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Curvature Conditions for Immersions of Submanifolds and Applications

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Abstract. Some curvature conditions about the geodesics emanating from a submanifold are obtained. These conditions are used to to study the topological and geometric properties of the ambient spaces which admit some minimal submanifolds.

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0. Introduction

Let M^n denote a complete and connected n-dimensional Riemannian manifold without boundary and S be a connected immersed submanifold without boundary. When S is a point, a well-known result of Ambrose [A2] says that the integral of the Ricci curvature along a geodesic $\gamma(t)$ emanating from S cannot be infinite if γ is free of conjugate points. It is natural to generalize this result to the case where the dimension of S is greater than zero. As shown later, the generalization is quite different and involves the second fundamental form of the submanifold and therefore is useful to study the nonexistence of minimal or totally geodesic submanifolds in an ambient space. We say that a geodesic $\gamma\colon [0, +\infty) \to M$ with $\gamma(0) \in S$ is an $S-path$ if $\gamma'(0) \perp S$ and there is no focal points to S along γ. Let $S_{\gamma'(0)}$ be the linear self-adjoint map associated with the second fundamental form α of S with respect to normal vector $\gamma'(0)$ and let $\mathcal{H}$ be the mean curvature vector of S at $\gamma(0)$. Given linearly independent tangent vectors v, w, let $K(v, w)$ be the sectional curvature associated to the plane generated by v and w. We have

THEOREM A. *Let S be a submanifold immersed in M with dimension $k \geqslant 1$. Assume that γ is an S-path and $v = v(t)$ is a parallel unit vector field along γ with $v(0) \in T_{\gamma(0)}S$. Then we have*

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$$\eta := \liminf_{t\to+\infty} \int_0^t K(v,\gamma')\,\mathrm{d}s + \langle S_{\gamma'(0)}v(0), v(0)\rangle \leqslant 0$$

and

$$\mu := \liminf_{t\to+\infty} \int_0^t \mathcal{RIC}_S(\gamma'(s))\,\mathrm{d}s + \langle \mathcal{H}, \gamma'(0)\rangle \leqslant 0,$$

where $\mathcal{RIC}_S(\gamma') = \sum_{i=1}^k K(\gamma', e_i)$ *and* $e_1, e_2, \ldots, e_k$ *are obtained by the parallel transport along* γ *of an orthonormal basis of* $T_{\gamma(0)}S$.

If $\eta = 0$, *then* $K(v(t), \gamma'(t)) \equiv 0$ *and* $\alpha(v(0), v(0)) \perp \gamma'(0)$. *If* $\mu = 0$, *then the function* $K(.,\gamma'(t))$ *vanishes when restricted to the parallel transport of the tangent space of* S *at* $\gamma(0)$. *Furthermore,* $\alpha(u,u) \perp \gamma'(0)$, *for all* $u \in T_{\gamma(0)}S$. *In particular,* $\mathcal{H} \perp \gamma'(0)$, *so in the codimension* 1 *case, the condition* $\mu = 0$ *implies that* S *is totally geodesic at* $\gamma(0)$ *and* $R(\gamma', .)\gamma'$ *vanishes along* γ, *where* R *is the curvature tensor.*

As a corollary we have the following:

COROLLARY 0.1. *Let* M *be a complete noncompact Kähler manifold. Assume that the holomorphic curvature* $H \geqslant 0$ *on* M *and* $H > 0$ *outside a compact set. Then* M *does not contain compact totally geodesic hypersurfaces.*

Remark 0.1. In the case of sectional curvature $K \geqslant 0$ a similar theorem was proved by Galloway and Rodriguez ([GR]).

Our second result is a generalization of a famous result of Frankel ([F1]) about the fundamental group of positively curved manifolds. Frankel proved that if M is a complete Riemannian manifold with positive curvature and S is a compact minimal hypersurface, then the natural homomorphism of fundamental group $\pi_1(S) \to \pi_1(M)$ is surjective. Note that, under these conditions, $\pi_1(M)$ is in fact finite and is trivial when the dimension of M is even. Using the definition of partial curvature in Definition 2.2, we generalize the above theorem and remark that, in our case, the fundamental group $\pi_1(S)$ can be infinite, since the curvature of S can be negative.

THEOREM B. *Assume that* S *is a compact minimal hypersurface in* M *and that any geodesic* γ *with* $\gamma'(0) \perp S$ *satisfies* $\mathrm{Ric}(\gamma', \gamma') \geqslant 0$ *and* $\mathrm{Ric}(\gamma'(0), \gamma'(0)) > 0$. *Then the natural homomorphism of fundamental groups* $\pi_1(S) \to \pi_1(M)$ *is surjective.*

The following result, which we will prove together with Theorem B, studies the fundamental group of a hypersurface in a Kähler manifold. It should be remarked that a result of Tsukamoto ([Ts]) says that the fundamental group of a compact Kähler manifold with positive holomorphic curvature is trivial. The next result considers the case $H \geqslant 0$ and M noncompact.

COROLLARY 0.2. *Assume that* S *is a compact totally geodesic hypersurface in the Kähler manifold* M *and that the holomorphic curvature* $H \geqslant 0$ *on* M *and* $H > 0$

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on S. Then the natural homomorphism of fundamental groups $\pi_1(S) \to \pi_1(M)$ is surjective.

The rest of this paper is organized as follows. In the first section we prove lemmas about ordinary differential equations and prove Theorem A, its corollary and other consequences. The proof of Theorem A motivates us to introduce some new definitions about curvatures, so in Section 2 we discuss further applications. In the third section we prove Theorem B and a related result in the case of radial curvatures. In the fourth section we extend the Cartan–Hadamard and Hopf–Rinow theorems to the radial case. We would like to mention that our generalized Hopf-Rinow theorem is now being used to prove a result about the existence of locally free totally geodesic $\mathbb{R}^2$-actions on spheres, in a joint work of one of the authors with F. Fang and S. Firmo. We present some examples in Section 5.

