projection of $L^\infty(G)$ onto $\mathcal{H}_\mu$; in the sequel we will obtain a generalized version of this result in the setting of the Choquet–Deny equation in a dual Banach space.

5 Vector Valued Harmonic Functions

Let E be a Banach space with a separable predual E_*. As is well known, the Borel structure in E_* defined by the norm topology is the same as that defined by the weak topology and also the same as the weak Borel structure generated by the functions $\langle \cdot, x \rangle$, $x \in E$. Thus a function f from a Borel space $(\mathcal{X}, \mathcal{A})$ to E_* is measurable (Borel) if and only if for each $x \in E$ the function $\mathcal{X} \ni t \to \langle f(t), x \rangle \in \mathbb{C}$ is Borel. We note that if $f \colon \mathcal{X} \to E_*$ is Borel, then the function $t \to \|f(t)\|$ is also Borel.

Given a complex measure σ on $(\mathcal{X}, \mathcal{A})$ and a Borel function $f \colon \mathcal{X} \to E_*$ such that $\int_{\mathcal{X}} \|f\| \, d|\sigma| < \infty$, by the integral $\int_{\mathcal{X}} f \, d\sigma$ we will mean the unique vector $\int_{\mathcal{X}} f \, d\sigma \in E_*$ with

$$\Big\langle \int_{\mathcal{X}} f \, d\sigma, x \Big\rangle = \int_{\mathcal{X}} \langle f(t), x \rangle \, \sigma(dt) \tag{5.1}$$

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for every $x \in E$.

The proper Borel structure in E to work with is that given by the weak* topology. This Borel structure is well known to be standard and coincides with the weak Borel structure generated by the functions $\langle x_*, \cdot \rangle$, $x_* \in E_*$. A function f from a Borel space $(\mathcal{X}, \mathcal{A})$ to E is therefore Borel if and only if for each $x_* \in E_*$ the function $\mathcal{X} \ni t \to \langle x_*, f(t) \rangle \in \mathbb{C}$ is Borel. If $f\colon \mathcal{X} \to E$ is Borel, then the function $t \to \|f(t)\|$ is also Borel.

Given a complex measure σ on $(\mathcal{X}, \mathcal{A})$ and a Borel function $f\colon \mathcal{X} \to E$ such that $\int_{\mathcal{X}} \|f\|\, d|\sigma| < \infty$, the integral $\int_{\mathcal{X}} f\, d\sigma$ is the unique vector $\int_{\mathcal{X}} f\, d\sigma \in E$ with

$$\Big\langle x_*, \int_{\mathcal{X}} f\, d\sigma \Big\rangle = \int_{\mathcal{X}} \langle x_*, f(t) \rangle\, \sigma(dt) \tag{5.2}$$

for every $x_* \in E_*$.

Let G be a locally compact group and $\mu \in M_1(G)$. A bounded (with respect to the norm on E) Borel function $h\colon G \to E$ will be called *μ-harmonic* if it satisfies equation (1.1) for every $g \in G$. The goal of this section is to extend Proposition 4.1 and Corollary 4.3 to such μ-harmonic functions. This will prove very useful in the following sections, in our study of the μ-harmonic vectors arising from a representation of G by isometries in E.

Let $\mathbb{B}(G, E)$ and $\mathbb{B}_i(E)$ denote the Banach spaces of bounded Borel functions $f\colon G \to E$ and bounded $\mathcal{B}^{(i)}$-measurable functions $f\colon G^\infty \to E$ (E-valued invariant random variables), resp., equipped with the sup-norms $\|\cdot\|_{\sup}$. The concept of a universally essentially bounded invariant random variable with values in E is introduced as in the classical case and the Banach space of equivalence classes of such random variables (with norm $\|\cdot\|_u$) will be denoted by $\mathbb{L}_i^\infty(\mu, E)$. Finally, let r_E stand for the right regular representation of G in $\mathbb{B}(G, E)$, as well as its extension to a representation of $M(G)$, and let $\mathbb{H}_\mu(E)$ denote the space of bounded E-valued μ-harmonic functions. The group acts on $\mathbb{B}(G, E)$, $\mathbb{B}_i(E)$, $\mathbb{L}_i^\infty(\mu, E)$, and $\mathbb{H}_\mu(E)$ in the same way as in the classical case.

It is clear that given $f \in \mathbb{B}_i(E)$, the function

$$h(g) = \int_{G^\infty} f(\omega)\, Q_g(d\omega) \tag{5.3}$$

is bounded μ-harmonic, and (5.3) defines an equivariant contraction R_E from $\mathbb{B}_i(E)$ into $\mathbb{H}_\mu(E)$. As in the classical case, we will use the same symbol R_E to denote the contraction from $\mathbb{L}_i^\infty(\mu, E)$ into $\mathbb{H}_\mu(E)$ defined by (5.3).

Lemma 5.1 *If $f\colon G^\infty \to E$ is a universally essentially bounded invariant random variable and $h = R_E f$, then the sequence $h \circ X_n$ converges weakly* u.a.e. to f and the sequence $\|h \circ X_n\|$ converges u.a.e. to $\|f\|$.*

Proof Given $x_* \in E_*$, consider the functions $h_{x_*}\colon G \to \mathbb{C}$ and $f_{x_*}\colon G^\infty \to \mathbb{C}$ defined by $h_{x_*}(g) = \langle x_*, h(g) \rangle$ and $f_{x_*}(\omega) = \langle x_*, f(\omega) \rangle$, resp. Then $h_{x_*} \in \mathbb{H}_\mu$ and $h_{x_*} = R f_{x_*}$. By Proposition 4.1, $f_{x_*} = \lim_{n\to\infty} h_{x_*} \circ X_n$ u.a.e. Since E_* is separable this easily implies that w*-$\lim_{n\to\infty} h \circ X_n = f$ u.a.e.

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The proof of the second statement invokes Proposition 4.2. Let $H(g) = \|h(g)\|$, $g \in G$. Then H is a bounded μ-subharmonic function and therefore there exists an invariant random variable $F\colon G^\infty \to [0,\infty)$ with $F = \lim_{n\to\infty} H \circ X_n$ u.a.e. We need to prove that $F(\omega) = \|f(\omega)\|$ u.a.e. It is easy to see that $\|f(\omega)\| \le F(\omega)$ u.a.e. Therefore to prove that $F(\omega) = \|f(\omega)\|$ u.a.e., it suffices to prove that

$$\int_{G^\infty} F(\omega) Q_g\,(d\omega) = \int_{G^\infty} \|f(\omega)\| Q_g\,(d\omega)$$

for every $g \in G$.

Let $\{x_{*i}\}_{i=1}^\infty$ be a sequence dense in the unit ball of E_*. For each $k = 1, 2, \dots$, define a function $H_k\colon G \to [0,\infty)$ by $H_k(g) = \max_{1\le i\le k} |\langle x_{*i}\,, h(g)\rangle|$. Let

$$S_{kn}(g) = \int_{G^\infty} H_k(\omega_n) Q_g\,(d\omega) = \int_G H_k(gg')\mu^n\,(dg'), \quad n = 0, 1, \dots \tag{5.4}$$

where $\omega_n = X_n(\omega)$, and let $S(g) = \sup_{k,n} S_{kn}(g)$. Note that the sequence $\{H_k\}_{k=1}^\infty$ is nondecreasing and each H_k is μ-subharmonic. Hence, $S_{kn} \le S_{k'n'}$ whenever $k \le k'$ and $n \le n'$. Moreover, the sequence $\{H_k\}_{k=1}^\infty$ converges pointwise to H while for any fixed k, $\lim_{n\to\infty} H_k(\omega_n) = \max_{1\le i\le k} |\langle x_{*i}\,, f(\omega)\rangle|$ u.a.e. Hence,

$$\begin{aligned} S(g) &= \sup_k \sup_n S_{kn}(g) = \sup_k \lim_{n\to\infty} S_{kn}(g) \\ &= \int_{G^\infty} \sup_k \max_{1\le i\le k} |\langle x_{*i}\,, f(\omega)\rangle|\; Q_g(d\omega) = \int_{G^\infty} \|f(\omega)\|\, Q_g(d\omega), \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} S(g) &= \sup_n \sup_k S_{kn}(g) = \sup_n \lim_{k\to\infty} S_{kn}(g) = \sup_n \int_{G^\infty} H(\omega_n) Q_g\,(d\omega) \\ &= \lim_{n\to\infty} \int_{G^\infty} H(\omega_n) Q_g\,(d\omega) = \int_{G^\infty} F(\omega) Q_g\,(d\omega). \end{aligned} \tag{5.6}$$

This completes the proof. ∎

Remark 5.2 It is a standard result of the theory of vector valued martingales [35, Proposition V-2-8] that when E is assumed separable, then the sequence $h \circ X_n$ converges in norm u.a.e. to f. This result can easily be deduced from the second statement of our lemma by mimicking a part of the argument in [35, p. 110].

