Can. J. Math., Vol. XXVII, No. 2, 1975, pp. 396–417

UNIFORM FINITE GENERATION OF THREE-DIMENSIONAL LINEAR LIE GROUPS

R. M. KOCH AND FRANKLIN LOWENTHAL

1. Introduction. A connected Lie group G is generated by one-parameter subgroups $\exp(tX_1), \ldots, \exp(tX_k)$ if every element of G can be written as a finite product of elements chosen from these subgroups. This happens just in case the Lie algebra of G is generated by the corresponding infinitesimal transformations $X_1, \ldots, X_k$; indeed the set of all such finite products is an arcwise connected subgroup of G, and hence a Lie subgroup by Yamabe's theorem [**9**]. If there is a positive integer n such that every element of G possesses such a representation of length at most n, G is said to be uniformly finitely generated by the one-parameter subgroups. In this case define the order of generation of G as the least such n; otherwise define it as infinity. Since the order of generation will, in general, depend upon the one-parameter subgroups, G may have many different orders of generation. Each such order of generation must be greater than or equal to the dimension of G by Sard's Theorem [**8**].

Computation of the order of generation of G for given $X_1, \ldots, X_k$ is analogous to finding the greatest wordlength needed to write each element of a finite group in terms of generators $g_1, \ldots, g_k$. In both cases it is natural to restrict attention to minimal generating sets. From now on, therefore, suppose that no subset of $\{X_1, \ldots, X_k\}$ generates the Lie algebra of G.

The only connected Lie groups of dimension two are $R \times R$, $R \times S^1$, $S^1 \times S^1$, and the $ax + b$ group (that is, the identity component of the real affine group on the line). Clearly the order of generation of a two-dimensional abelian Lie group is always two; a routine calculation shows that the same result holds for the $ax + b$ group [**4**].

In this paper we find the possible orders of generation for all linear Lie groups of dimension three. This problem has previously been solved for some three dimensional Lie groups when the generating set has two elements. The order of generation of $SO(3)$ and its universal covering group $SU(2)$ may be any integer greater than or equal to 3; it is determined by the Killing form of the pair of infinitesimal transformations [**3**; **5**]. The order of generation of $PSL(2, R)$ is infinite if both one-parameter subgroups are elliptic, 3 if exactly one is elliptic, 6 if both are hyperbolic with interlacing eigenvectors, and 4 in all other cases [**2**]. The same result holds for $SL(2, R)$ except that when the one-parameter subgroups are both hyperbolic with interlacing eigenvectors, the order of generation is 8 instead of 6 [**5**]. The order of generation of the isometry group of

Received September 24, 1973 and in revised form, August 14, 1974.

https://doi.org/10.4153/CJM-1975-048-0 Published online by Cambridge University Press

the Euclidean geometry $E(2)$ is infinite if both one-parameter subgroups are elliptic and 3 if one is elliptic and the other parabolic [**2**]. $E(2)$ is a three-dimensional subgroup of the complex affine group $\alpha z + \beta$ (α and β complex numbers). All other connected three-dimensional subgroups of this group have order of generation 3 except the subgroup $\alpha z + \beta$ where $\alpha > 0$, which cannot be generated by a pair of one-parameter subgroups [**4**]. It is, of course, evident that no abelian Lie group of dimension greater than two can be generated by a pair of one-parameter subgroups.

Let $n \neq \infty$ be the order of generation of a connected Lie group G generated by $\exp(tX_1), \ldots, \exp(tX_k)$. It is of interest to determine whether every element of G can be represented as a product of length n of the form $\exp(t_1X_{i_1}) \circ \ldots \circ \exp(t_nX_{i_n})$ where only the t_i's vary from element to element and the generators occur in some predetermined order. We shall find all possible expressions of this type for all three-dimensional linear Lie groups.

We would like to thank the referee for several helpful comments; in particular it was his idea to consider minimal generating sets with more than two elements.

