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A BASICALLY DISCONNECTED NORMAL SPACE Φ WITH $|\beta\Phi - \Phi| = 1$

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0. Definitions. All spaces considered are completely regular. $C^*(X)$ denotes the set of bounded continuous real-valued functions in X. A subspace S of X is called *C^*-embedded* in X if for every $f \in C^*(S)$ there is $\phi \in C^*(X)$ with $\phi \restriction S = f$.

A space X is called *almost compact* if $|\beta X - X| \leqq 1$; *basically disconnected* if every cozero-set has open closure; *extremally disconnected* if every open set has open closure; an *F-space* if every cozero-set is C^*-embedded; *small* if $|C^*(X)| = 2^\omega$; and *weakly Lindelöf* if every open cover has a subfamily $\mathscr{U}$ with $|\mathscr{U}| \leqq \omega$ and $\bigcup \mathscr{U}$ dense. A point p of a space X is called a *P-point* of X if every G_δ-set in X which contains p is a neighborhood of p.

$w(X)$ denotes the weight of X.

We identify cardinals with initial ordinals.

1. Introduction. In this note we complete the proof of the following theorem, begun in [**6**] and [**7**].

THEOREM. *Each of the following statements is equivalent to* CH:

(a) *Every small countably compact normal F-space is compact.*

(b) *Every small locally compact normal F-space is σ-compact.*

(c) *Every small F-space is weakly Lindelöf.*

To this end we construct the following example.

Example. There is an almost compact basically disconnected normal non-compact space Φ with $w(\beta\Phi) = \omega_2 \cdot 2^\omega$ such that the point of $\beta\Phi - \Phi$ is a P-point of $\beta\Phi$.

Remarks. (1) Every basically disconnected space is an F-space, [**3**, 14 N.4].

(2) Every almost compact space is locally compact and every almost compact normal space is countably compact.

(3) $|C^*(\Phi)| = |C^*(\beta\Phi)|$ so $|C^*(\Phi)| = \omega_2 \cdot 2^\omega$ since $\omega_2 \cdot 2^\omega \leqq |C^*(\beta\Phi)| \leqq (\omega_2 \cdot 2^\omega)^\omega = \omega_2 \cdot 2^\omega$.

In [**1**] it is shown that another statement about F-spaces, proved from CH in the literature, also is in fact equivalent to CH.

The example is of interest for yet another reason. Woods has asked if there is a real (= not requiring additional set theoretic axioms) example of a extremally disconnected locally compact space that is normal but not para-

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compact, [7]. Kunen and Parsons construct an example that is not real since it uses a weakly compact cardinal in [5]. Kunen constructs a real extremally disconnected normal space that is not paracompact (not even collectionwise Hausdorff) in [4]. His example is not locally compact. Our example comes close to answering Woods' question. It is not extremally disconnected but it is at least basically disconnected.

2. The example. (A) Let P be the subspace of all P-points (whether isolated or not) of $\omega_2 + 1$, the ordinals $\leqq \omega_2$ equipped with the order topology. Equivalently

$$P = \{\alpha \leqq \omega_2 \colon cf\alpha \neq \omega\}.$$

Throughout this section $^-$ is the closure operator in βP.

Definition. $\Phi = \beta P - \{\omega_2\}$.

Nonstandard Convention. We use the usual notation for intervals to denote the traces on P of intervals in $\omega_2 + 1$. For example, if $\alpha, \beta \leqq \omega_2$, then

$$(\alpha, \beta] = \{\xi \in P \colon \alpha < \xi \leqq \beta\}$$

(not necessarily $\alpha, \beta \in P$).

(B) We first show that Φ is almost compact and normal. This depends on the following easy observation, the proof of which is omitted; cf. [3, 6].

LEMMA. *The space X is almost compact and normal if and only if any two noncompact closed subsets of X have nonempty intersection.*

So let A and B be noncompact closed subsets of Φ. We use a well known type of argument to show that $A \cap B \neq \emptyset$.

Since $\omega_2 \in \bar{A} \cap \bar{B}$, and since $\{(\alpha, \omega_2]^- \colon \alpha < \omega_2\}$ is a neighborhood base in βP of ω_2, there is a strictly increasing ω_1-sequence $\langle \gamma_\xi \colon \xi < \omega_1 \rangle$ of ordinals $<\omega_2$ such that $(\gamma_\xi, \gamma_{\xi+1}]^-$ intersects both A and B for all $\xi < \omega_1$. Let $\sigma = \sup_{\xi<\omega_1}\gamma_\xi$. Then $\sigma \in P - \{\omega_2\} \subseteq \Phi$, hence $\sigma \in A \cap B$ since A and B are closed and since $\{(\gamma, \sigma]^- \colon \gamma < \sigma\}$ is a neighborhood base in Φ for σ.

(C) Clearly a (noncompact) almost compact space has precisely one compactification, [3, 6], hence

$$\beta\Phi = \beta P.$$

An immediate consequence is that Φ is basically disconnected: just note that X is basically disconnected if and only if βX is, for all X, [3, 6M.1].