Proposition 5.3 *R_E is an equivariant isometry of $\mathbb{L}_i^\infty(\mu, E)$ onto $\mathbb{H}_\mu(E)$. Moreover, for every $h \in \mathbb{H}_\mu(E)$ the sequence $h \circ X_n$ converges weakly* u.a.e. to $R_E^{-1}h$ and the sequence $\|h \circ X_n\|$ converges u.a.e. to $\|R_E^{-1}h\|$.*

Proof By Lemma 5.1 it suffices to prove the first statement.

We know that R_E is an equivariant contraction into $\mathbb{H}_\mu(E)$. Let $f\colon G^\infty \to E$ be a universally essentially bounded invariant random variable and $h = R_E f$. Using

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the first or the second statement of Lemma 5.1, it immediately follows that $\|f\|_u \leq \|h\|_{\sup}$. Hence, R_E is isometric. It remains to prove that R_E is surjective.

Given $h \in \mathbb{H}_\mu(E)$, for each $x_* \in E_*$ define a function $h_{x_*} \colon G \to \mathbb{C}$ by $h_{x_*}(g) = \langle x_*, h(g)\rangle$. Then $h_{x_*} \in \mathbb{H}_\mu$. Hence, by Proposition 4.1, the set

$$\Omega_{x_*} = \{\omega \in G^\infty \; ; \; \{h_{x_*}(\omega_n)\}_{n=0}^\infty \text{ converges}\}$$

is universally conull and there exists $f_{x_*} \in \mathbb{B}_i$ such that for every $\omega \in \Omega_{x_*}$, $f_{x_*}(\omega) = \lim_{n\to\infty} h_{x_*}(\omega_n)$, and that $h_{x_*} = Rf_{x_*}$. A routine argument using separability of E_* shows that $\Omega = \bigcap_{x_* \in E_*} \Omega_{x_*}$ is also universally conull. Now, for every $\omega \in \Omega$, the function $E_* \ni x_* \to f_{x_*}(\omega)$ is a bounded linear functional on E_*. It follows that there is a $\mathcal{B}^{(i)}$-measurable function $f \colon G^\infty \to E$ such that for every $\omega \in \Omega$, $f(\omega) = \text{w}^*\text{-}\lim_{n\to\infty} h(\omega_n)$, and $h = R_E f$. So R_E is indeed surjective. ∎

When E is a W^*-algebra, then under the norm $\|\cdot\|_u$, $\mathbb{L}_i^\infty(\mu, E)$ is a C^*-algebra. As in the classical case we will denote by $\diamond$ the product

$$h_1 \diamond h_2 = R_E[(R_E^{-1}h_1)(R_E^{-1}h_2)]$$

in $\mathbb{H}_\mu(E)$.

Corollary 5.4 *If E is a W^*-algebra, then for every $h \in \mathbb{H}_\mu(E)$, the sequence $\{h \circ X_n\}_{n=0}^\infty$ converges in the σ-strong* topology u.a.e. to $R_E^{-1}h$. Moreover, given $h_1, h_2 \in \mathbb{H}_\mu(E)$ and $g \in G$, the sequence $[r_E(\mu^n)(h_1h_2)](g)$ converges in the σ-strong* topology to $(h_1 \diamond h_2)(g)$.*

Proof Since the predual of E is separable, we may assume that E acts in a separable Hilbert space $\mathfrak{H}$. Since $\mathbb{H}_\mu(E)$ is a selfadjoint subspace of $\mathbb{B}(G, E)$ and each $h \in \mathbb{H}_\mu(E)$ is bounded, it suffices to prove convergence in the strong operator topology.

Given $h \in \mathbb{H}_\mu(E)$ let f be a representative of $R_E^{-1}h$. We need to show that there exists a universally conull invariant set Ω such that for every $\omega \in \Omega$ and $\xi \in \mathfrak{H}$, the sequence $h(\omega_n)\xi$ converges in norm to $f(\omega)\xi$.

Note that the function $h_\xi(g) = h(g)\xi$ is an $\mathfrak{H}$-valued bounded μ-harmonic function and the function $f_\xi(\omega) = f(\omega)\xi$, a universally essentially bounded $\mathfrak{H}$-valued invariant random variable with $h_\xi = R_{\mathfrak{H}} f_\xi$. Hence, by Lemma 5.1 the set

$$\Omega_\xi = \{\omega \in G^\infty \; ; \; \underset{n\to\infty}{\text{w-lim}}\, h(\omega_n)\xi = f(\omega)\xi \text{ and } \lim_{n\to\infty} \|h(\omega_n)\xi\| = \|f(\omega)\xi\| \,\}$$

is universally conull. But by an elementary result on weak convergence in $\mathfrak{H}$,

$$\Omega_\xi = \{\omega \in G^\infty \; ; \; \lim_{n\to\infty} h(\omega_n)\xi = f(\omega)\xi \,\}.$$

Separability of $\mathfrak{H}$ implies that

$$\Omega = \bigcap_{\xi \in \mathfrak{H}} \Omega_\xi = \{\omega \in G^\infty \; ; \; f(\omega) = \underset{n\to\infty}{\text{s-lim}}\, h(\omega_n)\}$$

is also universally conull. This proves the first statement.

To prove the second statement, given $h_1, h_2 \in \mathbb{H}_\mu(E)$, let f_1, f_2 be representatives of $R^{-1}h_1$ and $R^{-1}h_2$, resp. We know that $f_i(\omega) = \text{s-lim}_{n\to\infty} h_i(\omega_n)$ u.a.e. and therefore using boundedness of the h_i's, we also have $(f_1 f_2)(\omega) = \text{s-lim}_{n\to\infty}(h_1h_2)(\omega_n)$ u.a.e. Then a version of the Dominated Convergence Theorem yields, for every $g \in G$ and $\xi \in \mathfrak{H}$,

$$(h_1 \diamond h_2)(g)\xi = \int_{G^\infty} (f_1 f_2)(\omega)\xi \, Q_g(d\omega) = \lim_{n\to\infty} \int_{G^\infty} (h_1h_2)(\omega_n)\xi \, Q_g(d\omega)$$
$$= \lim_{n\to\infty} \int_{G^\infty} (h_1h_2)(gg')\xi \, \mu^n(dg') = \lim_{n\to\infty} [r_E(\mu^n)(h_1h_2)](g)\xi. \qquad \blacksquare$$

Remark 5.5 Write $\mathcal{H}_\mu(E)$ for the space of equivalence classes modulo λ of bounded E-valued μ-harmonic functions and $L_i^\infty(\mu, E)$ for $L^\infty(G^\infty, \mathcal{B}^{(i)}, Q_\lambda, E)$, the space of equivalence classes modulo Q_λ of Q_λ-essentially bounded E-valued invariant random variables. When G is σ-compact, there is no difficulty extending Proposition 5.3 and Corollary 5.4 to this setting, but we will not need it. An important difference between $\mathbb{H}_\mu(E)$ and $\mathcal{H}_\mu(E)$, and between $\mathbb{L}_i^\infty(\mu, E)$ and $L_i^\infty(\mu, E)$ (which is already present in the classical case $E = \mathbb{C}$) is that $\mathcal{H}_\mu(E)$ and $L_i^\infty(\mu, E)$ are duals of Banach spaces while $\mathbb{H}_\mu(E)$ and $\mathbb{L}_i^\infty(\mu, E)$ are, in general, not. Therefore, when E is a W^*-algebra, $\mathbb{L}_i^\infty(\mu, E)$ and $\mathbb{H}_\mu(E)$ will be C^*-algebras but, in general, not W^*-algebras, while $\mathcal{H}_\mu(E)$ and $L_i^\infty(\mu, E) = L_i^\infty(\mu) \otimes E$ remain W^*-algebras.

6 Harmonic Vectors

As in Section 5, E will denote a Banach space with a separable predual E_* and G a locally compact group. We will consider a representation π of G in E which is the adjoint of a strongly continuous representation π_* by isometries in E_*. Our goal is to obtain a Poisson formula for the μ-harmonic vectors in E (*cf.* §2) and relate it to the classical Poisson formula for the bounded μ-harmonic functions. Our main result will require the assumption that G be second countable, but the initial results of this section are valid for any locally compact group.

Observe that when $x \in E$ is a μ-harmonic vector, then the function $g \to \pi(g)x$ is a bounded E-valued μ-harmonic function. Hence, the space $\mathcal{H}_{\mu\pi}$ of the μ-harmonic vectors in E is isometrically isomorphic to a closed subspace of $\mathbb{H}_\mu(E)$ and therefore also to a closed subspace $\mathbb{L}_{\mu\pi}$ of $\mathbb{L}_i^\infty(\mu, E)$. We proceed to give a more explicit description of $\mathbb{L}_{\mu\pi}$.

Recall that G acts in a natural way on each of the function spaces $\mathbb{B}(G, E)$, $\mathbb{H}_\mu(E)$, $\mathbb{B}_i(E)$, $\mathbb{L}_i^\infty(\mu, E)$, and $L_i^\infty(\mu, E)$, and that we write gf for $g \in G$ applied to a function f. Now, the representation π induces a representation $\widehat{\pi}$ in each of the function spaces in question: $\widehat{\pi}(g)$ transforms a function f (or a class of functions when $f \in \mathbb{L}_i^\infty(\mu, E)$ or $f \in L_i^\infty(\mu, E)$) into the function $(\widehat{\pi}(g)f)(\,\cdot\,) = \pi(g)f(\,\cdot\,)$. It is clear that $\widehat{\pi}$ commutes with the natural action of G.