2. Classification of connected three-dimensional Lie groups. Recall the real affine group $A(n)$ acting on R^n; it is the set of all transformations from R^n to R^n of the form $v \to Av + l$ where $A \in GL(n, R)$ and $l \in R^n$. From now on denote such a transformation by $\langle A, l\rangle$. The affine group can be considered a subgroup of $GL(n + 1, R)$ by thinking of l as a column vector and identifying $\langle A, l\rangle$ with a matrix $\begin{bmatrix} A & l \\ 0 & 1 \end{bmatrix}$. The Lie algebra $a(n)$ of this group is thus the set of all matrices of the form $\begin{bmatrix} A & l \\ 0 & 0 \end{bmatrix}$ where A is an arbitrary $n \times n$ matrix and $l \in R^n$. Reversing the previous identification, we denote this element of the Lie algebra by $\langle A|l\rangle$. Notice that

$$[\langle A|l\rangle, \langle B|m\rangle] = \langle [A, B]|Am - Bl\rangle.$$

Let G_1 be a one-parameter subgroup of $GL(2, R)$; then $G = \{\langle A, l\rangle | A \in G_1, l \in R^2\}$ is a connected three-dimensional Lie group. Naturally, conjugate one-parameter subgroups give rise to isomorphic three-dimensional groups. By the theorem of the rational canonical form, each one-parameter subgroup of $GL(2, R)$ is conjugate to a subgroup generated by precisely one of the following infinitesimal generators:

$$\begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \quad |\alpha| < 1, \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} \quad \alpha \geqq 0.$$

The list of three-dimensional Lie algebras given early in Jacobson's book [**1**] implies

THEOREM 1. *Every three-dimensional real Lie algebra is isomorphic to precisely one of the following:*

https://doi.org/10.4153/CJM-1975-048-0 Published online by Cambridge University Press

(1) $R \times R \times R$,

(2_α) $\left\{ \langle A|l \rangle \in a(2) | A \text{ is a scalar multiple of } \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \right\}$ *for a fixed* α, $|\alpha| \leqq 1$,

(3) $\left\{ \langle A|l \rangle \in a(2) | A \text{ is a scalar multiple of } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$,

(4) $\left\{ \langle A|l \rangle \in a(2) | A \text{ is a scalar multiple of } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$,

(5_α) $\left\{ \langle A, l \rangle \in a(2) | A \text{ is a scalar multiple of } \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} \right\}$ *for a fixed* α, $\alpha \geqq 0$,

(6) $su(2)$,

(7) $sl(2, R)$.

THEOREM 2. *Every connected three-dimensional Lie group is isomorphic to precisely one of the following:*

(1) $R \times R \times R$ *or* $R \times R \times S^1$ *or* $R \times S^1 \times S^1$ *or* $S^1 \times S^1 \times S^1$,

(2_0) $(ax + b) \times R$ *or* $(ax + b) \times S^1$,

(2_α) $\left\{ \langle A, l \rangle \in A(2) | A = \begin{bmatrix} e^t & 0 \\ 0 & e^{\alpha t} \end{bmatrix} \right\}$ *for a fixed* α, $0 < |\alpha| \leqq 1$,

(3) $\left\{ \langle A, l \rangle \in A(2) | A = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \right\}$,

(4) $\left\{ \langle A, l \rangle \in A(2) | A = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \right\}$ *or*

$$\left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \right\} \Big/ \left\{ \begin{bmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| n \in Z \right\},$$

(5_0) $\tilde{E}(2)/kZ$ *where* k *is a fixed integer,* $k = 0, 1, 2, 3, \ldots$, *and* $\tilde{E}(2)$ *is the universal covering group of* $E(2)$,

(5_α) $\left\{ \langle A, l \rangle \in A(2) | A = e^{\alpha t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \right\}$ *for a fixed* α, $0 < \alpha$,

(6) $SU(2)$ *or* $SO(3)$,

(7) $\widetilde{SL}/kZ$ *where* k *is a fixed integer,* $k = 0, 1, 2, 3, \ldots$, *and* $\widetilde{SL}$ *is the universal covering group of* $SL(2, R)$.