1. Ricatti Equations and the Proof of Theorem A

Let M be a complete and connected n-dimensional Riemannian manifold. Let S be a connected Riemannian submanifold isometrically immersed in M. Along an S-path $\gamma(t)$ we take a parallel unit normal vector field $v(t)$ with $v(0) \in T_{\gamma(0)}S$ and define $\varphi(t) := K(v(t), \gamma'(t))$. Let $S_{\gamma'(0)}$ be the linear self-adjoint map associated with the second fundamental form of S. We have

LEMMA 1.1. *Let $\gamma\colon [0, B] \to M$ be a geodesic with $\gamma(0) \in S$ and $\gamma'(0) \perp S$. If there is no focal point to S along $\gamma|_{(0,B]}$, then for any $b \in (0, B]$ the following boundary value problem*

$$f''(t) + \varphi(t)f(t) = 0, \qquad f'(0) = A,\ f(0) = 1, \quad f(b) = 0 \tag{1.1}$$

has no solution on $[0, b]$, where $A = -\langle v(0), S_{\gamma'(0)}v(0)\rangle$.

Proof. Assume by contradiction that there exists a solution for (1.1). Set $X(t) = f(t)v(t)$. From the generalized Morse index I associated with S (see, for example, [A1] or [Ch], page 131) we have

$$\begin{aligned} 0 &< I(X, X) \\ &= -\int_0^b \{ff'' + f^2K(\gamma'(t), v(t))\}\,\mathrm{d}t - \langle v(0), f'(0)v(0)\rangle - \langle v(0), S_{\gamma'(0)}v(0)\rangle = 0. \end{aligned} \tag{1.2}$$

The above contradiction proves our lemma. □

Again with the hypotheses of Lemma 1.1 we analyze the equation with the initial value

$$f'' + \varphi(t)f = 0, \qquad f'(0) = A, \quad f(0) = 1. \tag{1.3}$$

Since Equation (1.3) is linear, we know that it has a solution $f\colon [0, +\infty) \to \mathbb{R}$. By Lemma 1.1 we have that $f(t) > 0$ for all t. So we can define $w(t) = f'(t)/f(t)$ which then satisfies the following Ricatti equation:

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$$w'(t) + w^2(t) + \varphi(t) = 0, \qquad w(0) = A.$$

LEMMA 1.2. *If* $\liminf_{t\to+\infty} \int_0^t \varphi(s)\,\mathrm{d}s \geqslant A$, *$\varphi$ is a continuous function and a is a positive constant, then there is no solution on* $[0, +\infty)$ *of the following problem*

$$w'(t) + aw^2(t) + \varphi(t) \leqslant 0, \qquad w(0) = A, \tag{1.4}$$

with $\int_0^\infty w^2(s)\,\mathrm{d}s = +\infty$.

Proof. Assume that $w\colon [0, +\infty) \to \mathbb{R}$ satisfies (1.4). Then we have

$$w(t) + a\int_0^t w^2(s)\,\mathrm{d}s + \int_0^t \varphi(s)\,\mathrm{d}s - A \leqslant 0 \tag{1.5}$$

Since $\liminf_{t\to+\infty} \int_0^t \varphi(s)\,\mathrm{d}s - A \geqslant 0$, given $\epsilon > 0$. There exists $T > 0$ such that, for $t > T$, it holds that $\int_0^t \varphi(s)\,\mathrm{d}s - A > -\epsilon$. For the sake of contradiction, assume that $V(t) = \int_0^t w^2(s)\,\mathrm{d}s \to +\infty$, as $t \to +\infty$. So we have

$$-w(t) \geqslant a\int_0^t w^2(s)\,\mathrm{d}s - \epsilon \geqslant \frac{a}{2}\int_0^t w^2(s)\,\mathrm{d}s,$$

for all t greater than certain $T' > T$. Then we obtain

$$V'(t) \geqslant \frac{a^2}{4}V^2(t), \quad \text{for all } t \geqslant T',$$

hence $-(1/V(t))' \geqslant a^2/4$. Thus for any $T_2 > T_1 > T'$, it holds that

$$\frac{1}{V(T_1)} - \frac{1}{V(T_2)} \geqslant \frac{a^2(T_2 - T_1)}{4}.$$

If we let T_2 tend to $+\infty$ we have a contradiction, since $V(t) \to +\infty$ as t goes to $+\infty$. Lemma 1.2 is proved. □

LEMMA 1.3. *Let φ, w, A satisfying* (1.4) *for all $t \geqslant 0$. Then*

$$\eta := \liminf_{t\to+\infty} \int_0^t \varphi(s)\,\mathrm{d}s - A \leqslant 0.$$

If, furthermore, $\eta = 0$ then $\varphi \equiv 0$ and $A = 0$.

Proof. Assume by contradiction that $\eta > 0$. Then there exist c and $T > 0$ such that for $t > T$ we have $\int_0^t \varphi(s)\,\mathrm{d}s > c > A$. By (1.5) we obtain $w(t) \leqslant A - c < 0$, which implies that $\int_0^{+\infty} w^2(s)\,\mathrm{d}s = +\infty$, and this is not possible by Lemma 1.2. So we have $\eta \leqslant 0$.

Now assume that $\eta = 0$. We assert that $w(s) \equiv 0$. If not, there exists $\epsilon > 0$ such that $(a\int_0^{+\infty} w^2(s)\,\mathrm{d}s) > \epsilon$. Since $\eta = 0$, there exists $T > 0$ so that for $t > T$ we have $\int_0^t \varphi(s)\,\mathrm{d}s - A > -\epsilon/2$. Inequality (1.5) implies that for $t > T$ we have $w(t) < -\epsilon/2$ and again $\int_0^{+\infty} w^2(s)\,\mathrm{d}s = +\infty$, thus we arrive at a contradiction. So we conclude that $w(s) \equiv 0$. By (1.4) we obtain $\varphi(t) \leqslant 0$ for all t and $A = 0$. Now $\eta = 0$, $A = 0$, $\varphi(t) \leqslant 0$, and the continuity of φ imply together that $\varphi(t) \equiv 0$. The proof is complete. □

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We are now in a position to complete the proof of Theorem A.

