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INJECTIVE HULLS OF TORSION FREE MODULES

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Introduction. In § 1, we begin with a basic theorem which describes a convenient embedding of a nonsingular left R-module into a complete direct product of copies of the left injective hull of R (Theorem 2). Several applications follow immediately. Notably, the injective hull of a finitely generated nonsingular left R-module is isomorphic to a direct sum of injective hulls of closed left ideals of R (Corollary 4). In particular, when R is left self-injective, every finitely generated nonsingular left R-module is isomorphic to a finite direct sum of injective left ideals (Corollary 6).

In § 2, where it is assumed for the first time that rings have identity elements, we investigate more generally the class of left R-modules which are embeddable in direct products of copies of the left injective hull Q of R. Such modules are called torsion free, and can also be characterized by the property that no nonzero element is annihilated by a dense left ideal of R (Proposition 12). When Q is a left quotient ring of R, we show that every finitely generated torsion free left R-module is torsionless if and only if Q is a right quotient ring of R (Corollary 13). An examination of the injective hull of a torsion free module follows, and provides convenient sufficient conditions for the embedding of torsion free modules into free modules. Specifically, if R has an artinian left quotient ring then every finitely generated torsionless R-module can be embedded in a free module (Theorem 18). Combining this with an earlier result we learn that if Q is an artinian left and right quotient ring of R, then every finitely generated torsion free R-module can be embedded in a free module (Corollary 20).

1. Throughout this section, all rings are associative, but do not necessarily contain identity elements. When a ring R fails to have an identity element, we let R^1 denote a ring with identity in which R is embedded in the usual manner. Except as indicated, all modules will be left R-modules and module homomorphisims will be written on the right. The injective hull of an R-module M is denoted $\hat{M}$. $Q = \hat{R}$ will always indicate the left injective hull of R. For all notions regarding injective modules over rings without identity elements we refer the reader to [**2**].

Let M be an R-module. For any submodule N of M and any subset K of M we set $(N{:}K) = \{r \in R \mid rK \subseteq N\}$, a left ideal of R. $Z(M) = \{m \in M \mid (0{:}m)$ is an essential left ideal of $R\}$ is a submodule, called the singular submodule of

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M. If $Z(M) = 0$ we say that M is *nonsingular*. We write $N \subseteq' M$ to indicate that N is an essential submodule of M. A submodule N of M is *closed* in M if it has no proper essential extensions in M. (The zero submodule is closed.) We remark that if N is a submodule of M and ${}_RN'$ is chosen maximal with respect to $N \cap N' = 0$, then N' is a closed submodule of M and $N + N'$ is an essential submodule of M (such N' exist by Zorn's lemma). Finally, an R-module M is (*essentially*) Γ-*generated* if it is (an essential extension of a submodule) generated by Γ elements, Γ a cardinal number.

LEMMA 1. *Let N be a submodule of an R-module M and let $m \in M$.*

(1) *If N is essential in M then $(N:m)$ is an essential left ideal of R.*

(2) *If M is nonsingular and $(N:m)$ is an essential left ideal of R, then N is an essential submodule of $N + R^1m$.* (See [3].)

THEOREM 2. *Suppose that M is a nonsingular R-module which is essentially Γ-generated. Then there exists an embedding $\mu: M \rightarrow \prod_{0 \leqslant \beta < \alpha} Q_\beta$ where* card $\alpha = \Gamma$, *each $Q_\beta = Q$, and there exist closed left ideals $I_\beta \subseteq R \subseteq Q_\beta$ with $\sum_{0 \leqslant \beta < \alpha} \oplus I_\beta$ an essential submodule of $M\mu$.*

Proof. We may assume that M is an essential extension of $\sum_{0 \leqslant \beta < \alpha} R^1 m_\beta$, where card $\alpha = \Gamma$ and $\{m_\beta\}_{0 \leqslant \beta < \alpha} \subseteq M$. For each ordinal $\gamma \leqq \alpha$, set

$$M_\gamma = \sum_{0 \leqq \beta < \gamma} R^1 m_\beta,$$

$$Q^\gamma = \prod_{0 \leqq \beta < \gamma} Q_\beta.$$

We proceed to establish the theorem for M_α by a transfinite induction.