It is easy to see that μ-harmonic functions of the form $\pi(\,\cdot\,)x$, where $x \in \mathcal{H}_{\mu\pi}$, are precisely those elements $h \in \mathbb{H}_\mu(E)$ which satisfy $g^{-1}h = \widehat{\pi}(g)h$ for every $g \in G$, *i.e.,*

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$h(gg') = \pi(g)h(g')$ for all $g, g' \in G$. Since the isomorphism R_E of Proposition 5.3 is equivariant with respect to both the natural action of G and the representation $\widehat{\pi}$, it follows that $\mathcal{H}_{\mu\pi}$ is isometrically isomorphic to

$$\mathbb{L}_{\mu\pi} = \{f \in \mathbb{L}_i^\infty(\mu, E) \, ; \, g^{-1}f = \widehat{\pi}(g)f \text{ for every } g \in G\}.$$

More precisely, we have the following.

Theorem 6.1 *The mapping $R_\pi \colon \mathbb{L}_{\mu\pi} \to E$ given by*

$$R_\pi f = (R_E f)(e) = \int_{G^\infty} f(\omega) Q_e(d\omega) \tag{6.1}$$

is an isometric isomorphism of $\mathbb{L}_{\mu\pi}$ onto $\mathcal{H}_{\mu\pi}$. Furthermore, given $x \in \mathcal{H}_{\mu\pi}$, the sequence $\pi(\omega_n)x$ converges weakly u.a.e. to $R_\pi^{-1}x$.*

Note that the elements of $\mathbb{L}_{\mu\pi}$ are equivalence classes modulo $\mathcal{N}_u$ of those universally essentially bounded $\mathcal{B}^{(i)}$-measurable functions $f \colon G^\infty \to E$ which have the property that for each $g \in G$, $f(g\omega) = \pi(g)f(\omega)$ u.a.e. Let

$$\mathbb{B}_{i\pi} = \{f \in \mathbb{B}_i(E) \, ; \, f(g\omega) = \pi(g)f(\omega) \text{ for every } g \in G \text{ and } \omega \in G^\infty\}.$$

Trivially, every element of $\mathbb{B}_{i\pi}$ represents an element of $\mathbb{L}_{\mu\pi}$. But it is also true that every element of $\mathbb{L}_{\mu\pi}$ can be represented by an element of $\mathbb{B}_{i\pi}$.

Corollary 6.2 *$\mathbb{L}_{\mu\pi}$ is the quotient of $\mathbb{B}_{i\pi}$ modulo $\mathcal{N}_u$.*

Proof It suffices to show that if $f \colon G^\infty \to E$ is a universally essentially bounded invariant random variable such that for each $g \in G$, $f(g\omega) = \pi(g)f(\omega)$ u.a.e., then there exists $f' \in \mathbb{B}_{i\pi}$ with $f = f'$ u.a.e. But with $x = R_\pi f$, the set $\{\omega \in G^\infty \, ; \, \text{w}^*\text{-}\lim_{n\to\infty} \pi(\omega_n)x \text{ exists}\}$ is obviously G-invariant, and by Theorem 6.1 it is universally conull. Define $f' \colon G^\infty \to E$ by

$$f'(\omega) = \begin{cases} \text{w}^*\text{-}\lim_{n\to\infty} \pi(\omega_n)x & \text{when the limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

■

Equation (6.1) can be viewed as a Poisson formula for the μ-harmonic vectors. However, the use of the subspace $\mathbb{L}_{\mu\pi}$ and the norm $\|\cdot\|_u$ has certain serious disadvantages.

First, $\mathcal{H}_{\mu\pi}$ is the dual of the quotient $E_*/J_{\mu\pi}$, because $\mathcal{H}_{\mu\pi} = J_{\mu\pi}^\perp$. So $\mathbb{L}_{\mu\pi}$ is also the dual of $E_*/J_{\mu\pi}$. But, in general, $\mathbb{L}_i^\infty(\mu, E)$ has no predual and therefore it is not clear how the predual $E_*/J_{\mu\pi}$ of $\mathbb{L}_{\mu\pi}$ is related to the Poisson formula and the boundary space $(G^\infty, \mathcal{B}^{(i)})$.

Secondly, as we pointed out in Section 4, when G is second countable, then for the purpose of representing the classical μ-harmonic functions, the badly behaved Borel

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space $(G^\infty, \mathcal{B}^{(i)})$ can be replaced by a standard Borel G-space or even a topological G-space. Recall that this regularization of the boundary relies on the fact that $L_i^\infty(\mu)$ is an abelian W^*-algebra.

Now, let β be any fixed finite measure equivalent to λ. Then $L_i^\infty(\mu, E)$ is canonically the dual of $L_i^1(\mu, E_*) = L^1(G^\infty, \mathcal{B}^{(i)}, Q_\beta, E_*)$, the space of equivalence classes of Q_β-integrable $\mathcal{B}^{(i)}$-measurable functions $f\colon G^\infty \to E_*$. As we will see, the Poisson formula for the μ-harmonic vectors becomes very satisfactory once the subspace $\mathbb{L}_{\mu\pi} \subseteq \mathbb{L}_i^\infty(\mu, E)$ is replaced by the weakly* closed subspace

$$L_{\mu\pi} = \{ f \in L_i^\infty(\mu, E) \, ; \, g^{-1} f = \widehat{\pi}(g) f \ \text{ for every } g \in G \} \subseteq L_i^\infty(\mu, E).$$

For the remainder of this article we will assume that G is second countable.

In the setting of Example 2 one readily obtains the following.

Proposition 6.3 *$L_{\mu\pi}$ equals the crossed product $L_i^\infty(\mu) \times_{\pi_l} G$ where π_l is the left regular representation in $L_i^\infty(\mu)$.*

Lemma 6.4 *Let $f\colon G^\infty \to E$ be a bounded invariant random variable such that for each $g \in G$, $f(g\omega) = \pi(g) f(\omega)$ for Q_λ-a.e. $\omega \in G^\infty$. Then there exists an invariant random variable $f'\colon G^\infty \to E$ such that $f = f'$ Q_λ-a.e., $\|f'\|_{\sup} \leq \|f\|_{\sup}$, and $\pi(g) f'(\omega) = f'(g\omega)$ for every $g \in G$ and $\omega \in G^\infty$.*

Proof Consider the Borel G-space $(G^\infty, \mathcal{B}^\infty)$. As a countable product of standard Borel spaces, $(G^\infty, \mathcal{B}^\infty)$ is standard. Moreover, Q_λ is a σ-finite invariant measure on $(G^\infty, \mathcal{B}^\infty)$ and the mapping $G \times G^\infty \ni (g, \omega) \to g\omega \in G^\infty$ is Borel.

Next, let E_f denote the closed ball $\bar{B}_E(0, \|f\|_{\sup})$ of radius $\|f\|_{\sup}$ and centre 0 in E. With the G-action given by π and with the weak* topology, E_f is a topological G-space and the mapping $G \times E_f \ni (g, x) \to \pi(g)x \in E_f$ is continuous. Therefore this mapping is also Borel (with respect to the product Borel structure on $G \times E_f$). Thus E_f is a standard Borel G-space with the mapping $G \times E_f \ni (g, x) \to \pi(g)x \in E_f$ Borel and f is a Borel function of G^∞ into E_f such that for every $g \in G$, $f(g\omega) = \pi(g) f(\omega)$ Q_λ-a.e. We are therefore in a position to apply [46, Proposition B.5, p. 198] to conclude that there exists a $\mathcal{B}^\infty$-measurable function $f''\colon G^\infty \to E_f$ such that $f'' = f$ Q_λ-a.e. and $f''(g\omega) = \pi(g) f''(\omega)$ for every $g \in G$ and $\omega \in G^\infty$.

Let

$$\begin{aligned} \Gamma &= \{\omega \in G^\infty \, ; \, f''(\omega) = f''(\vartheta^n(\omega)) \text{ for every } n \geq 0\} \\ &= \bigcap_{n=0}^{\infty} \vartheta^{-n}\big(\{\omega \in G^\infty \, ; \, f''(\omega) = f''(\vartheta(\omega))\}\big) \end{aligned}$$

where ϑ is the Markov shift, *cf.* §4. Note that $\Gamma \subseteq \vartheta^{-1}(\Gamma)$, Γ is G-invariant, $\Gamma \in \mathcal{B}^\infty$, and $Q_\lambda(G^\infty - \Gamma) = 0$ (the latter because ϑ preserves the measure class of Q_λ, *cf.* (4.5)).