Proof of Theorem A. $\eta \leqslant 0$ follows from Lemmas 1.1, 1.2 and 1.3. The case $\eta = 0$ is obtained also from this lemma. To estimate μ we consider an orthonormal basis $e_1, e_2, \dots, e_k$ of $T_{\gamma(0)}S$, and take its parallel transport $e_1(t), \dots, e_k(t)$ along γ. For any $t \geqslant 0$ let $W_t \subset T_{\gamma(t)}M$ be the parallel transport of $T_{\gamma(0)}S$ along γ. By Lemma 1.1 we obtain C^∞ functions $w_i\colon [0, +\infty) \to \mathbb{R}$, $i = 1, 2, \dots, k$ with

$$w_i' + w_i^2 + K(\gamma', e_i) = 0, \qquad w_i(0) = -\langle S_{\gamma'(0)} e_i, e_i\rangle. \tag{1.6}$$

Set $y(t) = \sum_{i=1}^k w_i$. So using the definition of $\mathcal{RIC}_S$, we have

$$y' + \sum_{i=1}^k w_i^2 + \mathcal{RIC}_S(\gamma') = 0. \tag{1.7}$$

Since

$$\sum_{i=1}^k w_i^2 \geqslant \frac{1}{k}\left(\sum_i w_i\right)^2,$$

we get

$$y' + \frac{1}{k}y^2 + \mathcal{RIC}_S(\gamma') \leqslant 0, \qquad y(0) = -\langle \mathcal{H}, \gamma'(0)\rangle. \tag{1.8}$$

Now we apply Lemma 1.3 to (1.8) and obtain $\mu \leqslant 0$. Assume that $\mu = 0$. Take any $w \in W_{t_0}$. We can choose the orthonormal basis $e_1, \dots, e_k$ in such a way that $e_1(t_0) = w$. By Lemma 1.3 we get $y(t) \equiv 0$ and $\mathcal{RIC}_S(\gamma'(t)) \equiv 0$. Thus (1.7) implies that $w_i(t) \equiv 0$ for all i and (1.6) implies that $K(\gamma'(t), e_i(t)) \equiv 0$. In particular, we have $K\big(\gamma'(t_0), w\big) = 0$. The system (1.6) also implies that $\langle S_{\gamma'(0)} e_i, e_i\rangle = 0$ for all i. Given a unit vector $u \in T_{\gamma(0)}S$, of course we can choose the orthonormal basis e_i in such a way that $e_1 = u$, hence $\alpha(u, u) \perp \gamma'(0)$ for all $u \in T_{\gamma(0)}S$. If $k = n - 1$ then $W_t = \big\{\gamma'(t)\big\}^\perp$, hence $R(\gamma', .)\gamma'$ vanishes. To see this just consider a basis of orthonormal eigenvectors for the restriction $(R(\gamma', .)\gamma')\colon W_t \to W_t$. At $\gamma(0)$ we have $\alpha(u, u) = 0$ for all $u \in T_{\gamma(0)}S$. If we consider a basis of orthonormal eigenvectors for $S_{\gamma'(0)}$ we conclude easily that $\alpha = 0$. The proof is complete. □

As a direct consequence of Theorem A we have

COROLLARY 1.1. *Let S be a k-dimensional closed submanifold of M. Assume that γ is an S-path. Let $v = v(t)$ be a parallel field along γ with $v(0) \in T_{\gamma(0)}S$. Suppose that the second fundamental form $\alpha(v(0), v(0)) = 0$ (respectively, that S is minimal at $\gamma(0)$). Then we have*

$$\eta := \liminf_{t\to+\infty} \int_0^t K(v, \gamma')\,\mathrm{d}s \leqslant 0,$$
$$\left(\text{respectively, } \liminf_{t\to+\infty} \int_0^t \mathcal{RIC}_S(\gamma'(s))\,\mathrm{d}s \leqslant 0\right).$$

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If this integral limit vanishes then $K(v, \gamma') \equiv 0$ *(respectively,* $K(\gamma', .)$ *vanishes on* W_t *for all* $t \geqslant 0$*, where* W_t *is the parallel transport of* $T_{\gamma(0)}S$ *along* γ.

Using this corollary we can complete the

Proof of Corollary 0.1. Assume by contradiction that S is a compact totally geodesic hypersurface with dimension $k \geqslant 1$. Since M is noncompact there exists a ray γ: $[0, +\infty) \to M$, with $\gamma(0) \in S$ and $d(\gamma(t), \gamma(0)) = t$, for all $t \geqslant 0$. This can be easily obtained by taking a sequence of points $p_i \to \infty$ and a sequence of minimizing geodesic segments joining p_i and S. The existence of such a ray follows from the compactness of the unit normal bundle of S. From the index theorem of Ambrose ([A1], see also [Ch], page 131) we know that γ is an S-path. Since $\gamma'(0) \perp S$ we have $J\gamma'(0) \in T_{\gamma(0)}S$, and we know that $J\gamma'$ is parallel along γ. So we apply Corollary 1.1 to the parallel field $v = J\gamma'$ and conclude that $\eta \leqslant 0$. Our hypotheses however lead to $\eta \geqslant 0$. So we conclude that $K(\gamma'(t), v(t)) \equiv 0$, and this contradicts the hypotheses of the corollary. □

2. Radial Curvatures

In this section we will give more applications of Theorem A. Throughout the section we assume that M is a complete and connected n-dimensional Riemannian manifold and S is a connected Riemannian submanifold isometrically immersed in M. Galloway and Rodríguez proved ([GR]) that if M is noncompact, has nonnegative sectional curvature, and positive sectional curvature outside a compact set, then M does not have compact minimal submanifolds. Theorem 2.1 below extends this result, and it also partially extends Theorem 1 in [Ka] for codimension greater than 1. We say that the k-Ricci curvature* at $p \in M$ is greater than c if for any $v \in T_pM$ and any k orthonormal vectors $e_1, \ldots, e_k$, which are perpendicular to v it holds that $\sum_{i=1}^{k} K(v, e_i) > c$.

THEOREM 2.1. *If M is complete and noncompact, the k-Ricci curvature of M is nonnegative, and is positive outside a compact set, then M does not admit k-dimensional compact minimal submanifolds.*

Proof. From the compactness of S there exists an S-path as in the proof of Corollary 0.1 above. So the conclusion follows directly from Corollary 1.1. □

To discuss more applications we need to introduce some more definitions. The notion of minimal radial curvature was first introduced by Klingenberg in [Kl] and was studied for example in [Mc1], [Mc2], [MS], [CX], [MM], [M]. We extend here such a definition to submanifolds. The notion of minimal S-radial curvature

*Our definition of k-Ricci is different from the one used in [Sh]. The definition here is weaker and we think that it is more natural.

appears – even without an explicit definition – for example in [E] and in [HK]. Given $p \in M$ we say that a minimal geodesic γ: $[0, a] \to M$ is a *minimal segment* between p and S if $\gamma(0) \in S$, $\gamma(a) = p$ and the distance $\mathrm{d}(p, S) = a$. Given linearly independent tangent vectors , w we denote by $K(, w)$ the sectional curvature associated to the plane generated by and w.