Let $\gamma \leqq \alpha$. If γ is not a limit ordinal, then by induction there exists an embedding $\mu_{\gamma-1}: M_{\gamma-1} \rightarrow Q^{\gamma-1}$ and closed left ideals $I_\beta \subseteq R \subseteq Q_\beta$ for $\beta < \gamma - 1$ with $\sum_{0 \leqslant \beta < -1} \oplus I_\beta$ an essential submodule of $M_{\gamma-1}\mu_{\gamma-1}$. If $M_{\gamma-1}$ is an essential submodule of M_γ, then since $Q^{\gamma-1}$ is injective, $\mu_{\gamma-1}$ extends to a monomorphism $\mu_\gamma': M_\gamma \rightarrow Q^{\gamma-1}$. Taking $I_{\gamma-1} = 0$ and setting μ_γ equal to μ_γ' composed with the canonical embedding of $Q^{\gamma-1}$ into $Q^{\gamma-1} \oplus Q_{\gamma-1} = Q^\gamma$, it is easy to check that μ_γ satisfies the desired requirements. If, on the other hand, $M_{\gamma-1}$ is not essential in $M_\gamma = M_{\gamma-1} + R^1 m_{\gamma-1}$, then by Lemma 1, $(M_{\gamma-1}:m_{\gamma-1})$ is not an essential left ideal of R. Hence, choosing a left ideal $I_{\gamma-1}$ maximal with respect to $I_{\gamma-1} \cap (M_{\gamma-1}:m_{\gamma-1}) = 0$, $I_{\gamma-1}$ is a closed left ideal of R and $I_{\gamma-1}m_{\gamma-1} \cong I_{\gamma-1}$ under the homomorphism $\mu_{\gamma-1}': I_{\gamma-1}m_{\gamma-1} \rightarrow Q_{\gamma-1}$ defined via $rm_{\gamma-1} \rightarrow r$, $r \in I_{\gamma-1}$ (this map is well-defined since $I_{\gamma-1} \cap (0:m_{\gamma-1}) = 0$). Next, note that $M_{\gamma-1} + I_{\gamma-1}m_{\gamma-1}$ (direct sum) is an essential submodule of M_γ. (Given $0 \neq m \in M_\gamma$, write $m = n_{\gamma-1} + am_{\gamma-1}$ with $n_{\gamma-1} \in M_{\gamma-1}$ and $a \in R^1$. There is no loss of generality in assuming that $a \notin (M_{\gamma-1}:m_{\gamma-1})$. By Lemma 1, $J = (I_{\gamma-1} + (M_{\gamma-1}:m_{\gamma-1}):a)$ is an essential left ideal of R, and so $Jm \neq 0$. Choose $0 \neq b \in J$ with $bm \neq 0$ and write $ba = c + d$, $c \in (M_{\gamma-1}:m_{\gamma-1})$, $d \in I_{\gamma-1}$. Then $0 \neq bm = bn_{\gamma-1} + bam_{\gamma-1} = (bn_{\gamma-1} + cm_{\gamma-1}) + dm_{\gamma-1} \in M_{\gamma-1} + I_{\gamma-1}m_{\gamma-1}$). Set $\mu_\gamma' = \mu_{\gamma-1} \oplus \mu_{\gamma-1}': M_{\gamma-1} + I_{\gamma-1}m_{\gamma-1} \rightarrow Q^{\gamma-1} \oplus Q_{\gamma-1} = Q^\gamma$.