Proof. If G is a simply connected Lie group with Lie algebra g and center $C(G)$, the most general connected Lie group with Lie algebra g is G/N where N is a discrete subgroup of $C(G)$; moreover $C(G/N) = C(G)/N$. If N_1 and N_2 are two such subgroups, G/N_1 and G/N_2 are isomorphic if and only if there is an automorphism σ of G such that $\sigma(N_1) = N_2$.

Clearly the first group listed after each number in Theorem 2 is the simply connected group corresponding to the algebra listed in Theorem 1. A simple calculation shows that the groups listed in 1, 2_α, 3, and 5_α have trivial center.

https://doi.org/10.4153/CJM-1975-048-0 Published online by Cambridge University Press

The center of $(ax + b) \times R$ is $\{0\} \times R$; its discrete subgroups are $\{0\} \times \lambda Z$ for fixed λ. If $\lambda \neq 0$, the automorphism $\sigma(g \times r) = g \times r/\lambda$ takes this subgroup to $0 \times Z$, so the only discrete subgroups that need be considered are $\{0\}$ and $\{0\} \times Z$, and these give rise to $\{ax + b\} \times R$ and $\{ax + b\} \times S^1$.

The group

$$\left\{ \langle A, l \rangle \in A(2) | A = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \right\}$$

is the same as

$$\left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \right\} \text{ and has center } \left\{ \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| b \in R \right\}$$

which in turn has discrete subgroups

$$\left\{ \begin{bmatrix} 1 & 0 & \lambda n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| n \in Z \right\};$$

if $\lambda \neq 0$, the automorphism

$$\sigma\left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & a/\sqrt{|\lambda|} & b/|\lambda| \\ 0 & 1 & c/\sqrt{|\lambda|} \\ 0 & 0 & 1 \end{bmatrix}$$

takes the latter group to

$$\left\{ \begin{bmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| n \in Z \right\}.$$

Hence the result listed in 4.

By a short calculation, the center of $E(2)$ is trivial, since $E(2)$ is topologically the same as $S^1 \times R^2$, $\pi_1(E(2)) = Z$. Thus the center of $\tilde{E}(2)$ is Z and the discrete subgroups of this group are kZ for $k = 0, 1, 2, \ldots$. Since $C(\tilde{E}(2)/kZ) = C(\tilde{E}(2))/kZ = Z_k$, distinct k's give rise to non-isomorphic groups.

The center of $SU(2)$ is $\pm I$, so the only groups with Lie algebra $su(2)$ are $SU(2)$ and $SU(2)/\pm I \cong SO(3)$.

Finally, the center of $PSL(2, R)$ is trivial. Since $PSL(2, R)$ is topologically the same as $S^1 \times R$, then $\pi_1(PSL(2, R)) = Z$. Thus $C(\widetilde{SL}(2, R)) = Z$. By the technique employed for $\tilde{E}(2)$ we find that the groups with Lie algebra $sl(2, R)$ are those listed in 7.

THEOREM 3. *The only non-linear three-dimensional connected Lie groups are*

$$\left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \right\} \Big/ Z \textit{ and } \widetilde{SL}(2, R)/kZ \textit{ for } k \neq 1, 2.$$

Proof. The groups $\tilde{E}(2)$ and $\tilde{E}(2)/kZ$ are respectively

$$\left\{\begin{bmatrix} t & 0 & 0 & 0 \\ 0 & \cos t & -\sin t & a \\ 0 & \sin t & \cos t & b \\ 0 & 0 & 0 & 1 \end{bmatrix}\right\} \text{ and } \left\{\begin{bmatrix} \cos t/k & -\sin t/k & 0 & 0 & 0 \\ \sin t/k & \cos t/k & 0 & 0 & 0 \\ 0 & 0 & \cos t & -\sin t & a \\ 0 & 0 & \sin t & \cos t & b \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right\}.$$

