PROJECTIVE MODULES OVER CENTRAL SEPARABLE ALGEBRAS

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In (**2**), M. Auslander and O. Goldman laid the foundations for the study of central separable algebras. For unexplained terminology and notation, see (**2**). Here we are interested in projective modules and the ideal structure of a central separable algebra A over some special commutative rings K. When K is a field, one consequence of Wedderburn's Theorem is that there is a unique (up to isomorphism) irreducible A-module. We show here that if K is a commutative ring with a finite number of maximal ideals (semi-local) and with no idempotents other than 0 and 1 or if K is the ring of polynomials in one variable over a perfect field, then there is a unique (up to isomorphism) indecomposable finitely generated projective A-module. An example in (**3**) shows that this result fails if one only assumes that K is a principal ideal domain. If K is either a semi-local ring with no idempotents other than 0 and 1 or a ring of polynomials in one variable over a perfect field, we can obtain the structure of a central separable K-algebra in terms of a unique (up to isomorphism) central separable K-algebra D which has no idempotents other than 0 and 1. In the Brauer group of K, D plays the same role that the division algebras do when K is a field. We also extend an important theorem on lifting automorphisms in separable algebras (**6**).

Theorem 1 and Corollary 1 are due to G. J. Janusz in the special case where K is local and noetherian. In the noetherian case, Theorem 1 can also be obtained from the results in (**3**). I would like to thank G. J. Janusz for helpful conversations.

THEOREM 1. *Let K be a semi-local ring with no idempotents other than* 0 *and* 1 *and let A be a central separable K-algebra, then any two indecomposable finitely generated projective A-modules are isomorphic.*

Proof. Let P and Q be indecomposable finitely generated projective A-modules. By the hypothesis on K, P and Q are free K-modules of rank n and m, respectively (see **2**, p. 399). Let N be the radical of A, then $\Pi = N \cap K$ is the radical of K and Π is the intersection of the maximal ideals of K. Also, A/N is a central separable (K/Π)-algebra and K/Π is a direct sum of fields. Since P and Q are free K-modules, P/NP and Q/NQ are free K/Π modules of (K/Π)-rank n and m, respectively. Assume that $n \geqq m$, then from the structure theory of semi-simple rings there are A/N modules $\bar{P}$ and $\bar{P}_1$ so

Received February 10, 1967.

https://doi.org/10.4153/CJM-1969-004-x Published online by Cambridge University Press

that $P/NP \cong \bar{P} \oplus \bar{P}_1$ and $Q/NQ \cong \bar{P}$, as (A/NA)-modules. There is, then, an A epimorphism from P onto Q/NQ. Since P is projective, this epimorphism induces an A-homomorphism f from P into Q. Now $Q = \text{image}\,(f) + NQ$; thus by Nakayama's lemma (**4**, p. 66), image $(f) = Q$ and f is onto. If we let $k(f)$ be all the elements in P sent to 0 in Q by f, then, since Q is projective, $k(f)$ is a direct summand of P and since P is indecomposable, $k(f) = 0$. Thus f is an isomorphism and the theorem is proved.

Remark. If K is any semi-local ring, then $K \cong K_1 \oplus \ldots \oplus K_n$ with each K_i a semi-local ring with no idempotents other than 0 and 1. If A is a central separable K-algebra, then $A \cong A_1 \oplus \ldots \oplus A_n$, where A_i is central separable over K_i. In this case, Theorem 1 implies that there are precisely n non-isomorphic indecomposable finitely generated projective A-modules.

THEOREM 2. *Let K be the ring of polynomials in one variable over a perfect field and let A be a central separable K-algebra, then any two indecomposable finitely generated projective A-modules are isomorphic.*

Proof. Let A and A' be equivalent in $B(K)$, the Brauer group of K. Then A^0 (the opposite algebra of A) is in the inverse class of A in $B(K)$, therefore $A \otimes_K A^0 \cong \text{Hom}_K(E, E)$ and $A' \otimes_K A^0 \cong \text{Hom}_K(E', E')$ for some finitely generated projective K-modules E and E'. Thus $A \cong \text{Hom}_{A^0}(E, E)$ and $A' \cong \text{Hom}_{A^0}(E', E')$. By Morita's Theorem (see **1**, Appendix), this implies that the category of left A-modules is equivalent to the category of left A'-modules. Thus it will suffice to show that in each class in $B(K)$ there is an element which satisfies the conclusion of the theorem.

By Theorem 7.5 of (**2**), if k is a perfect field and $K = k[x]$, then every class in $B(K)$ contains an element of the form $D \otimes_k k[x]$, where D is a division algebra over k. The obvious $k[x]$-homomorphism from the algebra $D \otimes_k k[x]$ to $D[x]$ is an isomorphism on $k[x]$, thus it is an isomorphism. It is well known (and easy to show) that $D[x]$ is a principal ideal ring and hence every finitely generated projective $D[x]$-module is free. Therefore, any two indecomposable finitely generated projective $D[x]$-modules are isomorphic, which proves the theorem.

Now we derive some consequences of the above results.