Another consequence is that the point in $\beta\Phi - \Phi$ is ω_2, which is a P-point in βP, hence in $\beta\Phi$.

(D) It remains to calculate $w(\beta\Phi)$. Denote the family of clopen sets of a space X by $\mathrm{CO}(X)$. Recall the following facts, which we will use without explicit reference.

Fact 1. $w(X) = |\mathrm{CO}(X)|$ if X is an infinite compact zero-dimensional space.
Fact 2. $|\mathrm{CO}(X)| \leqq w(X)^\omega$ if X is any Lindelöf space.

Since $\beta\Phi$ is zero dimensional, being basically disconnected, and since $\beta P = \beta\Phi$, we have

$$w(\beta\Phi) = |\mathrm{CO}(\beta\Phi)| = |\mathrm{CO}(P)|.$$

Claim. P is Lindelöf.

By transfinite induction one can easily show that the subspace $[0, \alpha]$ of P is Lindelöf for all $\alpha \leqq \omega_2$. In particular, $P = [0, \omega_2]$ is Lindelöf.

P has ω_2 isolated points and $[0, \omega)$ is a clopen discrete subset of P, hence $|\mathrm{CO}(P)| \geqq \omega_2 \cdot 2^\omega$. But evidently $w(P) = \omega_2$, hence $|\mathrm{CO}(P)| \leqq \omega_2{}^\omega = \omega_2 \cdot 2^\omega$.

This shows that $w(\beta\Phi) = \omega_2 \cdot 2^\omega$.

Corollary to proof. $|\Phi| = 2^{2^\omega}$.

$[0, \omega)$ is C^*-embedded in Φ and $[0, \omega)^- \subseteq \Phi$, hence $|\Phi| \geqq |\beta N| = 2^{2^\omega}$, [**3**, 9.3]. The argument above shows that $w([0, \alpha]^-) \leqq \omega_1{}^\omega = 2^\omega$ if $\alpha < \omega_2$. Since $|X| \leqq 2^{w(X)}$ for every Hausdorff X, it follows that

$$|\Phi| = |\cup \{[0, \alpha]^-: \alpha < \omega_2\}| \leqq \omega_2 \cdot 2^{2^\omega} = 2^{2^\omega}.$$

3. The theorem. Woods proved that CH implies (a) and (b) in [**7**, 1.1], and proved that CH implies the following statement

(*d*) *If X is a small compact F-space, then $S \subseteq X$ is C^*-embedded in X (if and) only if X is weakly Lindelöf,*

in [**6**, 2.3(3)]. Now observe that if Y is any space then βY is small if (and only if) Y is, and βY is an F-space if (and only if) Y is, [**3**, 14.25]. (This argument occurs in the proof of [**7**, 1.1]). Hence (*d*) implies (*c*).

Φ is a countably compact locally compact normal F-space which is not weakly Lindelöf, leave alone (σ-) compact, since the point of $\beta\Phi - \Phi$ is a P-point. Since Φ is small if (and only if) CH fails, it follows that each of (*a*), (*b*) and (*c*) implies CH.

Remarks. (*A*) The proof that (*d*) implies (*c*) can also be used to eliminate "compact" from (*d*).

(*B*) In addition to (*d*), the interested reader can add several other more technical statements to the list of equivalences of CH we gave, by looking up [**2**, 4.3 and 4.4] and [**7**, 2.2].

4. Remarks. (*A*) The subspace $X = P - \{\omega_2\}$ occurs in [**3**, 9L] as an example of a P-space that is not realcompact. Note that $\beta X = \beta P$, and that $\upsilon X = P$.

(*B*) The fact that (*a*) and (*b*) of the theorem are equivalent to CH answers a question of Woods, [**7**, 3.6].

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(C) The existence of Φ also shows that the condition that the F-space be small is essential in (a) and (b) of the theorem, even under CH. This answers a question implicit in [**7**, 3.5].

(D) It was known that (c) of the theorem is false without additional axioms, indeed is false without $2^\omega < 2^{\omega_1}$; the discrete space with ω_1 points is the appropriate example, [**7**].

(E) Consider the following statements:

(e) every countable subset of a countably compact normal F-space has compact closure;

(f) if Y is a dense C^*-embedded subset of $\beta\omega - \omega$, then $Y = \beta\omega - \omega$;

(g) if X is countably compact and normal, and if $\omega \subseteq X \subseteq \beta\omega$, then $X = \beta\omega$;

(h) if X is an infinite countably compact normal F-space, then $|X| \geqq 2^{2^\omega}$.

Using 6.7, 9.3, 14.25, 14.27 and 14N.5 of [**3**] one can easily prove that CH implies (e), since CH $\Leftrightarrow$ (a) [**7**, 2.1], that CH implies (f) [**2**, 4.6a], since CH $\Leftrightarrow$ (c), that (e) and (f) imply (g) and that (g) implies (h). So each of (e), (f), (g) and (h) follows from CH. For each of (e), (f), (g) and (h) it is open if the statement is equivalent to CH, is strictly weaker than CH but not true in ZFC, or is true in ZFC.

Added in print. I have recently shown that it is consistent with ZFC that (f) of 4(E) is false.

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