DEFINITION 2.1. Given $p \in M$, we say that the minimal S-radial curvature $K_S^{\min}(p) \geqslant c$ ($K_S^{\min}(p) \leqslant c$) if for any minimal segment γ between p and S, and any v orthogonal to the tangent vector γ' at p it holds that $K(\gamma', v) \geqslant c$ ($K(\gamma', v) \leqslant c$).

DEFINITION 2.2. We say that the minimal S-radial Ricci curvature $\mathrm{Ric}_S^{\min}(p) \geqslant c$ if any minimal segment γ between p and S satisfies $\mathrm{Ric}(\gamma', \gamma') > c$ at p, and that the S-radial parallel Ricci curvature $\mathcal{RIC}_S^{\min}(p) > c$ if any geodesic segment γ between p and S satisfies $\mathcal{RIC}_S(\gamma') > c$ at p.

It should be pointed out that $\mathcal{RIC}_S(\gamma')$ defined in Theorem A does not depend on the choice of the orthonormal basis $\{e_1(0), e_2(0), \ldots, e_k(0)\}$ of $T_{\gamma(0)}S$. Note that if the dimension of S is $n-1$ then $\mathrm{Ric}_S^{\min}(p) \geqslant c$ is equivalent to $\mathcal{RIC}_S^{\min}(p) \geqslant c$. In Section 5, we will see some examples of radial curvature bounded from below.

The following result, whose proof is very similar to that of Theorem A, shows that, if the distance from S has no upper bound on M, then the fact that S is minimal implies that radial curvatures tend to be nonpositive in some integral sense. Let $\rho(x) := \mathrm{d}(x, S)$. Precisely we have:

THEOREM 2.2. *Assume that M satisfies $\mathcal{RIC}_S^{\min}(x) \geqslant kK(\rho(x))$ for any $x \in M$, where $K(\rho)$ is a continuous function. Suppose that the distance function from S is unbounded. Then*

(a) $\liminf_{t\to+\infty} \int_0^t K(\rho)\,\mathrm{d}\rho < +\infty$;
(b) $v := \liminf_{t\to+\infty} \int_0^t K(\rho)\,\mathrm{d}\rho - \frac{1}{k}\sup_{p\in S}|\mathcal{H}(p)| \leqslant 0$;
furthermore, if $v = 0$ then we have $K(\rho) \equiv 0$ and S is minimal;
(c) *for any $j \geqslant 1$, if the j-Ricci curvature at x equals at least $jR(\rho(x))$ for some continuous function $R(\rho)$, then*

$$\liminf_{t\to+\infty} \int_0^t R(\rho)\,\mathrm{d}\rho < +\infty. \tag{2.1}$$

Of course the distance from S could be bounded even if M is noncompact. For example, let S be a line in a cylinder M. When S is minimal and $\mathcal{RIC}_S^{\min} \geqslant 0$ Theorem 2.2 implies that for any $r > 0$ it holds that

$$\inf\{\mathcal{RIC}_S^{\min}(x) \mid \rho(x) = r\} = 0 \tag{2.2}$$

(see, for example, the case that S is a meridian of a paraboloid).

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Proof of Theorem 2.2. First we prove (b). If $\sup_{p\in S}|\mathcal{H}(p)| = +\infty$ there is nothing to prove. So we assume that $\sup_{p\in S}|\mathcal{H}(p)| < +\infty$. We modify slightly Lemma 1.1:

LEMMA 2.1. *Let* $\gamma\colon [0,B] \to M$ *be a minimal segment between* S *and* $\gamma(B)$. *Then for any* $b \in (0,B]$ *the following boundary value problem*

$$f''(t) + K(t)f(t) = 0; \qquad f'(0) = A, \qquad f(0) = 1, \qquad f(b) = 0 \tag{2.3}$$

has no solution on $[0,b]$, *where* $A = (1/k)\sup_{p\in S}|\mathcal{H}(p)|$.

To prove Lemma 2.1, we take orthonormal parallel vector fields $v_i(t)$ along γ with $v_i(0) \in T_{\gamma(0)}S$, and let $X_i(t) = f(t)v_i(t)$. In a similar way as above, if there is a solution to (2.3) on $[0,b]$, then we have

$$\begin{aligned} 0 &< \sum_{i=1}^{n-1} I(X_i, X_i) \\ &\leqslant -\int_0^b \{kff'' + kf^2K(t)\}\,\mathrm{d}t - \sum_i \langle v_i(0), f'(0)v_i(0)\rangle - \sum_i \langle v_i(0), S_{\gamma'(0)}v_i(0)\rangle \\ &= -kA - \langle \mathcal{H}(\gamma(0)), \gamma'(0)\rangle \leqslant -kA + |\mathcal{H}(\gamma(0))| \leqslant 0. \end{aligned}$$

This contradiction shows Lemma 2.1.

We do not have necessarily an S-path in M, but we still have a sequence of minimal segments $\gamma_i\colon [0,B_i] \to M$ with $\gamma_i'(0) \perp S$ and $B_i \to +\infty$. Since $B_i \to +\infty$, we conclude from Lemma 2.1 that the boundary value problem

$$f'' + K(t)f = 0; \qquad f'(0) = A, \qquad f(0) = 1$$

has a positive solution f on $[0,+\infty)$. So we can finish the proof of (b) by setting $w = f'/f$ and using Lemma 1.3.

To prove (a) we need the following lemma:

LEMMA 2.2. *Let* $\gamma\colon [0,B] \to M$ *be a minimal segment between* S *and* $\gamma(B)$. *Then for any* $b \in (0,B]$ *the following boundary value problem*

$$f''(t) + K(t)f(t) = 0; \qquad f'(0) = 1, \qquad f(0) = 0, \qquad f(b) = 0$$

has no solution on $[0,b]$.