All other groups listed in Theorem 2 that are linear are clearly so. Let

$$G = \left\{\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\right\} \Big/ Z$$

and suppose $\rho: G \rightarrow GL(n, C)$ is one-to-one. Notice that the Lie algebra g of G is nilpotent: after a suitable conjugation we may suppose that each element of $\rho_*(g)$ has the form

$$\begin{bmatrix} \square & & & \\ & \square & & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & & & \square \end{bmatrix}$$

where the blocks are upper-triangular with constant entries on the diagonal. Clearly, then, whenever $X \in [\rho_* g, \rho_* g]$, the map from R to $GL(n, C)$, given by $t \rightarrow (\exp tX)$, is one-to-one. However, $\left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $\exp\begin{bmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the identity in G whenever $t \in Z$.

Every representation of $sl(2, R)$ can be written as a direct sum of irreducible representations, which are explicitly known, [**7**]. These representations all come from representations of $SL(2, R)$. Hence every representation of $\widetilde{SL}(2, R)$ vanishes on $2Z$; $\widetilde{SL}(2, R)/kZ$ possesses a faithful matrix representation only if $k = 1$ or 2.

3. Results. Let G be a connected Lie group with Lie algebra g; let $X_1, \ldots, X_k$ generate g. If $\sigma: G \rightarrow G$ is an automorphism, $\sigma_*(X_1), \ldots, \sigma_*(X_k)$ also generate g. Clearly the order of generation of G with respect to $X_1, \ldots, X_k$ is the same as the order of generation with respect to $\sigma_*(X_1), \ldots, \sigma_*(X_k)$. We shall classify completely minimal generating sets up to automorphisms of G.

Definition. Two minimal generating sets $\{X_1, \ldots, X_k\}$ and $\{Y_1, \ldots, Y_k\}$ of the Lie algebra g of G are equivalent if there exist real numbers $r_1, \ldots, r_k$

https://doi.org/10.4153/CJM-1975-048-0 Published online by Cambridge University Press

a permutation τ of $\{1, \ldots, k\}$, and an automorphism σ of G such that $Y_i = r_i\sigma_*(X_{\tau(i)})$.

THEOREM 4. *On the next two pages is a list of all connected three-dimensional linear Lie groups G, their minimal generating sets up to equivalence, and the orders of generation of G with respect to these sets. When the order of generation n is finite, the last column indicates those expressions of length n which yield all elements of G.* (For instance, $XYXY$ means every element of G can be written in the form $\exp(t_1X) \circ \exp(t_2Y) \circ \exp(t_3X) \circ \exp(t_4Y)$.)

4. Proof. Let $\{X_1, \ldots, X_k\}$ be a minimal generating set for a three-dimensional Lie algebra with $k = 2$ or 3. When $k = 2$ the elements $X_1, X_2, [X_1, X_2]$ are linearly independent. When $k = 3$ the elements X_1, X_2, X_3 are linearly independent and each pair $\{X_i, X_j\}$ generates a two-dimensional subalgebra.

For the moment set aside groups locally isomorphic to subgroups of $A(2)$. The theorem is trivial for abelian groups. In the case of two generators, the groups $SU(2)$ and $SO(3)$ were considered in [**3**] and [**5**] and the groups $PSL(2, R)$ and $SL(2, R)$ were considered in [**2**] and [**5**]. It is easy to see that $su(2)$ contains no two-dimensional subalgebras, so $SU(2)$ and $SO(3)$ possess no minimal generating sets with three elements.

Suppose $\{X, Y, Z\}$ is a minimal generating set for $PSL(2, R)$ or $SL(2, R)$. Let $B \in GL(2, R)$; the map $A \to BAB^{-1}$ is an automorphism of $SL(2, R)$ which preserves $\{\pm I\}$ and so induces an automorphism of $PSL(2, R)$. The induced Lie algebra automorphism maps X to BXB^{-1}; therefore we may assume X is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, or $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. If $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, $[X, Y] = \begin{bmatrix} c & -2a \\ 0 & -c \end{bmatrix}$; X and Y generate a two-dimensional subalgebra only if $c = 0$. Similarly Z has the form $\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$ so X, Y, Z do not generate $sl(2, R)$. If $X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, $[X, Y] = \begin{bmatrix} -(b+c) & 2a \\ 2a & b+c \end{bmatrix}$; then X, Y, and $[X, Y]$ are linearly independent unless Y is a multiple of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Similarly Z must be a multiple of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and X, Y, Z cannot generate $sl(2, R)$