Let

$$\Delta = \bigcap_{n=0}^{\infty} \vartheta^{-n}(G^\infty - \Gamma) = \{\omega \in G^\infty - \Gamma \,;\, \vartheta^n(\omega) \in G^\infty - \Gamma \text{ for every } n \geq 1\}.$$

Then $\vartheta^{-1}(\Delta) = \Delta$, *i.e.*, $\Delta \in \mathcal{B}^{(i)}$, and Δ is G-invariant. Note also that if $\omega \in (G^\infty - \Gamma) \cap (G^\infty - \Delta)$, then there exists $k(\omega) \geq 1$ such that $\vartheta^{k(\omega)}(\omega) \in \Gamma$, and if $\vartheta^i(\omega) \in \Gamma$ and $\vartheta^j(\omega) \in \Gamma$ for some $i, j \geq 1$, then $f''(\vartheta^i(\omega)) = f''(\vartheta^j(\omega))$.

Define a function $f' \colon G^\infty \to E$ by

$$f'(\omega) = \begin{cases} 0 & \text{for } \omega \in \Delta, \\ f''(\omega) & \text{for } \omega \in \Gamma, \\ f''(\vartheta^{k(\omega)}\omega) & \text{for } \omega \in G^\infty - (\Gamma \cup \Delta). \end{cases}$$

It is straightforward to verify that f' is an invariant random variable equal to f Q_λ-a.e. and satisfying $\|f'\|_{\sup} \leq \|f\|_{\sup}$ as well as $f'(g\omega) = \pi(g)f'(\omega)$ for every $g \in G$ and $\omega \in G^\infty$. ∎

Lemma 6.5 *Given $f_1, f_2 \in \mathbb{B}_{i\pi}$ the following conditions are equivalent:*

(i) $f_1 = f_2$ *u.a.e.,*
(ii) $f_1 = f_2$ Q_λ*-a.e.,*
(iii) $f_1 = f_2$ Q_e*-a.e.*

Moreover, for every $f \in \mathbb{B}_{i\pi}$, $\|f\|$ is constant u.a.e.

Proof Note that the set $\{\omega \in G^\infty \,;\, f_1(\omega) = f_2(\omega)\}$ is G-invariant. The equivalence follows immediately from this observation and equations (4.1) and (4.2). The second statement follows from Corollary 4.4 and the observation that the function $F(\omega) = \|f(\omega)\|$ is G-invariant. ∎

By Corollary 6.2 and Lemmas 6.4 and 6.5, there exists a surjective mapping $\Phi \colon \mathbb{L}_{\mu\pi} \to L_{\mu\pi}$ such that $\Phi[f]_u = [f]_{Q_\lambda}$ for every $f \in \mathbb{B}_{i\pi}$ where $[\,\cdot\,]_u$ and $[\,\cdot\,]_{Q_\lambda}$ denote equivalence classes modulo $\mathcal{N}_u$ and Q_λ, resp. It is clear that Φ is a linear mapping. It is also an isometry because, by Lemma 6.5, the essential sup-norm of $f \in \mathbb{B}_{i\pi}$ modulo $\mathcal{N}_u$ is the same as the essential sup-norm modulo Q_λ-null sets. It follows that there exists an isometry R_π of $L_{\mu\pi}$ onto $\mathcal{H}_{\mu\pi}$ such that

$$R_\pi[f]_{Q_\lambda} = (R_E f)(e) = \int_{G^\infty} f(\omega)\, Q_e(d\omega) \tag{6.2}$$

for every $f \in \mathbb{B}_{i\pi}$.

Theorem 6.6 *R_π is an isometric weak* homeomorphism of $L_{\mu\pi}$ onto $\mathcal{H}_{\mu\pi}$.*

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Proof It remains to prove that R_π and R_π^{-1} are weakly* continuous. By Lemma 4.8 it is enough to prove this for R_π. Using the Krein–Smulian theorem, it suffices to show that the restriction of R_π to the closed unit ball of $L_{\mu\pi}$ is weakly* continuous at zero. Let F_j be a net in the closed unit ball of $L_{\mu\pi}$, converging weakly* to 0. It is enough to show that $\lim_j \langle x_*\,, R_\pi F_j\rangle = 0$ for all x_* in a dense subset D of E_*.

Using an approximate identity in $L^1(G) \subseteq M(G)$, one can see that the set

$$D = \Big\{ \int_G \pi_*(g)x_*\, \sigma(dg)\ ;\ \sigma \in L^1(G), x_* \in E_* \Big\}$$

is dense in E_*. Using Lemma 6.4, for each i choose a representative f_j of F_j in $\mathbb{B}_{i\pi}$. Let $y_* = \int_G \pi_*(g)x_*\,\sigma(dg) \in D$. Then

$$\begin{aligned}
(6.3)\qquad \langle y_*\,, R_\pi F_j\rangle &= \int_{G^\infty} \langle y_*\,, f_j(\omega)\rangle\, Q_e(d\omega)\\
&= \int_{G^\infty} \Big(\int_G \langle \pi_*(g)x_*\,, f_j(\omega)\rangle \sigma\,(dg) \Big)\, Q_e\,(d\omega)\\
&= \int_{G^\infty} \Big(\int_G \langle x_*\,, \pi(g^{-1}) f_j(\omega)\rangle \sigma\,(dg) \Big)\, Q_e\,(d\omega)\\
&= \int_{G^\infty} \Big(\int_G \langle x_*\,, f_j(g^{-1}\omega)\rangle \sigma\,(dg) \Big)\, Q_e\,(d\omega)\\
&= \int_{G^\infty} \langle x_*\,, f_j(\omega)\rangle\, Q_{\tilde{\sigma}}\,(d\omega).
\end{aligned}$$

Let β be a finite measure equivalent to λ. Then $Q_{\tilde{\sigma}} \ll Q_\beta$. Let s be a version of the Radon–Nikodym derivative of the restriction of $Q_{\tilde{\sigma}}$ to $\mathcal{B}^{(i)}$ with respect to the restriction of Q_β to $\mathcal{B}^{(i)}$. Then

$$\langle y_*\,, R_\pi F_j\rangle = \int_{G^\infty} \langle s(\omega)x_*\,, f_j(\omega)\rangle Q_\beta(d\omega).$$

Since the function $\omega \to s(\omega)x_*$ defines an element of $L_i^1(\mu, E_*) = L_i^\infty(\mu, E)_*$, and w*-$\lim_j F_j = 0$, it becomes clear that $\lim_j \langle y_*\,, R_\pi F_j\rangle = 0$. ■

Remark 6.7 Let $(\mathcal{X}, \mathcal{A}, \alpha)$ be a μ-boundary which is a standard Borel space with the mapping $G \times \mathcal{X} \ni (g, t) \to gt \in \mathcal{X}$ Borel, and let Φ be the equivariant isomorphism of $L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$ onto $L_i^\infty(\mu)$. It can be shown [20] that there exists a $\mathcal{B}^{(i)}$-measurable function $F\colon G^\infty \to \mathcal{X}$ which induces Φ in the sense that $\Phi f = f \circ F$, $f \in L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$, and is itself equivariant when restricted to a suitable G-invariant subset $\Omega \in \mathcal{B}^{(i)}$. The measure $\rho = FQ_e$ is the Poisson kernel on the μ-boundary $(\mathcal{X}, \mathcal{A}, \alpha)$:

$$(R\Phi f)(g) = \int_{\mathcal{X}} f(gt)\rho\,(dt)\ \ (\mathrm{mod}\ \lambda)$$

for every $f \in L^\infty(\mathcal{X}, \mathcal{A}, \alpha)$. [6]

Using the mapping F, one can obtain a version of the Poisson formula (6.2) on the standard μ-boundary $(\mathcal{X}, \mathcal{A}, \alpha)$. This can be particularly useful when a "natural" realization of the μ-boundary is known. For example, when G is almost connected and μ is spread out, then the natural pointwise realization of the μ-boundary turns out to be a homogeneous space G/H of G [24]; $\mathcal{H}_{\mu\pi}$ is then isomorphic to the space of equivariant maps from G/H to E, modulo the family of null sets of G/H.

Remark 6.8 Theorem 6.6 applies, in particular, to the classical Choquet–Deny equation in $E = L^\infty(G)$. In this case $L_{\mu\pi}$ must therefore be isomorphic to $L_i^\infty(\mu)$. The isomorphism $\Psi = R_\pi^{-1}R \colon L_i^\infty(\mu) \to L_{\mu\pi}$ (where R is as in Proposition 4.5) can be described explicitly as follows. Given $f \in \mathbb{B}_i$, let $\dot{f}$ denote the function $\dot{f} \colon G^\infty \times G \to \mathbb{C}$ given by $\dot{f}(\omega, g) = f(g\omega)$. Using the fact that the mapping $G^\infty \times G \ni (\omega, g) \to g\omega \in G^\infty$ is measurable with respect to the σ-algebras $\mathcal{B}^\infty \times \mathcal{B}$ and $\mathcal{B}^\infty$, it follows that for a fixed ω, $f(\omega, \cdot\,) \in \mathbb{B}(G)$, and for every $\nu \in L^1(G) \subseteq M(G)$, the function $\omega \to \int_G \dot{f}(\omega, g)\,\nu(dg) \in \mathbb{C}$ is $\mathcal{B}^{(i)}$-measurable. Hence, if we define $\ddot{f} \colon G^\infty \to L^\infty(G)$ by $\ddot{f}(\omega) = [\dot{f}(\omega, \cdot\,)]_\lambda$, then $\ddot{f} \in \mathbb{B}_i(L^\infty(G))$. It is immediate that $\ddot{f} \in \mathbb{B}_{i\pi}(L^\infty(G))$, so that $[\ddot{f}]_{Q_\lambda} \in L_{\mu\pi}$. Moreover, $R_\pi[\ddot{f}]_{Q_\lambda} = R[f]_{Q_\lambda}$. Therefore Ψ is given by $\Psi[f]_{Q_\lambda} = [\ddot{f}]_{Q_\lambda}$ for every $f \in \mathbb{B}_i$. We also note that Ψ is equivariant with respect to the natural actions of G on $L_i^\infty(\mu)$ and $L_{\mu\pi}$ (the left regular representations).