To prove (c) we need the following lemma:

LEMMA 2.3. *Let* $\gamma\colon [0,B] \to M$ *be a minimal segment between* S *and* $\gamma(B)$. *Then for any* $b \in (0,B]$ *the following boundary value problem*

$$f''(t) + R(t)f(t) = 0; \qquad f'(0) = 1, \qquad f(0) = 0, \qquad f(b) = 0$$

has no solution on $[0,b]$.

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The proofs of Lemmas 2.2 and 2.3 are similar and easier than the proof of Lemma 2.1. So we will not present them here. Since the distance from S is unbounded we obtain as above that the boundary value problem

$$f''(t) + K(t)f(t) = 0; \qquad f'(0) = 1, \qquad f(0) = 0 \tag{2.4}$$

has a positive solution on $[0, +\infty)$. Set as above $w = \frac{f'}{f}$. Fix $t_0 > 0$ and let $w(t_0) = w_0$. We get the Ricatti equation

$$w' + w^2 + K(t) = 0; \qquad w(t_0) = w_0,$$

for all $t \geqslant t_0$. By Lemma 1.3 we obtain

$$\liminf_{t\to+\infty} \int_{t_0}^{t} K(\rho)\,\mathrm{d}\rho \leqslant w_0,$$

and (a) is proved. The proof of (c) is completely similar. $\square$

Set $S_t = \{x \in M \mid \mathrm{d}(x, S) = t\}$. An easy consequence of Theorem A is

COROLLARY 2.1. *Let S be minimal and compact without boundary. Assume that M is noncompact and satisfies $K_S^{\min} \geqslant 0$, and in addition that $K_S^{\min}(p) > 0$ for all p in some sphere S_t. Then S is a point, hence M has finite topological type and at most one end.*

By Theorem A we have that S must be a point. To complete the proof of Corollary 2.1, we use the result of Machigashira and Shiohama (see [Mc2], [MS]), which says that in this case M has finite topological type and admits no line. So M has at most one end.

Corollary 2.2 below is a direct consequence of Lemma 2.1 by making $K(t) \equiv c$ and noting that the function $f(t) = \cos t\sqrt{c} + (A/\sqrt{c})\sin t\sqrt{c}$ is a solution for (2.3) with $b = 1/\sqrt{c}\cot^{-1}\big(-(A/\sqrt{c})\big)$.

COROLLARY 2.2. *Let S be an isometric immersion with dimension $k \geqslant 1$. Suppose that the image of S is a closed set and that the k-Ricci curvature is bounded below by $kc > 0$. Let $A = (1/k)\sup_{p\in S}|\mathcal{H}(p)|$. Then S is $\big(1/\sqrt{c}\cot^{-1}\big(-(A/\sqrt{c})\big)\big)$-dense in M.*

To the best of our knowledge Corollary 2.2 above is new, even if we assume that $\mathrm{Ric} \geqslant (n-1)c > 0$ and that S has constant $|\mathcal{H}|$.

We recall that the radius $\mathrm{rad}(X)$ of a compact metric space X is the radius of the smallest ball which contains X, that is, $\mathrm{rad}(X) = \min_{x\in X}\max_{y\in X}\mathrm{d}(x, y)$. It follows for example from [JX] that there is no compact minimal immersion contained in an open hemisphere of S^n. Corollary 2.3 below extends this fact and follows directly from Corollary 2.2.

COROLLARY 2.3. *Assume that the k-Ricci curvature of M is bounded from below by a positive constant $kc > 0$. Given $p, q \in M$ and $0 < r < \mathrm{d}(p, q) - (\pi/2\sqrt{c})$, there is no* *compact without boundary minimal immersed submanifold of dimension k contained in* $B(p, r)$. *In particular, if M has* Ricci $\geqslant n-1$ *and the metric radius* $\operatorname{rad}(M) > \pi/2$ *then there not exist any compact minimal immersed hypersurface in* $B(p, \operatorname{rad}(M) - (\pi/2)) \subset M$.

By making $R(t) \equiv c$ in Lemma 2.3 we obtain:

COROLLARY 2.4. *Let S be an isometric immersion without boundary with dimension* $k \geqslant 1$, *and whose image is a closed set. If* $\mathrm{Ric}_S^{\min} \geqslant (n-1)c > 0$, *then S is* $\pi/\sqrt{c}$-*dense in M. In particular, M is compact if S is compact.*

If we consider circles in a Euclidean 2-sphere, it is easy to conclude that the sestimates in Corollaries 2.2 to 2.4 are sharp.

Corollary 2.5 below follows easily from Lemma 1.1 and will be used in Example 5.1.

COROLLARY 2.5. *Assume that S is an isometric totally geodesic immersion and its dimension* $k \geqslant 1$. *Suppose that for any* $p \in M$ *there exists a minimal segment* γ *between p and S, and a parallel vector field* $v(t)$ *along* $\gamma(t)$ *with* $v(0) \in T_{\gamma(0)}S$, *such that* $K(\gamma'(t), v(t)) \geqslant c > 0$ *for all t. Then S is* $\pi/2\sqrt{c}$-*dense in M.*

3. The Fundamental Group

To prove Theorem B we need the following general lemma for the existence of variations. Frankel in [F2] constructed without proof some similar variation, and for the sake of clearness we present a detailed version of it below.

LEMMA 3.1. *Let* $\gamma\colon [0, d] \to M$ *be a geodesic. Consider a parallel field V along* γ *with V orthogonal to* γ, *and smooth curves* $\mu, \eta\colon [-\delta, \delta] \to M$ *containing, respectively,* $p = \gamma(0)$ *and* $q = \gamma(d)$, *with* $\mu'(0) = V(0)$ *and* $\eta'(0) = V(d)$. *Then there exists a variation* $f(t, s)$ *of* γ *with* $\partial f/\partial s(t, 0) = V(t)$ *and* $\partial f/\partial s(0, s) = \mu'(s)$, $\partial f/\partial s(d, s) = \eta'(s)$, *for* $|s|$ *sufficiently small.*

Proof. Consider a tubular neighborhood U of the extended geodesic $\gamma\colon (-\epsilon, d+\epsilon) \to M$, and extend V to U by the parallel transport along geodesics which are orthogonal to γ.