μ_γ' is clearly a monomorphism. Since $M_{\gamma-1} + I_{\gamma-1}m_{\gamma-1}$ is an essential submodule of M_γ and Q^γ is injective, μ_γ' extends to a monomorphism $\mu_\gamma: M_\gamma \to Q^\gamma$. It is easy to check that $\sum_{0\leqslant\beta<\gamma} \oplus I_\beta$ is an essential submodule of $M_\gamma\mu_\gamma$. If γ is a limit ordinal, then define $\mu_\gamma = \bigcup_{\beta<\gamma}\mu_\beta: M_\beta \to Q^\beta$ (note that $Q^\gamma \supset \bigcup_{\beta<\gamma}Q^\beta$). The definition of μ_γ is consistent since μ_β extends μ_δ for every $\delta < \beta$. The remaining parts of the induction hypothesis are equally easy to verify. Thus we have a monomorphism $\mu_\alpha: M_\alpha \to Q^\alpha$, and since M_α is an essential submodule of M, we can take μ to be any extension of μ_α to M.

Remark. Reviewing the proof, we observe that since $\bigcup_{0\leqslant n<\infty}Q^n = \sum_{0\leqslant n<\infty} \oplus Q_n$, a countably generated nonsingular module can in fact be embedded in a countable direct sum of copies of Q.

COROLLARY 3. *If M is an essentially n-generated nonsingular R-module, then there is a monomorphism μ of M into Q^n, and closed left ideals $I_1, \ldots, I_n$* of R, with $I_1 \oplus \ldots \oplus I_n \subseteq' M\mu \subseteq Q^n$ *and each* $I_k \subseteq R \subseteq Q = Q_k$.

COROLLARY 4. *If M is a finitely generated nonsingular R-module, then there exist closed left ideals $I_1, \ldots, I_n$* of R *with* $\hat{M} \cong \hat{I}_1 \oplus \ldots \oplus \hat{I}_n$.

Proof. By Corollary 3, we have $\hat{M} \cong (M\mu)^\wedge \cong$ the injective hull of $I_1 \oplus \ldots \oplus I_n \cong \hat{I}_1 \oplus \ldots \oplus \hat{I}_n$.

COROLLARY 5. *If M is an n-generated nonsingular R-module, then there exist n-generated R-submodules $J_1, \ldots, J_n$ of Q with M isomorphic to an essential submodule of $J_1 \oplus \ldots \oplus J_n$.*

Proof. In the situation described by Corollary 3, we choose $\hat{I}_k$ to be an injective hull of I_k contained in $Q_k = Q$. By [**4**, p. 91], there exists an isomorphism ν of $(M\mu)^\wedge \subseteq Q^n$ onto $\hat{I}_1 \oplus \ldots \oplus \hat{I}_n$ which leaves $I_1 \oplus \ldots \oplus I_n$ elementwise fixed. We then have $I_1 \oplus \ldots \oplus I_n \subseteq M\mu\nu \subseteq' \hat{I}_1 \oplus \ldots \oplus \hat{I}_n$ with each $I_k \subseteq \hat{I}_k \subseteq Q_k$. Then $M\mu\nu\pi_k \subseteq \hat{I}_k \subseteq Q_k$ for each k, where π_k is the canonical projection of Q^n onto Q_k. Since each $M\mu\nu\pi_k$ is n-generated and $M\mu\nu \subseteq M\mu\nu\pi_1 \oplus \ldots \oplus M\mu\nu\pi_n$, the conclusion follows.

COROLLARY 6. *If R is a left self-injective ring, then every finitely generated nonsingular module is isomorphic to a finite direct sum of injective left ideals.*

Proof. A closed left ideal of a left self-injective ring is clearly injective, and so the conclusion follows from Corollary 3.

We remark that the hypothesis actually used in the proof of the previous Corollary is that every closed nonsingular left ideal of R is injective.

Recall that a module is *finite dimensional* if it contains no infinite direct sums of nonzero submodules. A finite dimensional module is essentially finitely generated. A ring R will be called finite dimensional if ${}_RR$ is finite dimensional.

COROLLARY 7. *If R is a finite dimensional left self-injective ring, then every nonsingular module is isomorphic to a direct sum of left ideals.*

Proof. Over a finite-dimensional ring, a direct sum of nonsingular injective modules is injective [**8**, Theorem 2.5]. This corollary thus follows from Theorem 2.

COROLLARY 8. *A left self-injective left nonsingular ring is regular* [**4**, p. 106].