Remark 6.9 Let $\mathcal{F}_{\mu\pi} \subseteq \mathcal{H}_{\mu\pi}$ denote the subspace of trivial solutions of the Choquet–Deny equation in E. Recall that $x \in \mathcal{F}_{\mu\pi}$ if and only if $\pi(g)x = x$ for every $g \in G_\mu$. Let $F_{\mu\pi} = R_\pi^{-1}\mathcal{F}_{\mu\pi}$. We claim that given $f \in \mathbb{B}_{i\pi}$, $[f]_{Q_\lambda} \in F_{\mu\pi}$ if and only if f is constant Q_e-a.e.

Indeed, necessity follows from the last statement of Theorem 6.1: if $x = R_\pi[f]_{Q_\lambda}$, then

$$f(\omega) = \underset{n\to\infty}{\mathrm{w}^*\text{-}\lim}\, \pi(\omega_n)x \quad Q_e\text{-a.e.}$$

But

$$\int_{G^\infty} \|\pi(\omega_n)x - x\|\, Q_e(d\omega) = \int_G \|\pi(g)x - x\|\, \mu^n\,(dg) = 0.$$

So for each n, $\pi(\omega_n)x = x$ Q_e-a.e. and therefore $f(\omega) = x$ Q_e-a.e. Conversely, suppose that there is a set $\Omega \in \mathcal{B}^{(i)}$ with $Q_e(\Omega) = 1$ and $f(\omega) = x$ for every $\omega \in \Omega$. Using (4.7) and (4.2) we obtain that $Q_g(\Omega) = 1$ for μ-a.e. $g \in G$. Hence,

$$\pi(g)x = \int_{G^\infty} \pi(g)f(\omega)\,Q_e(d\omega) = \int_{G^\infty} f(g\omega)\,Q_e(d\omega) = \int_{G^\infty} f(\omega)\,Q_g(d\omega) = x$$

for μ-a.e. $g \in G$. Since the mapping $g \to \pi(g)x$ is continuous with respect to the weak* topology on E, this implies that $\pi(g)x = x$ for every $g \in G_\mu$.

[6]This is shown in [20] for so-called continuous μ-boundaries. The result for standard μ-boundaries requires a little extra work relying on the theory of pointwise realizations of L^∞-spaces and homomorphisms between them [29].

The above description of $F_{\mu\pi}$ simplifies when $G = G_\mu$ because in this case the constant function $f(\omega) = x$, $x \in \mathcal{F}_{\mu\pi}$, belongs to $\mathbb{B}_{i\pi}$. Therefore in this case $F_{\mu\pi}$ consists of equivalence classes modulo Q_λ of such constant functions.

For a general μ, $\mathcal{H}_{\mu\pi} = \mathcal{F}_{\mu\pi}$ if and only if every $f \in \mathbb{B}_{i\pi}$ is constant modulo Q_e. This will be the case whenever the classical Choquet–Deny equation in $L^\infty(G)$ has only trivial solutions. However, recall that for a certain type of representations we have $\mathcal{H}_{\mu\pi} = \mathcal{F}_{\mu\pi}$, regardless of what $\mathcal{H}_\mu$ is (*cf.* Proposition 2.2, Corollaries 2.3 and 2.4, and Proposition 2.5). Thus for certain types of representations it is always true that every bounded $\mathcal{B}^{(i)}$-measurable equivariant function $f\colon G^\infty \to E$ is constant modulo Q_e (and modulo Q_λ when $G = G_\mu$). This seems to be an interesting and completely unexplored property of the μ-boundaries.

Example 1 (continued) The solutions of the Choquet–Deny equation in $E = M(\mathcal{X})$ are in one-to-one correspondence with equivalence classes modulo Q_λ of bounded $\mathcal{B}^{(i)}$-measurable equivariant functions $f\colon G^\infty \to M(\mathcal{X})$. Suppose that $\sigma = R_\pi\varphi$ is a μ-stationary measure, *i.e.*, $\sigma \in \mathcal{H}_{\mu\pi} \cap M_1(\mathcal{X})$. Using the last statement of Theorem 6.1, it is easy to see that $\varphi(\omega) \in M_1(\mathcal{X})$ Q_λ-a.e. More precisely, $\varphi = [f]_{Q_\lambda}$ where $f \in \mathbb{B}_{i\pi}$ is such that $f(\omega) \in M_1(\mathcal{X})$ for every ω in a universally conull G-invariant set $\Omega \in \mathcal{B}^{(i)}$. Let $\mathbb{B}_{i\pi 1}$ denote the subset of $\mathbb{B}_{i\pi}$ consisting of such functions. Thus μ-stationary measures are in one-to-one correspondence with equivalence classes of the elements of $\mathbb{B}_{i\pi 1}$.

In general, the set of μ-stationary measures, and therefore also $\mathbb{B}_{i\pi 1}$, can be empty (*cf.* Corollary 2.3). It is never empty when $\mathcal{X}$ is compact. In this case the existence of an equivariant function $f \in \mathbb{B}_{i\pi 1}$ can be also deduced using [46, Proposition 4.3.9] and the fact that the action of G on every μ-boundary is amenable. The fact that the existence of a μ-stationary measure on a locally compact second countable G-space implies the existence of an equivariant function $f \in \mathbb{B}_{i\pi 1}$ was first observed and exploited by Furstenberg [10]. $\Diamond$

Suppose that E is a W^*-algebra and $\pi(g) \in \operatorname{Aut}(E)$ for every $g \in G$. Then $L_{\mu\pi}$ is a weakly* closed $*$-subalgebra of $L_i^\infty(\mu, E) = L_i^\infty(\mu) \otimes E$, and so both $L_{\mu\pi}$ and $\mathcal{H}_{\mu\pi}$ are themselves W^*-algebras when the multiplication, $\diamond$, in $\mathcal{H}_{\mu\pi}$ is given by

$$x_1 \diamond x_2 = R_\pi[(R_\pi^{-1}x_1)(R_\pi^{-1}x_1)].$$

A simple computation and Corollary 5.4 yield the following.

Corollary 6.10 *If E is a W^*-algebra and $\pi(g) \in \operatorname{Aut}(E)$ for every $g \in G$, then for every $x \in \mathcal{H}_{\mu\pi}$ the sequence $\pi(\omega_n)x$ converges Q_λ-a.e. in the σ-strong* topology to $R_\pi^{-1}x$. Moreover, given $x_1, x_2 \in \mathcal{H}_{\mu\pi}$, the sequence $\pi(\mu^n)(x_1x_2)$ converges in the σ-strong* topology to $x_1 \diamond x_2$.*

Evidently, the product $\diamond$ coincides with the product of E whenever the Choquet–Deny theorem is true for μ. We note that in the setting of Example 2, since $\mathcal{H}_{\mu\pi}$ contains a copy of $\mathcal{H}_\mu$, the product $\diamond$ coincides with the product of $B(L^2(G))$ if and only if the Choquet–Deny theorem is true.

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7 Approximation Properties in the Predual and Projections

The goal of this concluding section is to present some additional properties of the space of μ-harmonic vectors and the Poisson formula. We will continue to work in the setting of Section 6. Recall that G is assumed second countable.

We begin with a few results which link the predual $L_{\mu\pi*}$ of $L_{\mu\pi}$ to the predual of E. Let $R_{\pi*}\colon E_* \to L_{\mu\pi*}$ denote the preadjoint of R_π and $(E_*)_1$ and $(L_{\mu\pi*})_1$ the unit balls in E_* and $L_{\mu\pi*}$, resp. Our first result summarizes general properties of weakly* continuous isometries between dual Banach spaces.