COROLLARY 1. *If K is a semi-local ring with no idempotents other than* 0 *and* 1 *or a ring of polynomials in one variable over a perfect field, then every class in $B(K)$, the Brauer group of K, is represented by a unique (up to isomorphism) central separable K-algebra D with no idempotents other than* 0 *and* 1. *Moreover, if A is equivalent to D in $B(K)$, then $A \cong M_n(D)$ for a uniquely determined integer n.*

Proof. Let D and D' be central separable K-algebras without idempotents except for 0 and 1 and equivalent in $B(K)$. Then if D^0 is the opposite algebra

https://doi.org/10.4153/CJM-1969-004-x Published online by Cambridge University Press

of D, there are finitely generated projective K-modules E and E' with $D \otimes_K D^0 \cong \mathrm{Hom}_K(E, E)$, $D' \otimes_K D^0 \cong \mathrm{Hom}_K(E', E')$. Thus $D \cong \mathrm{Hom}_{D^0}(E, E)$ and $D' \cong \mathrm{Hom}_{D^0}(E', E')$. Since D and D' have no idempotents other than 0 and 1, E and E' are indecomposable finitely generated projective D^0-modules so are isomorphic over D^0 by Theorems 1 and 2. Thus we conclude that $D \cong D'$.

Now let A be any central separable K-algebra equivalent to D in $B(K)$. Arguing as above, there is a finitely generated projective D^0-module E with $A \cong \mathrm{Hom}_{D^0}(E, E)$. But E is a direct sum of a finite number of indecomposable D^0-modules which are all isomorphic to D^0 by Theorems 1 and 2 since D^0 has no idempotents other than 0 and 1; hence it is indecomposable when viewed as a module over itself. Thus E is a free D^0-module and $A \cong M_n(D)$. By the first paragraph, D is uniquely determined by A and therefore n is uniquely determined by A.

COROLLARY 2. *With the hypothesis of Corollary* 1, *there is a decomposition of A into mutually isomorphic indecomposable projective A-ideals, and any finitely generated projective A-module is isomorphic to a direct sum of projective left ideals of A. If Ae is an indecomposable projective direct summand of A generated by the idempotent e, then eAe is equivalent to A in $B(K)$, eAe has no idempotents other than* 0 *and* 1, *and* $A = M_n(eAe)$.

Proof. We can write $A = Ae_1 \oplus \ldots \oplus Ae_n$, where the Ae_i are projective indecomposable left ideals in A and the e_i are orthogonal idempotents. By Theorems 1 and 2, all the left ideals which appear in any such decomposition are isomorphic. Let Ae be such an ideal, then $\mathrm{Hom}_A(Ae, Ae) = (eAe)^0$, therefore eAe is a central separable K-algebra with no idempotents other than 0 and 1 and by Morita's Theorem, $\mathrm{Hom}_{(eAe)^0}(Ae, Ae) = A$ (see **11**, §1). Thus eAe is in the same class as A, namely, $B(K)$, and by Corollary 1, $A \cong M_n(eAe)$.

Now we prove a result on extending automorphisms (see also **6**, Theorem 1).

COROLLARY 3. *Let K be a semi-local ring without idempotents other than* 0 *and* 1 *or let K be the ring of polynomials in one variable over a field of characteristic zero. Let A be a central separable K-algebra, B a separable subalgebra with no central idempotents except for* 0 *and* 1, *then any algebra monomorphism from B into A extends to an inner automorphism of A.*

Proof. In the semi-local case, the result is just Theorem 1 of (**6**), and the argument is analogous to the one which follows. Let k be a field of characteristic $= 0$. We need to make use of the concept and properties of the separable closure of a ring which are developed in (**10**, §1). We refer the interested reader to this paper for the relevant definitions and facts we employ here. Let $\bar{k}$ be the algebraic closure of k, then, using the terminology of (**10**), $\bar{k}[x]$ is a separably closed ring. This fact is proved in (**9**, p. 106). Since $\bar{k}[x]$ is a locally strongly separable $k[x]$-algebra, $\bar{k}[x]$ is the separable closure of $k[x]$. Let S be any commutative finitely generated projective separable $k[x]$-algebra with no idempotents other than 0 and 1. By the results in (**10**), S is isomorphic to a $k[x]$-subalgebra of $\bar{k}[x]$. One concludes from the remark in (**10**, p. 466) that $S \cong l[x]$ for some finite separable field extention l of k. By Theorem 2, any two finitely generated projective modules over the central separable $l[x]$-algebra Y are Y-isomorphic if and only if they are $l[x]$-isomorphic. Let $Y = B \otimes_K A^0$ $(K = k[x])$, and define left Y-module structures on A via $(b \otimes a)x = bxa$ and $(b \otimes a)x = \tau(b)xa$. We call the resulting modules A_1 and A_2. Then A_1 and A_2 are projective K-modules and, therefore, are projective Y-modules. Also, if S is the centre of B, then by the preceding paragraph, $S \cong l[x]$ for a finite separable field extension l of k. Also, S may be viewed as the centre of Y therefore A_1 and A_2 are isomorphic as $l[x]$-modules and thus there is a Y-isomorphism h sending A_1 onto A_2. Then $h(1a) = h(1)a$ for all $a \in A$. Thus $h(1)$ is a unit in a; $h(b) = h(1)b = \tau(b)h(1)$ therefore

$$\tau(b) = h(1)bh(1)^{-1}.$$

Thus τ extends to an inner automorphism of A, proving the corollary.

In (**3**, p. 46), an example of a principal ideal domain K and a central separable K-algebra D, for which the conclusion of Theorem 1 fails, is presented. If k is a field which is not perfect, then the mapping from $B(k)$ to $B(k[x])$ is not onto, this fact is pointed out in (**2**, p. 389). If $\bar{k}$ is an algebraically closed field of characteristic p $(p \neq 0)$, then $\bar{k}[x]$ is not separably closed since the discriminant of the polynomial $y^p - y - x$ is a unit in $\bar{k}[x]$; see (**10**, Theorem 2.2). If K is a ring with trivial class group, then the problem of finding a unique indecomposable finitely generated projective module over a central separable K-algebra A is equivalent to finding a unique representative with no idempotents other than 0 and 1 in the class of A in $B(K)$, this remark is a consequence of the results found in (**12**).

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