Proof. By Corollary 6, every finitely generated left ideal is injective; hence, it is a direct summand of R^1, and therefore is generated by an idempotent.

COROLLARY 9. *Let R be a finite dimensional ring and M a nonsingular left R-module. Then M is finite-dimensional if and only if M is essentially finitely generated* [**7**, p. 226].

Proof. If M is essentially finitely generated, then M can be embedded in Q^n for some n. Since ${}_RR$ is finite-dimensional, so is ${}_RQ$, and hence so is every submodule of Q^n.

We next turn our attention to a sufficient condition for the singular submodule to split off.

COROLLARY 10. *If R is a regular left self-injective ring and M is an essentially finitely generated R-module, then $M = Z(M) \oplus N$, where N is isomorphic to a finite direct sum of finitely generated injective left ideals of R.*

Proof. By [**5**, p. 338], $Z(M/Z(M)) = 0$, since $Z(R) = 0$, and so $M/Z(M)$ is seen to be essentially finitely generated and nonsingular. By Corollary 6, $M/Z(M)$ is isomorphic to a finite direct sum of finitely generated injective left ideals. Since an injective left ideal is a direct summand of R^1, $M/Z(M)$ is projective. So $M = Z(M) \oplus N$ with $N \cong M/Z(M)$.

2. Henceforth, we will assume as a blanket hypothesis that R *has an identity element.*

A module ${}_RM$ is a *rational* extension of a submodule N if, given any $m \in M$ and $0 \neq m' \in M$, $(N:m)m' \neq 0$. S is a *left quotient ring* of R if S is a ring, R is a subring of S, and ${}_RS$ is a rational extension of ${}_RR$. For R-modules S and M, M will be called *S-torsionless* if ${}_RM$ can be embedded in a direct product of copies of S; equivalently, M is S-torsionless if for every $0 \neq m \in M$ there exists $f \in \mathrm{Hom}_R(M, S)$ with $mf \neq 0$. An R-torsionless R-module is called simply *torsionless.*

A left ideal D of R is *dense* if ${}_RR$ is a rational extension of D. For the purpose at hand, we require the following facts. Let D be a left ideal of R and S a left quotient ring of R. (Refer to [**4**] for details.)

(1) R is a dense left ideal of R.

(2) D is a dense left ideal of R if and only if $Dq \neq 0$ for all $0 \neq q \in Q$.

(3) $\bigcap_{i=1}^{n}(D:s_i)$ is a dense left ideal of R for any $s_1, \ldots, s_n \in S$ and dense left ideal D of R.

(4) If A and B are S-modules and the only element of B which is annihilated by a dense left ideal of R is the zero element, then $\mathrm{Hom}_R(A, B) \subseteq \mathrm{Hom}_S(A, B)$.

(This is easily proved by observing that for any $s \in S$, $a \in A$, and $f \in \mathrm{Hom}_R(A, B)$, $(sa)f - s(af)$ is annihilated by $(R:s)$.)

THEOREM 11. *For R a ring with identity element and left quotient ring S, the following conditions are equivalent.*
(1) *S is a right quotient ring of R.*
(2) *Every finitely generated R-submodule of S is torsionless.*
(3) *Every finitely generated S-torsionless R-module is torsionless.*

Proof. (1) $\Rightarrow$ (2). Assume that S is a right quotient ring of R. Let $M = Rs_1 + \ldots + Rs_n$ be a submodule of S. Set $D = \bigcap_{i=1}^{n}[R:s_i]$, where $[R:s_i] = \{r \in R | s_i r \in R\}$. Since S is a right quotient ring of R, D is a dense right ideal of R. Define $f: M \rightarrow \prod_{d \in D} R_d$, where each $R_d = R$, via $mf = \{md\}_{d \in D}$, $m \in M$. f is clearly an R-homomorphism, and $m \in \ker f$ if and only if $mD = 0$. Since D is a dense right ideal of R, $mD = 0$ implies that $m = 0$. So f is a monomorphism.