Theorem 7.1 *$R_{\pi*}E_* = L_{\mu\pi*}$ and $R_{\pi*}((E_*)_1)$ is norm dense in $(L_{\mu\pi*})_1$.*

Theorem 7.2 *If E is a W^*-algebra and $\pi(g) \in \operatorname{Aut}(E)$ for every $g \in G$, then $R_{\pi*}N_E$ is norm dense in $N_{L_{\mu\pi}}$, where N_E and $N_{L_{\mu\pi}}$ denote the sets of normal states on E and $L_{\mu\pi}$, resp.*

Proof Since R_π preserves positivity and maps the unit of $L_{\mu\pi}$ to the unit of E, it is clear that $R_{\pi*}N_E \subseteq N_{L_{\mu\pi}}$. The density follows by a routine application of the Hahn–Banach theorem. ■

Recall that in the case of the classical Choquet–Deny equation in $L^\infty(G)$, $R_{\pi*}\varphi = \varphi * Q_e$ for every $\varphi \in L^1(G) \subseteq M(G)$. So Theorem 7.2 says that probability measures of the form $\varphi * Q_e$, where φ is a probability measure in $L^1(G) \subseteq M(G)$, are norm dense in the set of probability measures in $L^1_i(\mu)$. When μ is spread out, it can be shown that this is equivalent to the condition that the convex hull of the orbit of Q_e under the natural action of G be dense in the set of probability measures in $L^1_i(\mu)$. An abstraction of this property of the action of G on the μ-boundary is the concept of a strongly approximately transitive action, which plays an important role in the theory of μ-boundaries [21–24]. When μ is not necessarily spread out, Theorem 7.2 implies that the action of G on the μ-boundary is approximately transitive, a concept first introduced by Connes and Woods in the context of the theory of von Neumann algebras [6].

Our next result is a generalization of [9, Théorème 1].

Theorem 7.3 *If $x_* \in E_*$, then*

$$\|R_{\pi*}x_*\| = \inf_{n\geq 1}\Big\| \frac{1}{n}\sum_{i=1}^{n} \pi_*(\widetilde{\mu}^i)x_*\Big\| = \lim_{n\to\infty}\Big\| \frac{1}{n}\sum_{i=1}^{n} \pi_*(\widetilde{\mu}^i)x_*\Big\|.$$

Proof The sequence $\| \sum_{i=1}^n \pi_*(\widetilde{\mu}^i)x_*\|$, $n = 1, 2, \dots$ is subadditive. Hence, the second equality is elementary. Moreover, from the identity $\pi(\mu)R_\pi = R_\pi$, it immediately follows that

$$\|R_{\pi*}x_*\| \leq \lim_{n\to\infty}\Big\| \frac{1}{n}\sum_{i=1}^{n} \pi_*(\widetilde{\mu}^i)x_*\Big\|.$$

It remains to establish the opposite inequality.

Let $\varepsilon > 0$ be given. Then for every $n \geq 1$ we can find $x_n \in E$ with $\|x_n\| \leq 1$ and

$$|\langle x_* , \tfrac{1}{n}\sum_{i=1}^{n} \pi(\mu^i)x_n\rangle| = |\langle \tfrac{1}{n}\sum_{i=1}^{n} \pi_*(\widetilde{\mu}^i)x_* , x_n\rangle| \geq \Big\| \tfrac{1}{n}\sum_{i=1}^{n} \pi_*(\widetilde{\mu}^i)x_*\Big\| - \varepsilon.$$

Define z_n by $z_n = \frac{1}{n}\sum_{i=1}^{n} \pi(\mu^i)x_n$. Clearly, z_n is a sequence in the closed unit ball of E. Since this ball is weakly* compact there is a convergent subnet z_{n_j} and, obviously, $z = \text{w}^*\text{-}\lim_j z_{n_j} \in \mathcal{H}_{\mu\pi}$. Hence, there is ζ in the closed unit ball of $L_{\mu\pi}$ with $z = R_\pi\zeta$. Thus

$$\begin{aligned}\lim_{n\to\infty}\Big\| \tfrac{1}{n}\sum_{i=1}^{n} \pi_*(\widetilde{\mu}^i)x_*\Big\| - \varepsilon &= \lim_j\Big\| \tfrac{1}{n_j}\sum_{i=1}^{n_j} \pi_*(\widetilde{\mu}^i)x_*\Big\| - \varepsilon \\ &\leq \lim_j\Big| \Big\langle x_* , \tfrac{1}{n_j}\sum_{i=1}^{n_j} \pi(\mu^i)x_{n_j}\Big\rangle \Big| \\ &= |\langle x_* , R_\pi\zeta\rangle| = |\langle R_{\pi*}x_* , \zeta\rangle| \leq \|R_{\pi*}x_*\|.\end{aligned}$$

Since ε is arbitrary, we are done. ∎

We will now prove the existence of a norm 1 projection of E onto $\mathcal{H}_{\mu\pi}$ and explore some of the consequences of this result.

Theorem 7.4 *If $\mathcal{H}_{\mu\pi} \neq \{0\}$, then there exists a projection K, of norm 1, of E onto $\mathcal{H}_{\mu\pi}$, which commutes with every weakly* continuous linear operator $T\colon E \to E$ commuting with π. If E is a W^*-algebra and $\pi(g) \in \operatorname{Aut}(E)$ for every $g \in G$, then K can be chosen completely positive.*

Proof Let $\{n_j\}$ be an ultranet in $\mathbb{N}$ with $n_j \to \infty$. Since each closed ball in E is weakly* compact, the ultranet $\frac{1}{n_j}\sum_{i=1}^{n_j} \pi(\mu^i)x$ converges in the weak* topology, for every $x \in E$. We can thus define $K\colon E \to E$ by $Kx = \text{w}^*\text{-}\lim_j \frac{1}{n_j}\sum_{i=1}^{n_j} \pi(\mu^i)x$. It is easy to see that K has all the desired properties. To prove complete positivity use [36, Theorem 6.5]. ∎

Remark 7.5 Let K denote the projection constructed in the proof of Theorem 7.4. When G_μ is compact, by a version of the Ito–Kawada theorem [40, Theorem 2, p. 138], the sequence $\frac{1}{n}\sum_{i=1}^{n} \mu^i$ converges weakly* to the normalized Haar measure ω_{G_μ} of G_μ; therefore when G_μ is compact, $K = \pi(\omega_{G_\mu})$. On the other hand, when G_μ is not compact, Lemma 2.1 implies that the kernel of K will always contain the norm closed π-invariant subspace E_0 consisting of those vectors $x \in E$ for which the function $g \to \langle x_* , \pi(g)x\rangle$ belongs to $C_0(G)$ for every $x_* \in E_*$.

Example 3 Consider the classical Choquet–Deny equation in $E = L^\infty(G)$. It is clear from Remark 7.5 that the projection K constructed in the proof of Theorem 7.4

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is weakly* continuous if and only if G_μ is compact, and that when G_μ is not compact, then $K(C_0(G)) = \{0\}$. In fact, when G_μ is not compact, then not only the projection constructed in the proof of Theorem 7.4, but any bounded projection of $L^\infty(G)$ onto $\mathcal{H}_\mu$ which commutes with every weakly* continuous linear operator $T\colon L^\infty(G) \to L^\infty(G)$ commuting with π must necessarily vanish on $C_0(G)$ and therefore cannot be weakly* continuous.

Let us prove a slightly stronger result. Let $\mathrm{LUC}(G)$ denote the set of bounded left uniformly continuous functions $f\colon G \to \mathbb{C}$ and let $S\colon \mathrm{LUC}(G) \to \mathcal{H}_\mu$ be any bounded linear operator commuting with the left regular representation. Then $S(\mathrm{LUC}(G)) \subseteq \mathrm{LUC}(G)$, in particular, $S(C_0(G)) \subseteq \mathrm{LUC}(G)$. Hence, the mapping $C_0(G) \ni f \mapsto (Sf)(e)$ is a well-defined bounded linear functional on $C_0(G)$, and so there exists $\sigma \in M(G)$ such that $(Sf)(e) = \int_G f\,d\sigma$ for every $f \in C_0(G)$. Using the assumption that S commutes with the left regular representation, it follows that $S\restriction C_0(G) = \pi(\sigma)\restriction C_0(G)$ (where π is the right regular representation). Next, since for every $f \in C_0(G)$, $\pi(\mu)Sf = Sf$, *i.e.*, $\pi(\mu * \sigma)f = \pi(\sigma)f$, we obtain $\mu * \sigma = \sigma$, which is a version of the Choquet–Deny equation in $M(G)$ where the left, instead of right, regular representation is used. When G_μ is not compact, Corollary 2.3 yields $\sigma = 0$ and so S vanishes on $C_0(G)$. Hence we obtain the following.

Proposition 7.6 *If $S\colon \mathrm{LUC}(G) \to \mathcal{H}_\mu$ is a bounded linear operator commuting with the left regular representation, then S vanishes on $C_0(G)$.*

$\Diamond$

Example 2 (continued) Let us identify the elements of $L^\infty(G)$ with the corresponding multiplication operators in $L^2(G)$. Moreover, let

$$B_0(L^2(G)) = \{A \in B(L^2(G))\ ; \text{for every } T \in \mathcal{T}(L^2(G)) \\ \text{the function } g \mapsto \operatorname{tr}[T(\pi(g)A)] \text{ vanishes at infinity}\}.$$

So $B_0(L^2(G))$ is a norm-closed subspace containing the ideal of compact operators, as well as $C_0(G)$.