In short, $X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. If $Y = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, $[X, Y] = \begin{bmatrix} 0 & 2b \\ -2c & 0 \end{bmatrix}$; the latter is a linear combination of X and Y precisely when $b = 0$ or $c = 0$. So we may assume

$$Y = \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix}, \qquad Z = \begin{bmatrix} b & 0 \\ 1 & -b \end{bmatrix}.$$

G	X	Y	Z	Order of generation	Expressions giving all of G
(1) $R \times R \times R$ $R \times R \times S^1$ $R \times S^1 \times S^1$ $S^1 \times S^1 \times S^1$	Any linearly independent set			3	All permutations of XYZ
(2_0) $(ax + b) \times R$	$\langle 1\|0\rangle \times 0$	$\langle 0\|1\rangle \times 1$	—	4	$XYXY$, $YXYX$
	$\langle 1\|0\rangle \times 0$	$\langle 1\|1\rangle \times 1$	—	4	$XYXY$, $YXYX$
	$\langle 1\|0\rangle \times 0$	$\langle 0\|1\rangle \times 0$	$\langle 0\|0\rangle \times 1$	3	All permutations of XYZ
	$\langle 1\|0\rangle \times 0$	$\langle 0\|1\rangle \times 0$	$\langle 1\|0\rangle \times 1$	3	All permutations of XYZ
	$\langle 1\|0\rangle \times 0$	$\langle 1\|1\rangle \times 0$	$\langle 0\|0\rangle \times 1$	3	None
$(ax + b) \times S^1$	$\langle 1\|0\rangle \times 0$	$\langle 0\|1\rangle \times 1$	—	4	$XYXY$, $YXYX$
	$\langle 1\|0\rangle \times 0$	$\langle 1\|1\rangle \times \lambda$, $\lambda \neq 0$	—	4	$YXYX$, $YXYX$
	$\langle 1\|0\rangle \times 0$	$\langle 0\|1\rangle \times 0$	$\langle 0\|0\rangle \times 1$	3	All permutations of XYZ
	$\langle 1\|0\rangle \times 0$	$\langle 0\|1\rangle \times 0$	$\langle 1\|0\rangle \times \lambda$, $\lambda \neq 0$	3	All permutations of XYZ
	$\langle 1\|0\rangle \times 0$	$\langle 1\|1\rangle \times 0$	$\langle 0\|0\rangle \times 1$	3	None
(2_1) $\left\{ \langle A, l\rangle \in A(2) \mid A = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \right\}$	$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle 0 \middle\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle 0 \middle\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$	3	All permutations of XYZ
	$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle 0 \middle\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$	3	None
	$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \middle\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$	3	None
(2_α) $\left\{ \langle A, l\rangle \in A(2) \mid A = \begin{bmatrix} e^t & 0 \\ 0 & e^{\alpha t} \end{bmatrix} \right\}$	$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle 0 \middle\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$	—	4	$XYXY$, $YXYX$
for $0 < \|\alpha\| \leqq 1, \alpha \neq 1$	$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \middle\| \begin{bmatrix} -1 \\ -\alpha \end{bmatrix} \right\rangle$	—	4	None
	$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle 0 \middle\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle 0 \middle\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$	3	All permutations of XYZ
	$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle 0 \middle\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$	3	None
	$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \middle\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle 0 \middle\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$	3	None
(3) $\left\{ \langle A, l\rangle \in A(2) \mid A = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \right\}$	$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle 0 \middle\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$	—	4	$XYXY$, $YXYX$
	$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$	—	4	None
(4) $\left\{ \langle A, l\rangle \in A(2) \mid A = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \right\}$	$\left\langle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle 0 \middle\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$	—	4	$XYXY$, $YXYX$
(5_0) $\tilde{E}(2)/kZ$ for $k = 0, 1, 2, 3, \ldots$	$\left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle 0 \middle\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$	—	3	XYX
	$\left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	—	∞	—
(5_α) $\{ \langle A, l\rangle \in A(2) \mid A =$ $e^{\alpha t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \}$	$\left\langle \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle 0 \middle\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$	—	3	XYX
for $\alpha < 0$	$\left\langle \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle$	$\left\langle \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} \middle\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$	—	3	XYX, YXY

