Canad. Math. Bull. Vol. 17 (2), 1974.

A NOTE ON LIE ISOMORPHISMS

BY
WALLACE S. MARTINDALE, 3RD

The purpose of this note is to remove the assumption of characteristic different from 3 from a recent result of ours ([1], Theorem 11) so as to obtain

MAIN THEOREM. *Let S be a prime ring with* 1, *of characteristic different from* 2 *and containing two non-zero idempotents e_1 and e_2 whose sum is* 1. *Let ϕ be a Lie isomorphism of S onto a prime ring R with* 1. *Let Q be the complete ring of right quotients of R, let C be the center of Q, and let $T=RC$. Then ϕ is of the form $\sigma+\tau$, where σ is either an isomorphism or the negative of an anti-isomorphism of S into T and τ is an additive mapping of S into C which maps commutators into zero.*

For notation, definitions, and background results we refer the reader to our paper [1]. We remark parenthetically that the ring $T=RC$ coincides with what we have subsequently called the central closure of R (see [2]). We recall that the ring T is (anti)isomorphic to the ring $T_r(T_l)$ of right (left) multiplications of T (acting on T); T_l is isomorphic to T', the opposite ring of T. Because of its importance in this note we restate [1], Theorem 5 as a

LEMMA. *Let R be a prime ring with* 1 *and let $T=RC$. Then $T'\otimes_c T\simeq T_lT_r$, according to the rule:*

$$a\otimes b \to a_lb_r, \qquad a\in T', \qquad b\in T.$$

As the Main theorem is already known to be true in case char. $S\neq 2, 3$ we assume henceforth without loss of generality that char. $S=3$ (and thus char. $R=3$). The only place where char. $\neq 3$ was used in [1] was in the proof of [1], Theorem 6. This result in turn was used only in the proof of [1], Theorem 7 in order to assert that $\phi(e_i)=c_i+f_i$, f_i an idempotent in T, $c\in C$, $i=1, 2$. Therefore the validity of the Main Theorem will have been established when we complete the proof of the following.

THEOREM. *If e is an idempotent $\neq 0$, 1 in S, then $\phi(e)=c+f$, f an idempotent in $T=RC$, $c\in C$.*

Proof. For $s\in S$ it is easily verified that $[[[se]e]e]=[se]$ (here $[xy]$ means $xy-yx$). Setting $x=\phi(s)$, $a=\phi(e)$, and applying ϕ to this equation, we obtain

$$[[[xa]a]a] = [xa] \tag{1}$$

for all $x \in R$ (and hence for all $x \in T$). (1) may be written as

$$(2) \qquad x(a^3-a) = (a^3-a)x$$

for all $x \in T$, making use of the fact that char. $R=3$. In other words we have

$$(3) \qquad a^3 = a+\lambda, \qquad \lambda \in C.$$

We next choose a non-zero element u of $eS(1-e)$ and note that $eu-ue=u$. Setting $b=\phi(u)$ and applying ϕ to this equation we obtain

$$(4) \qquad ab-ba = b \neq 0 \qquad (\text{thus } ab = b+ba).$$

For $s \in S$ one easily verifies that $[[[se]u]e]=0$. Setting $x=\phi(s)$ and applying ϕ we have

$$(5) \qquad [[[xa]b]a] = 0$$

for all $x \in R$ (and hence for all $x \in T$). (5) may be rewritten as

$$(6) \quad (aba)_r - a_l(ab)_r - b_l(a^2)_r + (ab)_l a_r - a_l(ba)_r + (a^2)_l b_r + (ba)_l a_r - (aba)_l = 0$$

or, via the Lemma, as

$$(7) \quad 1\otimes aba - a\otimes ab - b\otimes a^2 + ab\otimes a - a\otimes ba + a^2\otimes b + ba\otimes a - aba\otimes 1 = 0$$

Partial replacement of ab by $b+ba$ from (4) enables us to rewrite (7) as

$$(8) \quad 1\otimes aba - a\otimes(ab+ba) + a^2\otimes b + b\otimes(a-a^2) + ba\otimes 2a - aba\otimes 1 = 0.$$

At this point we suppose that $1, a, a^2$ are C-independent. If $b=\alpha+\beta a+\gamma a^2$, $\alpha, \beta, \gamma \in C$, then $ab-ba=0$, a contradiction of (4). Therefore $1, a, a^2, b$ are C-independent. Suppose

$$(9) \qquad ba = \alpha+\beta a+\gamma a^2+\delta b, \qquad \alpha, \beta, \gamma, \delta \in C.$$

Then $aba=\alpha a+\beta a^2+\gamma a^3+\delta ab$ and, by making use of (3), (4), and (9), we see that

$$(10) \qquad aba = (\gamma\lambda+\delta\alpha)+(\alpha+\gamma+\delta\beta)a+(\beta+\delta\gamma)a^2+(\delta+\delta^2)b.$$

Partial substitution of (9) and (10) in (8) yields

$$(11) \quad 1\otimes\{aba+2\alpha a-(\gamma\lambda+\delta\alpha)\}+a\otimes\{2\beta a-ab-ba-(\alpha+\gamma+\delta\beta)\}$$
$$+a^2\otimes\{b+2\gamma a-(\beta+\delta\gamma)\}+b\otimes\{-a^2+(2\delta+1)a-(\delta+\delta^2)\} = 0.$$

It follows in particular that $-a^2+(2\delta+1)a-(\delta+\delta^2)=0$, a contradiction to the independence of $1, a, a^2$. Therefore we must assume that $1, a, a^2, b, ba$ are C-independent.

Now suppose

$$(12) \qquad aba = \alpha+\beta a+\gamma a^2+\delta b+\mu ba, \qquad \alpha, \beta, \gamma, \delta, \mu \in C.$$

https://doi.org/10.4153/CMB-1974-047-x Published online by Cambridge University Press

Partial substitution of (12) in (8) gives

$$1\otimes(aba-\alpha)-a\otimes(ab+ba+\beta)+a^2\otimes(b-\gamma)+b\otimes(a-a^2-\delta)+ba\otimes(2a-\mu) = 0.$$

In particular $a-a^2-\delta=0$, again contradicting the independence of 1, a, a^2. Therefore 1, a, a^2, b, ba, aba are C-independent. But this clearly violates (8), and so we must conclude that 1, a, a^2 are C-dependent. Since $a \notin C$ we may thus write

$$a^2 = \alpha a+\beta, \qquad \alpha, \beta \in C, \tag{13}$$

whence

$$a^3 = (\alpha^2+\beta)a+\alpha\beta. \tag{14}$$

Equating (3) and (14) yields

$$\alpha^2+\beta = 1. \tag{15}$$

Using (13) and (15) and char. $R=3$, one verifies directly that $f=a+2(1-\alpha)$ is an idempotent, and thus $a=f+c$, where $c=1-\alpha \in C$.

ACKNOWLEDGEMENT. The research involved in this paper was partially supported by the National Science Foundation (GP-12090).

REFERENCES

1. W. S. Martindale, 3rd, *Lie isomorphisms of prime rings*, Trans. Amer. Math. Soc. **142** (1969), 437–455.

2. W. S. Martindale, 3rd, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12** (1969), 576–584.

UNIVERSITY OF MASSACHUSETTS,
AMHERST, MASSACHUSETTS