Proposition 7.7 *When G_μ is not compact, then every bounded linear operator $S\colon B(L^2(G)) \to \mathcal{H}_{\mu\pi}$ which commutes with every weakly* continuous operator commuting with π, vanishes on $B_0(L^2(G))$. In particular, any bounded projection of $B(L^2(G))$ onto $\mathcal{H}_{\mu\pi}$ which commutes with every weakly* continuous operator commuting with π vanishes on $B_0(L^2(G))$ and therefore cannot be weakly* continuous (in fact, is even singular).*

Proof Given $\xi, \eta \in L^2(G)$, let $\Psi_{\xi\eta}\colon B(L^2(G)) \to \mathrm{LUC}(G) \subseteq B(L^2(G))$ denote the linear mapping which associates with each $A \in B(L^2(G))$ the multiplication operator defined by the left uniformly continuous function $h_{\xi\eta A}(g) = \langle(\pi(g)A)\xi\,,\eta\rangle$. It is easy to see that when A_α is a norm bounded net in $B(L^2(G))$ which converges to zero weakly*, the net $h_{\xi\eta A_\alpha}$ converges to zero uniformly on compacta. This implies, via

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a routine argument, that $\Psi_{\xi\eta}$ is weakly* continuous. It is also straighforward to see that $\Psi_{\xi\eta}$ commutes with π.

Let e_j be a net of positive integrable functions on G forming a bounded approximate identity in $L^1(G)$. Since for a given $f \in \mathrm{LUC}(G)$, $\Psi_{\sqrt{e_j}\sqrt{e_j}} f = e_j^\star * f$, where $\star$ denotes the involution in $L^1(G)$, it follows that the net $\Psi_{\sqrt{e_j}\sqrt{e_j}} \restriction \mathrm{LUC}(G)$ converges in the strong operator topology to the identity operator on $\mathrm{LUC}(G)$. Now, if $T \in B(B(L^2(G)))$ commutes with every weakly* continuous operator commuting with π, then $T\Psi_{\sqrt{e_j}\sqrt{e_j}} = \Psi_{\sqrt{e_j}\sqrt{e_j}} T$, and it follows that $T(\mathrm{LUC}(G)) \subseteq \mathrm{LUC}(G)$. This applies, in particular, to our operator $S\colon B(L^2(G)) \to \mathcal{H}_{\mu\pi}$. Therefore

$$S(\mathrm{LUC}(G)) \subseteq \mathrm{LUC}(G) \cap \mathcal{H}_{\mu\pi} \subseteq \mathcal{H}_{\mu}.$$

Thus, using Proposition 7.6, we conclude that $S(C_0(G)) = \{0\}$. But

$$\Psi_{\xi\eta}(B_0(L^2(G))) \subseteq C_0(G)$$

for each $\xi, \eta \in L^2(G)$. Hence,

$$\Psi_{\xi\eta} S(B_0(L^2(G))) = S\Psi_{\xi\eta}(B_0(L^2(G))) \subseteq S(C_0(G)) = \{0\}.$$

Since this holds for arbitrary $\xi, \eta \in L^2(G)$, $S(B_0(L^2(G))) = \{0\}$, as claimed. ■

We wish to point out that Proposition 7.7 does not rely on the Hilbert space setting: the proof can be modified to yield an analogous result for $L^p(G)$, $1 < p < \infty$, with the obvious definition of $B_0(L^p(G))$. ♢

Corollary 7.8 *Suppose that E is a W^*-algebra and $\pi(g) \in \mathrm{Aut}(E)$ for every $g \in G$. If $\mathcal{H}_{\mu\pi} \neq \{0\}$ and E is injective, then $\mathcal{H}_{\mu\pi}$ is also injective.*

Proof We will show that $L_{\mu\pi}$ is injective. Since $L_i^\infty(\mu, E) = L_i^\infty(\mu) \otimes E$ is injective [41, p. 120], it suffices to construct a norm 1 projection Λ of $L_i^\infty(\mu, E)$ onto $L_{\mu\pi}$.

Let η be any absolutely continuous probability measure on G. The formula

$$Jf = \int_G \pi(g)^{-1} f(g)\,\eta(dg)$$

defines a contraction J of $L^\infty(G, E)$ into E such that $J[\pi(\,\cdot\,)x]_\lambda = x$ for every $x \in E$. Define $\widehat{K} = KJ$ where K is the projection described in Theorem 7.4. Then $\widehat{K}$ is a contraction of $L^\infty(G, E)$ onto $\mathcal{H}_{\mu\pi}$ such that $\widehat{K}[\pi(\,\cdot\,)x]_\lambda = x$ for every $x \in \mathcal{H}_{\mu\pi}$.

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Next, (5.3) defines a contraction, which we will denote again by R_E, of $L_i^\infty(\mu, E)$ into $L^\infty(G, E)$ (in fact, R_E is an isometry onto $\mathcal{H}_\mu(E)$, *cf.* Remark 5.5). Note that $R_E\varphi = [\pi(\cdot)R_\pi\varphi]_\lambda$ for every $\varphi \in L_{\mu\pi}$. Put $\Lambda = R_\pi^{-1}\widehat{K}R_E$. ■

Our final result involves the concept of an amenable action of a locally compact group on a W^*-algebra. The concept of an amenable action was first introduced by Zimmer [45] in the context of a measure class preserving action of a locally compact second countable group on a standard Borel space. It was subsequently shown [1] that Zimmer's definition is equivalent to the following: Let Γ be a locally compact second countable group and $\mathcal{X}$ a standard Borel Γ-space such that the mapping $\Gamma \times \mathcal{X} \ni (g, x) \to gx$ is Borel. Let α be a σ-finite quasiinvariant measure on $\mathcal{X}$. Then Γ acts on $L^\infty(\mathcal{X}, \alpha)$ and the resulting representation of Γ is the adjoint of the natural strongly continuous representation of Γ in $L^1(\mathcal{X}, \alpha)$. Consider the tensor product $L^\infty(\Gamma) \otimes L^\infty(\mathcal{X}, \alpha)$ equipped with the Γ-action which is the tensor product of the action of Γ on $L^\infty(\Gamma)$ by left translations and the action on $L^\infty(\mathcal{X}, \alpha)$. The action of Γ on $(\mathcal{X}, \alpha)$ is *amenable* if there exists a norm 1 equivariant projection of $L^\infty(\Gamma)\otimes L^\infty(\mathcal{X}, \alpha)$ onto $L^\infty(\mathcal{X}, \alpha)$. Motivated by the theory of W^*-crossed products, this definition was extended to the case when Γ acts on an arbitrary W^*-algebra [2]: Let E be a W^*-algebra and $\gamma\colon \Gamma \to \mathrm{Aut}(E)$ a representation of Γ which is the adjoint of a strongly continuous representation of Γ in E_*. The resulting action of Γ on E is called *amenable* if there exists a norm 1 equivariant projection P of $L^\infty(\Gamma) \otimes E$ onto E where the action of Γ on $L^\infty(\Gamma) \otimes E$ is the tensor product of the action of Γ on $L^\infty(\Gamma)$ by left translations and the action γ on E. We note that by [41, p. 116], P is necessarily a conditional expectation, *i.e.,* P preserves positivity and for all $z \in L^\infty(\Gamma) \otimes E$ and $x \in E$, we have $P(z(1 \otimes x)) = (Pz)x$ and $P((1 \otimes x)z) = xPz$. When Γ is amenable, γ is automatically amenable, but many actions of nonamenable groups are also amenable.

Recall that in the case of the classical Choquet–Deny equation in $L^\infty(G)$ the action of G on G by left translations gives rise to the natural action of G on any μ-boundary. Zimmer [45] showed that when μ is spread out, this boundary action is always amenable (granted that the μ-boundary is a standard Borel G-space). Connes and Woods [7] sketched a proof of amenability of the action of G on the space-time boundary of a random walk with time-dependent transition probabilities (the μ-boundaries discussed here can be viewed as a special case of such space-time boundaries). A simple proof of amenability of the action of G on boundaries of arbitrary random walks on amenable G-spaces, given in [15], is based on a version of Theorem 7.4. Here we adapt this proof to the setting of the Choquet–Deny equation in a W^*-algebra E.

Let Γ and γ be as described above in the definition of amenablility of the action on the W^*-algebra E, and let us assume that γ and π commute. Then $\mathcal{H}_{\mu\pi}$ is a Γ-invariant subspace of E, and by Corollary 6.10, $\gamma(g)\restriction\mathcal{H}_{\mu\pi} \in \mathrm{Aut}(\mathcal{H}_{\mu\pi})$ for every $g \in \Gamma$. Thus Γ acts on $\mathcal{H}_{\mu\pi}$.