G	X	Y	Z	Order of generation	Expressions giving all of G
(6) $SU(2)$	$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$	$\begin{bmatrix} i\cos\theta & i\sin\theta \\ i\sin\theta & i\cos\theta \end{bmatrix}$ $0 < \theta \leqq \frac{\pi}{2}$	—	$n+2$ for $\pi/(n+1) \leqq \theta < \pi/n$	Both possibilities
$SO(3)$	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -\cos\theta & 0 \\ \cos\theta & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{bmatrix}$	—	$n+2$ for $\pi/(n+1) \leqq \theta < \pi/n$	Both possibilities
(7) $SL(2, R)$	elliptic: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	elliptic: $\begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}$ $-1 < \alpha < 0$	—	∞	—
	elliptic: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	parabolic: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	—	3	XYX
	elliptic: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	hyperbolic: $\begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}$ $0 < \alpha \leqq 1$	—	3	XYX
	parabolic: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	parabolic: $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	—	4	$XYXY$, $YXYX$
	parabolic: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	hyperbolic: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	—	4	$XYXY$, $YXYX$
	hyperbolic: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	hyperbolic (fixed points non-interlacing) $\begin{bmatrix} \alpha & -1 \\ 1 & -\alpha \end{bmatrix}$ $\alpha > 1$	—	4	None
	hyperbolic: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	hyperbolic (fixed points interlacing) $\begin{bmatrix} \alpha & 1 \\ 1 & -\alpha \end{bmatrix}$ $a \geqq 0$	—	8	$XYXYXYXY$, $YXYXYXYX$
	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$	6	None
$PSL(2, R)$	elliptic: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	elliptic: $\begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}$ $-1 < \alpha < 0$	—	∞	—
	elliptic: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	parabolic: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	—	3	XYX
	elliptic: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	hyperbolic: $\begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}$ $0 < \alpha \leqq 1$	—	3	XYX
	parabolic: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	parabolic: $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	—	4	$XYXY$, $YXYX$
	parabolic: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	hyperbolic: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	—	4	$XYXY$, $YXYX$
	hyperbolic: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	hyperbolic (fixed points non-interlacing) $\begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix}$ $\alpha > 1$	—	4	None
	hyperbolic: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	hyperbolic (fixed points interlacing) $\begin{bmatrix} \alpha & 1 \\ 1 & -\alpha \end{bmatrix}$ $\alpha \geqq 0$	—	6	$XYXYXY$, $YXYXYX$
	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$	4	None

https://doi.org/10.4153/CJM-1975-048-0 Published online by Cambridge University Press

Notice that $[Y, Z] = \begin{bmatrix} 1 & -2b \\ -2a & -1 \end{bmatrix}$; this is a linear combination of Y and Z precisely when $ab = -\frac{1}{4}$. Conjugate X, Y and Z using $\begin{bmatrix} 2a & 0 \\ 0 & 1 \end{bmatrix}$; X maps to X, Y to $\begin{bmatrix} a & 2a \\ 0 & -a \end{bmatrix}$, and Z to $\begin{bmatrix} b & 0 \\ -2b & -b \end{bmatrix}$. So we may assume

$$X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}.$$

Notice that $\exp(tX)$ and $\exp(tY)$ generate the two-dimensional group $\left\{\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \middle| a > 0\right\}$; similarly $\exp(tX)$ and $\exp(tZ)$ generate $\left\{\begin{bmatrix} a & 0 \\ b & a^{-1} \end{bmatrix} \middle| a > 0\right\}$ and $\exp(tY)$ and $\exp(tZ)$ generate

$$\left\{\begin{bmatrix} a+b & a \\ c & c+b \end{bmatrix} \middle| b > 0, a + c = \frac{1-b^2}{b}\right\}.$$