(2) $\Rightarrow$ (3). If M is a finitely generated submodule of $\prod_{i \in I} S_i$, each $S_i = S$, then $M \subseteq \prod_{i \in I} M\pi_i$ where π_i is the canonical projection map into S_i. Each $M\pi_i$ is torsionless because it is a finitely generated R-submodule of S. Finally, M, being a submodule of a direct product of torsionless modules, is torsionless.

(3) $\Rightarrow$ (1). (3) implies (2) trivially. We assume that (2) holds and let $M = Rs_1 + \ldots + Rs_n$ be a finitely generated, hence torsionless, left R-submodule of S. Set $[R:M] = \{r \in R | Mr \subseteq R\}$ and $l[R:M] = \{s \in S | s[R:M] = 0\}$; $[R:M]$ is a right ideal of R and $l[R:M]$ is a left ideal of S. We claim that $l[R:M] \cap M = 0$. Suppose that $0 \neq m \in l[R:M] \cap M$. Since ${}_RM$ is torsionless, there exists $f \in \mathrm{Hom}_R(M, R)$ with $mf \neq 0$. Extend f to $f' \in \mathrm{Hom}_R(S, Q)$. (Here, as always, Q denotes the left injective hull of R, and for convenience we can assume that $S \subseteq Q$. This will automatically be supposed in the sequel without mention.) Then, as was remarked prior to this theorem, $f' \in \mathrm{Hom}_S(S, Q)$. Hence, for each $s \in S$, $sf' = s(1f)$, and since $M(1f) = Mf \subseteq R$, $1f \in [M:R]$. So $mf = m(1f) = 0$, which is a contradiction, establishing that $l[R:M] \cap M = 0$ for any finitely generated R-submodule M of S.

Finally, suppose that $s \in S$, $0 \neq t \in S$ are given. Set $M = Rs + Rt$. Then $[R:M] = [R:s] \cap [R:t]$, so $t[R:M] \subseteq t[R:s]$. Were it the case that $t[R:s] = 0$, we would have $t \in l[R:M] \cap M = 0$, contradicting the assumption that $t \neq 0$. Hence, $t[R:s] \neq 0$, proving that S is a right quotient ring of R.

The preceding theorem appeared in [**1**] for nonsingular rings. We have followed the proof given there, suitably modified. One should observe that in the proof that (1) $\Rightarrow$ (2), the fact that S was a left quotient ring of R was not used. Also, in the proof of the converse, we needed the hypothesis only for two-generator submodules of S.[1]

[1]It has been kindly brought to our attention by the referee that for the case where S is the maximal left quotient ring of R, the previous theorem has been independently proved by Kanzo Masaike, *On quotient rings and torsionless modules*, Science Reports of Tokyo U. of Education (to appear).

Next, a word on a related situation. Define an R-module M to be *torsion free* if for any $0 \neq m \in M$ there exists $a \in R$ with $\mathrm{Hom}_R(Ram, R) \neq 0$; see [**6**]. We have proved in the course of Theorem 2 that a nonsingular module is torsion free in this sense.

PROPOSITION 12. *For an R-module M, the following conditions are equivalent.*
(1) *M is torsion free.*
(2) *M is Q-torsionless.*
(3) *No nonzero element of M is annihilated by a dense left ideal of R.*

Proof. (1) $\Rightarrow$ (2). For each $0 \neq m \in M$, choose $0 \neq f_m \in \mathrm{Hom}_R(Ram, R)$, for some $a \in R$. f_m extends to $g_m \in \mathrm{Hom}_R(M, Q)$. Define $\nu: M \rightarrow \prod_{0 \neq m \in M} Q_m$, where each $Q_m = Q$, via $x\nu = \{xg_m\}_{0 \neq m \in M}$. ν is easily seen to be the desired monomorphism.

(2) $\Rightarrow$ (3). This follows directly from property (2) of dense left ideals.