Corollary 7.9 *If γ and π commute, Γ acts amenably on the W^*-algebra E, and $\mathcal{H}_{\mu\pi} \neq \{0\}$, then Γ acts amenably on $\mathcal{H}_{\mu\pi}$.*

Proof Let $P\colon L^\infty(\Gamma) \otimes E \to E$ be the projection appearing in the definition of amenability of the action of Γ on E, and let K be the projection described in Theorem 7.4. Then $KP\restriction(L^\infty(\Gamma) \otimes \mathcal{H}_{\mu\pi})$ is a norm 1 equivariant projection of $L^\infty(\Gamma) \otimes \mathcal{H}_{\mu\pi}$ onto $\mathcal{H}_{\mu\pi}$. ■

Amenability of the natural action of $\Gamma = G$ on a μ-boundary $(\mathcal{X}, \alpha)$ follows as a special case of Corollary 7.9 because $L^\infty(\mathcal{X}, \alpha) \cong \mathcal{H}_\mu$ and G always acts amenably on $L^\infty(G)$.

We note that in the case of the "noncommutative" Choquet–Deny equation considered in Example 2, the action of $\Gamma = G$ on $B(L^2(G))$ associated with the left regular representation is amenable if and only if G is amenable, by [3, Corollaire 3.7]. The trivial example $\mu = \delta_e$ indicates that, in general, when G fails to be amenable, the action of G on the W^*-algebra $\mathcal{H}_{\mu\pi}$ of the μ-harmonic operators need not be amenable.

Note added in proof Some time after submission of the present paper, the authors discovered the article [14] by Izumi where Example 2 is studied and the structure formula $L_i^\infty(\mu) \times_{\pi_l} G$ is obtained for the special case of a countable discrete group G. Izumi asks (Problem 4.3) if the latter holds for an arbitrary locally compact second countable group. Our Proposition 6.3 answers this question in the affirmative. ◊

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References

[1] S. Adams, G.A. Elliott, and T. Giordano, *Amenable actions of groups.* Trans. Amer. Math. Soc. **344**(1994), no. 2, 803–822.

[2] C. Anantharaman-Delaroche, *Action moyennable d'un groupe localement compact sur une algèbre de von Neumann.* Math. Scand. **45**(1979), no. 2, 289–304.

[3] ———, *Action moyennable d'un groupe localement compact sur une algèbre de von Neumann. II.* Math. Scand. **50**(1982), no. 2, 251–268.

[4] R. Azencott, *Espaces de Poisson des groupes localement compacts.* Lecture Notes in Mathematics 148, Springer-Verlag, Berlin, 1970.

[5] C.-H. Chu and A. T. Lau, *Harmonic functions on groups and Fourier algebras.* Lecture Notes in Mathematics 1782, Springer-Verlag, Berlin 2002.

[6] A. Connes and E. J. Woods, *Approximately transitive flows and ITPFI factors.* Ergodic Theory Dynam. Systems **5**(1985), no. 2, 203–236.

[7] ———, *Hyperfinite von Neumann algebras and Poisson boundaries of time dependent random walks.* Pacific J. Math. **137**(1989), no. 2, 225–243.

[8] N. Dunford and J. T. Schwartz, *Linear Operators. I.* Wiley, New York 1988.

[9] Y. Derriennic, *Lois "zéro ou deux" pour les processus de Markov. Applications aux marches aléatoires.* Ann. Inst. H. Poincaré Sect. B (N.S.) **12**(1976), no. 2, 111–129.

[10] H. Furstenberg, *Boundary theory and stochastic processes on homogeneous spaces.* In: Harmonic Analysis on Homogeneous Spaces, Proc. Sympos. Pure Math. 26. American Mathematical Society, Providence, RI, 1973, pp. 193–229.

[11] ———, *Random walks and discrete subgroups of Lie groups.* Advances in Probability and Related Topics 1, Dekker, 1971, pp. 3–63.

[12] Y. Guivarc'h, *Extension d'un théorème de Choquet-Deny à une classe de groupes non abéliens.* Astérisque 4, Soc. Math. France, Paris, 1973, pp. 41–59.

[13] E. Hewitt and L. J. Savage, *Symmetric measures on Cartesian products.* Transactions Amer. Math. Soc. **80**(1955), 470–501.

[14] M. Izumi, *Non-commutative Poisson boundaries.* Contemp. Math. **347**(2004), 69–81.

[15] W. Jaworski, *Poisson and Furstenberg Boundaries of Random Walks.* Ph.D. Thesis, Queen's University, 1991.

[16] ———, *Poisson and Furstenberg boundaries of random walks.* C. R. Math. Rep. Acad. Sci. Canada **13**(1991), no. 6, 279–284.

[17] ———, *Countable amenable identity excluding groups.* Canad. Math. Bull. **47**(2004)), no. 2, 215–228.

[18] ———, *Probability measures on almost connected amenable locally compact groups and some related ideals in group algebras.* Illinois J. Math. **45**(2001), no. 1, 195–212.

[19] ———, *Ergodic and mixing probability measures on* [SIN] *groups.* J. Theoret. Probab. **17**(2004), 741–759.

[20] ———, *The asymptotic σ-algebra of a recurrent random walk on a locally compact group.* Israel J. Math. **94**(1996), 201–219.

[21] ———, *Strongly approximately transitive group actions, the Choquet-Deny theorem, and polynomial growth.* Pacific J. Math. **165**(1994), no. 1, 115–129.

[22] ———, *Strong approximate transitivity, polynomial growth, and spread out random walks on locally compact groups.* Pacific J. Math. **170**(1995), no. 2, 517–533.

[23] ———, *A Poisson formula for solvable Lie groups.* J. Anal. Math. **68**(1996), 183–208.

[24] ———, *Random walks on almost connected locally compact groups: boundary and convergence.* J. Anal. Math.**74**(1998), 235–273.

[25] V. A. Kaimanovich, *The Poisson boundary of polycyclic groups.* In: Probability Measures on Groups and Related Structures XI. World Sci. Publ., River Edge, NJ, 1995 pp. 182–195.

[26] ———, *The Poisson formula for groups with hyperbolic properties.* Ann. of Math.**152**(2000), no. 3, 659–692.

[27] V. A. Kaimanovich and A. M. Vershik, *Random walks on discrete groups: boundary and entropy.* Ann. Probab. **11**(1983), no. 3, 457–490.

[28] F. Ledrappier, *Some asymptotic properties of random walks on free groups.* In: Topics in Probability and Lie Groups: Boundary Theory. CRM Proc. Lecture Notes 28, American Mathematical Society. Providence, RI, 2001 pp. 117–152, .

[29] G. W. Mackey, *Point realizations of transformation groups.* Illinois J. Math. **6**(1962), 327–335.

[30] P.-A. Meyer, *Probabilités et potentiel.* Hermann, Paris, 1966.

[31] A. Mukherjea, *Limit theorems for probability measures on non-compact groups and semigroups.* Z. Wahrscheinlichkeitstheorie Verw. Gebiete **33**(1975/1976), no. 4, 273–284.

[32] M. Neufang, *Abstrakte harmonische Analyse und Modulhomomorphismen über von Neumann-Algebren.* Ph.D. Thesis, Universität des Saarlandes, 2000.

[33] M. Neufang, *A quantized analogue of the convolution algebra $L_1(G)$.* Preprint, `http://mathstat.carleton.ca/~mneufang`

[34] J. Neveu, *Mathematical Foundations of the Calculus of Probability.* Holden-Day, San Francisco, 1965.

[35] ———, *Discrete-Parameter Martingales.* North-Holland, Amsterdam, 1975.

[36] V.I. Paulsen, *Completely Bounded Maps and Dilations.* Pitman Research Notes in Mathematics Series 146, Longman, Scientific & Technical, Harlow, 1986.

[37] A. Pirkovskii, *Biprojectivity and biflatness for convolution algebras of nuclear operators.* Canad. Math. Bull. **47**(2004), no. 3, 445–455.

[38] D. Revuz, *Markov Chains.* Second edition. North Holland Mathematical Library 11, North Holland Publishing, Amsterdam, 1984.

[39] J. Rosenblatt, *Ergodic and mixing random walks on locally compact groups.* Math. Ann **257**(1981), no. 1, 31–42.

[40] M. Rosenblatt, *Markov Processes: Structure and Asymptotic Behaviour.* Die Grundlehren der Mathematischen Wissenschaften 184. Springer-Verlag, New York, 1971.

[41] S. Stratila, *Modular Theory in Operator Algebras.* Abacus Press, Tunbridge Wells, 1981.

[42] V. S. Varadarajan, *Groups of automorphisms of Borel spaces.* Trans. Amer. Math. Soc. **109**(1963), 191–220.

[43] G. Willis, *Probability measures on groups and some related ideals in group algebras.* J. Funct. Anal. **92**(1990), no. 1, 202–263.

[44] ———, *Factorization in finite-codimensional ideals in group algebras.* Proc. London Math. Soc. **82**(2001), no. 3, 676–700.

[45] R. J. Zimmer, *Amenable ergodic group actions and an application to Poisson boundaries of random walks.* J. Funct. Anal. **27**(1978), no. 3, 350–372.

[46] ———, *Ergodic Theory and Semisimple Lie Groups.* Monographs in Mathematics 81. Birkhäuser Verlag, Basel, 1984.

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