Now consider a closed tubular neighborhood Ω of μ, where Ω consists of points whose distance from μ does not exceed $\nu > 0$. If δ and ν are sufficiently small we can assume that $\Omega \subset U$. Extend μ' to a field W in Ω by parallel transport along geodesics which are orthogonal to μ. This construction implies that W agrees with V along $\gamma \cap \Omega$. Let $\alpha\colon U \to [0, 1]$ be a smooth function such that $\alpha(x) = 0$, if $\mathrm{d}(x, \mu) \leqslant \nu/4$ and $\alpha(x) = 1$, if $\mathrm{d}(x, \mu) \geqslant 3\nu/4$. Set $X = \alpha V + (1-\alpha)W$. Clearly X agrees with V along γ and agrees with μ' along μ. Similarly we can modify X in such a way that X agrees with η' along η. The variation $f(t, s)$ can be given by the integral curves of X starting at $\gamma(t)$. □

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Now let us prove Theorem B and Corollary 0.2.

Proof of Theorem B and Corollary 0.2. Let $\pi: \tilde{M} \to M$ be the universal covering of M.

CLAIM. $\pi^{-1}(S)$ *is connected.*

Assume by contradiction that $S_1 \cup S_2 = \pi^{-1}(S)$ where S_1 is a connected component of $\pi^{-1}(S)$ and S_2 is the nonempty union of the other connected components. Both S_1 and S_2 are submanifolds which are also closed sets in $\tilde{M}$. We assert that there exist two points $\tilde{x} \in S_1$ and $\tilde{y} \in S_2$ such that $\mathrm{d}(\tilde{x}, \tilde{y}) = \mathrm{d}(S_1, S_2)$. To prove this we consider the continuous function on S_1 defined by $f(x) = \mathrm{d}(x, S_2)$. First we show that $f(x_1) = f(x_2)$ for any $x_1, x_2 \in S_1$ with $\pi(x_1) = \pi(x_2)$. In fact, let $\tilde{\sigma}$ be a continuous path joining x_1 and x_2 in S_1. Then $\sigma = \pi \circ \tilde{\sigma}$ is a loop in S. We can look at σ as an isometry of $\tilde{M}$ such that $\pi \circ \sigma = \pi$. So it is easy to see that S_1 and S_2 are invariant by σ. Then

$$f(x_2) = \mathrm{d}(x_2, S_2) = \mathrm{d}(\sigma x_1, \sigma S_2) = \mathrm{d}(x_1, S_2) = f(x_1).$$

So we can define a function $\varphi: S \to \mathbb{R}$ by $\varphi(x) = \mathrm{d}(\tilde{x}, S_2)$, where $\tilde{x}$ is any point in S_1 with $\pi(x) = x$. Since S is compact, there exists $x \in S$ such that $\varphi(x)$ is minimal. For the corresponding $\tilde{x} \in S_1$ we have that $\mathrm{d}(\tilde{x}, S_2)$ is also minimal. The existence of $\tilde{y} \in S_2$ with $\mathrm{d}(\tilde{x}, \tilde{y}) = \mathrm{d}(S_1, S_2)$ is trivial.

Now assume the hypotheses of Theorem B. Let us consider some minimal geodesic $\gamma: [0, d] \to \tilde{M}$ joining $\tilde{x}$ to $\tilde{y}$. We can choose an orthonormal basis $e_1(0), e_2(0), \ldots, e_{n-1}(0)$ of $T_{\tilde{x}}S_1$ and its parallel transport $e_1(t), e_2(t), \ldots, e_{n-1}(t)$ along γ. Let $S_{\gamma'(0)}$ and $S_{\gamma'(d)}$ be the linear self adjoint maps associated with the second fundamental forms of S_1 and S_2. Because of Lemma 3.1 we can choose any smooth curves $\mu_i \subset S_1$ and $\eta_i \subset S_2$ with $\mu_i'(0) = e_i(0)$ and $\eta_i'(0) = e_i(d)$. Lemma 3.1 provides corresponding variations. From the fact that S_1 and S_2 are minimal, we have

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^{n-1} I(e_i, e_i) \\ &= \sum_{i=1}^{n-1} \langle e_i(d), S_{\gamma'(d)} e_i(d) \rangle - \sum_{i=1}^{n-1} \langle e_i(0), S_{\gamma'(0)} e_i(0) \rangle - \int_0^b \left\{ \sum_{i=1}^{n-1} K(\gamma', e_i) \right\} \mathrm{d}t \\ &= -\int_0^b \left\{ \sum_{i=1}^{n-1} K(\gamma', e_i) \right\} \mathrm{d}t < 0. \end{aligned}$$

This contradiction proves our claim.

Now assume the hypotheses of Corollary 0.2. We consider γ as above and the parallel field $v = J\gamma'$. Then we have

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$$\begin{aligned}0 &\leqslant I(v, v)\\ &= \langle v(d), S_{\gamma'(d)}v(d)\rangle - \langle v(0), S_{\gamma'(0)}v(0)\rangle - \int_0^b K(\gamma', v)\,\mathrm{d}t\\ &= -\int_0^b K(\gamma', J\gamma')\,\mathrm{d}t < 0,\end{aligned}$$

and we have a contradiction. So the claim is proved.

For any element $[\eta] \in \pi_1(M)$, there is a representative loop ξ with base point $p \in S$. We can lift ξ to a curve $\tilde{\xi}$ in $\tilde{M}$ such that both initial point $\tilde{\xi}_0$ and end point $\tilde{\xi}_1$ belong to $\pi^{-1}(p)$. Since $\pi^{-1}(S)$ is connected there exists a curve $\tilde{v}$ in $\pi^{-1}(S)$ connecting $\tilde{\xi}_0$ and $\tilde{\xi}_1$ which is homotopic to $\tilde{\xi}$. So $\pi \circ \tilde{v}$ is a loop in S and $[\pi \circ \tilde{v}] = [\eta]$. This proves our conclusion. $\square$

Remark 3.1. Note that the proof of Corollary 0.2 shows that we can replace the hypothesis that S is totally geodesic by the hypothesis: the second fundamental form $\alpha(JN, JN) = 0$ for all unit vectors $N \perp S$.

4. The Nonpositive Case – the Cartan–Hadamard Theorem

Let $\nu(S)$ be the normal fibre bundle of the submanifold S. The next result extends the classical Hopf–Rinow Theorem ([HR]). It will be needed to prove below a generalization of the Cartan–Hadamard Theorem. It is also being used to prove a result about the existence of locally free totally geodesic $\mathbb{R}^2$-actions on spheres, in a joint work of one of the authors with F. Fang and S. Firmo.