(3) $\Rightarrow$ (1). Let T be a torsion submodule of M; $\mathrm{Hom}_R(T, Q) = 0$ (see [**6**]). If $0 \neq x \in T$, then $(0:x)$, by hypothesis, is not a dense left ideal; i.e., there exists $0 \neq q \in Q$ with $(0:x)q = 0$. Define $f \in \mathrm{Hom}_R(Rx, Rq)$ via $(rx)f = rq$. f then extends to a nonzero element of $\mathrm{Hom}_R(T, Q)$. Thus it must be that $T = 0$.

COROLLARY 13. *Assume that Q is a left quotient ring of R. Then Q is a right quotient ring of R if and only if every finitely generated torsion free R-module is torsionless.*

Proof. This follows from Proposition 12 and Theorem 11.

COROLLARY 14. *${}_RQ$ is torsionless if and only if every torsion free R-module is torsionless.*

Proof. This follows from the fact that direct products and submodules of torsionless modules are torsionless.

LEMMA 15. *Let S be a left quotient ring of R and M an R-torsion free S-module. Then M is R-injective if and only if M is S-injective.*

Proof. Assume that M is R-injective and that $f \in \mathrm{Hom}_S(J, M)$, with J a left ideal of S. Extend f to $f' \in \mathrm{Hom}_R(S, M)$. Then by the remark preceding Theorem 11, $f' \in \mathrm{Hom}_S(S, M)$, proving that M is S-injective.

Conversely, suppose that M is S-injective and let $f \in \mathrm{Hom}_R(I, M)$ be given with I a left ideal of R. f extends to a homomorphism $g \in \mathrm{Hom}_R(SI, M)$ by defining $(\sum_{i=1}^{n} s_i x_i)g = \sum_{i=1}^{n} s_i (x_i g)$, for any $s_1, \ldots, s_n \in S$, $x_1, \ldots, x_n \in I$. (g is well-defined because $\bigcap_{i=1}^{n}(R:s_i)$ is a dense left ideal of R.) Again, $g \in \mathrm{Hom}_S(SI, M)$, and since M is S-injective and SI is a left ideal of S, g extends to $g' \in \mathrm{Hom}_S(S, M)$. $g'|_R$ is then the desired extension of f, which proves that M is R-injective.

This lemma has appeared often and in many guises, frequently for S a classical left quotient ring of R.

LEMMA 16. *Let S be a left quotient ring of R and J a left ideal of S. Then J is an essential (dense) left ideal of S if and only if $J \cap R$ is an essential (dense) left ideal of R.*

Proof. Let I be a nonzero R-submodule of S. Then $SI \cap J \neq 0$. So there exist $s_1, \ldots, s_n \in S$, $x_1, \ldots, x_n \in I$ with $0 \neq \sum_{i=1}^n s_i x_i \in J$. Then $D = \bigcap_{i=1}^n (R:s_i)$ is dense, and so $0 \neq D(\sum_{i=1}^n s_i x_i) \subseteq I \cap J$. This proves that J is an essential R-submodule of S, and clearly, then, so is $J \cap R$. The converse is obvious. The remainder of the proof is left as an exercise.

PROPOSITION 17. *Let S be a left quotient ring of R and M a torsion free R-module. Then SM is a rational extension of M, and SM is S-nonsingular if and only if M is R-nonsingular. If $S = Q$ and M is essentially finitely generated and nonsingular over R, then QM is the R-injective hull of M and is a finitely generated Q-module. If $S = Q$ is a finite-dimensional ring and M is any nonsingular R-module, then QM is the R-injective hull of M.*

Proof. Of course, we assume here that $S \subseteq Q$, and by SM we mean the S-module generated by M inside a fixed direct product of copies of Q in which M is embedded.

Let $x \in SM$ and $0 \neq y \in SM$ be given. Write $x = \sum_{i=1}^t s_i m_i$, where each $s_i \in S$ and each $m_i \in M$. $D = \bigcap_{i=1}^t (R:s_i)$ is then a dense left ideal of R, and $Dx \subseteq M$. Also, $Dy \neq 0$ since SM is still R-torsion-free.

Next, assume that $Z_S(SM) \neq 0$. Then $Z_S(SM) \cap M \neq 0$. Hence, there exists $0 \neq m \in M$ and J an essential left ideal of S with $Jm = 0$. By Lemma 16, $J \cap R$ is an essential left ideal of R, and $(J \cap R)m = 0$, so $Z(_RM) \neq 0$. Conversely, suppose that $Z(_RM) \neq 0$. Let $0 \neq m \in M$ with $Im = 0$ for some essential left ideal I of R. Since $SI \cap R \supseteq I$, SI is, by the previous lemma, an essential left ideal of S. Since $SIm = 0$, $m \in Z_S(SM)$.

Now assume that $S = Q$ and that M is essentially finitely generated and nonsingular over R. Then by Theorem 2, we may assume that $M \subseteq Q^n$ for some positive integer n. QM is an R-essential extension of M, by the first paragraph; hence, QM is essentially finitely generated over Q. By the previous paragraph, QM is a nonsingular Q-module. Since Q is left self-injective [**4**, p. 95], Corollary 6 may be applied to learn that QM is Q-injective and Q-finitely generated. It follows from Lemma 15 that QM is the R-injective hull of M.

Finally, let $S = Q$ be a finite dimensional ring and M a nonsingular R-module. Then, as in the preceding paragraph, QM is an R-essential extension of M and is Q-nonsingular. By Theorem 2, we may assume that QM contains a direct sum of nonsingular closed left ideals $I_\alpha (\alpha \in A)$ of R as an essential R-submodule. It follows that $\sum_{\alpha \in A} QI_\alpha \subseteq' QM$ with the sum direct. By the hypothesis on Q, each left ideal QI_α of Q is essentially finitely generated and nonsingular; hence, by Corollary 6, it is Q-injective. Applying [**8**, Theorem 2.5], we learn that $\sum_{\alpha \in A} \oplus QI_\alpha$ is Q-injective, and so, therefore, is $QM = \sum_{\alpha \in A} \oplus QI_\alpha$.

As an application of these results we obtain a convenient sufficient condition for finitely generated nonsingular modules to be embeddable in free modules.

THEOREM 18. *If R has an artinian left quotient ring S then every finitely generated torsionless R-module can be embedded in a free module.*

Proof. Set $M^* = \mathrm{Hom}_R(M, R)$, M a finitely generated torsionless R-module. Then there is a natural embedding of M into $\prod_{f \in M^*} R_f$, where each $R_f = R$, under the map μ defined by $m \to \{mf\}_{f \in M^*}$. We may regard $\prod_{f \in M^*} R_f \subseteq \prod_{f \in M^*} S_f \subseteq \prod_{f \in M^*} Q_f$, where each $S_f = S$, $Q_f = Q$. Each $f \in M^*$ extends to $f' \in \mathrm{Hom}_R(SM, Q_f)$, and by the remark preceding Theorem 11, $f' \in \mathrm{Hom}_S(SM, Q_f)$. We then have an S-homomorphism $\mu' : SM \to \prod_{f \in M^*} Q_f$ via $x\mu' = \{xf'\}_{f \in M^*}$, $x \in SM$.

Then μ' extends μ, and since μ is a monomorphism and SM is an R-essential extension of M, μ' is also a monomorphism. Also, SM is an artinian S-module since it is finitely generated over S. Now among all finite intersections $\bigcap_{i=1}^{n} \ker f_i'$, $f_i \in M^*$, choose one which is minimal. Since μ' is a monomorphism, $\bigcap_{f \in M^*} \ker f' = 0$. Hence, it must be that for the minimal choice, $\bigcap_{i=1}^{n} \ker f_i' = 0$. But then $\bigcap_{i=1}^{n} \ker f_i = 0$, and this implies that M can be embedded in $\sum_{i=1}^{n} \oplus R_{f_i}$.

Combining this theorem with Theorem 11 and Corollary 13 yields the following results.

COROLLARY 19. *If R has a left artinian left and right quotient ring S, then every finitely generated S-torsionless R-module can be embedded in a free R-module.*

COROLLARY 20. *If Q is a left artinian left and right quotient ring of R, then every finitely generated torsion free R-module can be embedded in a free module.*

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