PROPOSITION 4.1. *Let N be a connected Riemannian manifold and S a complete C^2 submanifold of N without boundary. Assume that for any point $q \in N$ there exists $p \in S$ with $\mathrm{d}(p, q) = \mathrm{d}(q, S)$ and assume further that $\exp^{\perp}(v)$ is defined for all $v \in \nu(S)$. Then N is complete.*

Proof. The proof below, with slight modifications, is presented in [Cm] for the case that S is a point.

CLAIM. *For any point $q \in N$ there exists a minimal segment between q and S.*

Set $r = \mathrm{d}(q, S)$. Take a point $p \in S$ with $\mathrm{d}(p, q) = r$. Consider a small normal ball $B(p, \delta)$. Choose x_0 in the sphere $S(p, \delta)$ such that $\mathrm{d}(x_0, q)$ is minimal. It is easy to see that $r = \delta + \mathrm{d}(x_0, q)$. So the normal geodesic γ with $\gamma(0) = p$ and $\gamma(\delta) = x_0$ is a minimal segment between x_0 and S. Then γ is orthogonal to S and satisfies $\mathrm{d}(\gamma(s), S) = s$, for $0 \leqslant s \leqslant \delta$. By hypothesis $\gamma(t)$ is defined for all $t \geqslant 0$. We assert that $\gamma(r) = q$. For this, consider the equation

$$\mathrm{d}(\gamma(s), q) = r - s. \tag{4.1}$$

This equation is true for $s = 0$. Let s_0 be the maximum value of $0 \leqslant s \leqslant r$ such that s satisfies (4.1). Assume by contradiction that $s_0 < r$. We prove that for a small $\delta' > 0$ Equation (4.1) remains satisfied for $s_0 + \delta'$, obtaining a contradiction. Let $B(\gamma(s_0), \delta')$ be a normal neighborhood of $\gamma(s_0)$. Let x_0' be in the boundary of $B\big(\gamma(s_0), \delta'\big)$ with the minimal distance from q. Then we have $d\big(\gamma(s_0), q\big) = \delta' + \mathrm{d}(x_0', q)$.

Then

$$\mathrm{d}(p, x_0') \geqslant r - \mathrm{d}(x_0', q) = r - \mathrm{d}(\gamma(s_0), q) + \delta' = s_0 + \delta',$$

hence $\mathrm{d}(p, x_0') = s_0 + \delta'$ and $x_0' = \gamma(s_0 + \delta')$. So we have

$$\mathrm{d}(\gamma\big(s_0 + \delta'\big), q\big) = \mathrm{d}(x_0', q) = \mathrm{d}\big(\gamma(s_0), q\big) - \delta' = r - s_0 - \delta',$$

and $s_0 + \delta'$ satisfies (4.1), thus arriving to a contradiction. The claim is proved.

Now we prove that bounded closed subsets of N are compact, hence N is complete. Since S is complete, the normal bundle $\nu(S)$ with the canonical (flat) metric is also complete. Let $A \subset N$ be bounded and closed. Since A is bounded, the above claim implies that there exists a ball $B(S, r) = \{(p, v) \in \nu(S) | \mathrm{d}((p, v), S) \leqslant r\}$ such that A is contained in the image $\exp^{\perp}(B(S, r))$. Take a point $(p, v) \in B(S, r)$ such that $\exp^{\perp}(p, v) \in A$. If D is the diameter of A, then for any $(x, w) \in B(S, r)$ such that $\exp \perp(x, w) \in A$ we have that $\mathrm{d}(p, x) \leqslant D + 2r$. Then there exists a compact subset $L \subset B(S, r)$ such that $\exp^{\perp}(L) \supset A$. Thus A is a closed subset of the compact set $\exp^{\perp}(L)$, hence A is compact. Proposition 4.1 is proved. □

Next we want to extend the classical Cartan–Hadamard Theorem to the radial case.

THEOREM 4.1. *Let S be a compact without boundary totally geodesic submanifold of M. Assume that the natural homomorphism i_*: $\pi_1(S) \to \pi_1(M)$ is onto (in particular, if M is simply connected) and that M satisfies $K(\gamma', v) \leqslant 0$ for all γ orthogonal to S and all v tangent to M at $\gamma(t)$. Then* $\exp^{\perp}$ *is a diffeomorphism.*

Proof. We proceed similarly as in the proof of the classical Cartan–Hadamard Theorem. First we investigate the existence of focal points of S. For this, take a geodesic γ: $[0, +\infty) \to M$ with $\gamma'(0) \in \nu(S)$. Let J be a Jacobi field along γ with $J(0) \in T_{\gamma(0)}S$ and $J'(0) + S_{\gamma'(0)}J(0) \in \nu(S)$, where $S_{\gamma'(0)}$ here is the linear and self-adjoint map associated with the second fundamental form of S. Since S is totally geodesic we have $S_{\gamma'(0)}J(0) = 0$. Consider the function $f(t) = \langle J(t), J(t)\rangle$. Then we have $f' = 2\langle J', J\rangle$ and

$$f'' = 2\langle J', J'\rangle + 2\langle J'', J\rangle = 2\langle J', J'\rangle - 2\langle R(\gamma', J)\gamma', J\rangle \geqslant 0,$$

because of one of our hypotheses. Since $f'(0) = 0$ and $f''(t) \geqslant 0$ we conclude that f is nondecreasing, hence $|J(t)| \geqslant |J(0)|$ and S is free of focal points in M.

So we have that $\exp^{\perp}: \nu(S) \to M$ is a local diffeomorphism. Thus we can define a Riem annian metric in $\nu(S)$ in such a way that $\exp^{\perp}$ becomes a local isometry. The rays γ in $\nu(S)$ which are orthogonal to S are geodesics in such a metric. By

Proposition 4.1 $\nu(S)$ with the induced metric is complete. Then we have that $\exp^{\perp}$ is a covering map.

Now we prove that $\exp^{\perp}$ is a diffeomorphism. Of course $\exp^{\perp}$ is onto. First we assert that $(\exp^{\perp})_*$ is onto. Fix $p \in S$ and take a loop $\eta \subset M$ at p. Since i_* is onto, η is homotopic to a loop $\mu \subset S$ at p. Since $(\exp^{\perp})_*(\mu) = \mu$ we have that $(\exp^{\perp})_*$ is onto. Finally we assume that $\exp^{\perp}(v) = \exp^{\perp}(w)$. We connect v and w by a curve σ. Then $\pi \circ \sigma$ is a loop in M. Since $(\exp^{\perp})_*$ is onto this loop must be lifted to a loop at v. Since the lifting of $\pi \circ \sigma$ starting at v is unique, we conclude that $v = w$. Thus $\exp^{\perp}$ is a diffeomorphism. $\square$

Remark 4.1. If we consider an Euclidean sphere $S^n \subset \mathbb{R}^{n+1}$ we see that the fact that S is totally geodesic is essential in Theorem 4.1.

Remark 4.2. The proof of Theorem 4.1 shows something more. We can replace the curvature condition by the fact that S is free of focal points.

QUESTION. Does Theorem 4.1 remain true if we replace the condition on i_* with the condition $\pi_1(M) = \pi_1(S)$?

5. Examples

First we present examples of manifolds with bounded minimal radial curvature and parallel minimal S-radial Ricci curvature.

EXAMPLE 5.1. Let $M = \mathbb{C}P^m$ be the projective space of $\mathbb{C}^{m+1}$ with the invariant Riemannian metric defined in [Wg]. The sectional curvatures of M have been known well by the work of Wong ([Wg]). Denote by S the totally geodesic submanifold of real dimension m which is isometric to a real projective space $\mathbb{R}P^m$. Given $x \in S$, $y \in M$, it has been proved in [Wg] that $1 \leqslant K(T_1, T_2) \leqslant 4$, for all $T_1, T_2 \in T_yM$ and $K(T_1, T_2) = 1$ for all $T_1, T_2 \in T_xS$. Then

$$\mathcal{RIC}_S^{\min} \geqslant m + 3. \tag{5.1}$$

The proof of (5.1) is outlined below, to understand the details a reading of [Wg] is needed. S is a maximal totally real submanifold of M. For any $x \in M \backslash S$ there exists a minimal segment γ: $[0, d] \to M$ between x and S with $\gamma(0) \in S$ and $\gamma(d) = x$. We can choose an orthonormal basis $\{e_1, e_2, \ldots, e_m\}$ in $T_{\gamma(0)}S$ such that the section P spanned by $\gamma'(0)$ and e_1 is a unitary section at $\gamma(0)$ in the sense of [Wg]. So the sectional curvature of P is 4 and is constant after parallel transport along γ. (5.1) follows because the other sections have sectional curvatures not less than 1. Using this and Corollary 2.2 we obtain that S must be $\pi/(2\sqrt{m+3})$-dense in M. Using the properties of the section P and Corollary 2.5 we can conclude that S is $\pi/4$-dense in M. It is well-known that the maximal distance $\pi/4$ from S is attained. From (5.1) we also have $\mathrm{Ric}_S^{\min} \geqslant m + 3 + (m - 1) = 2m + 2$.

EXAMPLE 5.2. Let S, N be complete manifolds and $M = S \times N$ be a Riemannian product. Assume that N satisfies the sectional curvature $K \geqslant c$ ($K \leqslant c$). Then M satisfies $K_S^{\min} \geqslant \min\{c, 0\}$ ($K_S^{\min} \leqslant \max\{c, 0\}$), where $S = S \times \{n_0\}$.

In fact, take $p \in M$ and a minimal segment γ: $[0, a] \to M$ between $p = (s, n)$ and S with $q = \gamma(0) \in S$ and $\gamma(a) = p$. We assume that $K \geqslant c$ in N (the other case is similar). Set $v = \gamma'(a)$. Since $\gamma'(0) \in v(S)$ we have that $\gamma'(0)$ is tangent to N, hence γ is contained in N. Now consider a unit vector $w \perp v$. If w is tangent to N we have $\langle R(v, w)v, w\rangle \geqslant c$. If w is tangent to $S \times \{n\}$ then we have $\langle R(v, w)v, w\rangle = 0$. Assume that w is a sum of a vector u_S which is tangent to S with a vector u_N which is tangent to N. Then we have

$$\langle R(v, w)v, w\rangle = \langle R(v, u_S)v, u_S\rangle + \langle R(v, u_N)v, u_N\rangle + 2\langle R(v, u_N)v, u_S\rangle.$$

We have that $R(v, u_N)v$ is tangent to N, hence $\langle R(v, u_N)v, u_S\rangle = 0$. Clearly

$$\langle R(v, u_S)v, u_S\rangle = 0 \quad \text{and} \quad \langle R(v, u_N)v, u_N\rangle \geqslant c|u_N|^2.$$

If $c \geqslant 0$ we have $c|u_N|^2 \geqslant 0$. Since $|u_N| \leqslant 1$, if $c < 0$ we have $c|u_N|^2 \geqslant c$. So we conclude that $K_S^{\min} \geqslant \min\{0, c\}$.

EXAMPLE 5.3. Let S^k be a complete manifold with metric g_S and $\mathbb{R}^l$ be an Euclidean space with the standard metric. Let $M = S^k \times \mathbb{R}^l \times [0, +\infty)$ be the Riemannian manifold whose metric is

$$g_M = \mathrm{d}r^2 + f^2(r)g_S + g^2(r)g_{\mathbb{R}^l},$$

where $f(r)$ and $g(r)$ are smooth functions on $[0, +\infty)$ and positive on $(0, +\infty)$ and satisfy $f(0) = 1$, $f'(0) = 0$; and $g(r) = r - (K/6)r^3 + o(r^3)$ when r is close to zero. Then some standard calculations show that

$$K_S^{\min}(x) \geqslant \min\left\{-\frac{f''(r(x))}{f(r(x))}, -\frac{g''(r(x))}{g(r(x))}\right\}$$

and

$$\mathrm{Ric}_S^{\min}(x) = -\frac{kf''(r(x))}{f(r(x))} - \frac{lg''(r(x))}{g(r(x))}.$$

This kind of models could be used to apply the volume comparison theorems in [HK] and [E].

We point out that in [W] there is an another example of a similar type of Example 5.3.

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