The order of generation of $PSL(2, R)$ is at least 4, for $\pm\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ cannot be written as a product of three elements of the requisite types. Indeed $\pm\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ does not belong to the two-dimensional subgroups described earlier, so the only possible representations involve X, Y and Z. Instead of writing $\exp(t_1X) \circ \exp(t_2Y) \circ \exp(t_3Z)$ write XYZ; notice that $XYZ = XYZX$, since $\exp(0 \cdot X) = I$; thus $\pm\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ would have one of the forms XYZ, XZY, YXZ, YZX, ZXY, ZYX and therefore one of the forms $(XY)(ZX)$, $(XZ)(YX)$, $(XY)(XZ)$, $(XY)(ZX)$, $(XZ)(XY)$, $(XZ)(YX)$. But

$$\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} c & 0 \\ d & c^{-1} \end{bmatrix} = \begin{bmatrix} * & * \\ * & a^{-1}c^{-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c & 0 \\ d & c^{-1} \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} = \begin{bmatrix} ac & * \\ * & * \end{bmatrix}$$

and neither can equal $\pm\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Since the order of generation of each two dimensional Lie group is 2, and

$$\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} c & 0 \\ d & c^{-1} \end{bmatrix} = \begin{bmatrix} ac + bd & bc^{-1} \\ a^{-1}d & a^{-1}c^{-1} \end{bmatrix}$$

can be written as a product of 4 elements then every $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in $SL(2, R)$ with $D > 0$ can be so generated. Similarly

$$\begin{bmatrix} c & 0 \\ d & c^{-1} \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} = \begin{bmatrix} ac & bc \\ ad & bd + a^{-1}c^{-1} \end{bmatrix}$$

and thence $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be written as a product of 4 elements if $A > 0$.

https://doi.org/10.4153/CJM-1975-048-0 Published online by Cambridge University Press

Finally

$$\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} c+d & c \\ e & e+d \end{bmatrix} = \begin{bmatrix} a(c+d)+be & ac+b(e+d) \\ a^{-1}e & a^{-1}(e+d) \end{bmatrix}$$

where $a > 0$, $d > 0$, $c + e = (1 - d^2)/d$; if $C < 0$, $\begin{bmatrix} 0 & -C^{-1} \\ C & 0 \end{bmatrix}$ can be written thus. In short, whenever $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in SL(2, R)$, then one of $\pm\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be written as a product of 4 elements, and so the order of generation of $PSL(2, R)$ is 4.

No fixed expression of length 4 generates $PSL(2, R)$. Indeed such an expression would involve all three generators; and some generator would be repeated. Without loss of generality we can take this repeated generator to be X, for conjugation by $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ is an automorphism interchanging X and Y and taking Z to $-Z$ and conjugation by $\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$ is a similar automorphism interchanging X and Z. The expression we seek would thus have one of the forms $(XY)(ZX)$, $(XZ)(YX)$, $(XY)(XZ)$, $(XZ)(XY)$, $(YX)(ZX)$, $(ZX)(YX)$ whereas some elements of $PSL(2, R)$ cannot be written as $\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} c & 0 \\ d & c^{-1} \end{bmatrix}$ or $\begin{pmatrix} c & 0 \\ d & c^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$.

An alternate geometric proof along the lines developed in [**2**] can also be given.

The order of generation of $SL(2,R)$ is at least 6, for $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ cannot be written as a product of five elements. Indeed an expression of length five would involve some generator only once. Since all automorphisms of $SL(2, R)$ leave $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ fixed, we can assume this generator is Z. Thus the product of length five would have the form

$$\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} c & 0 \\ d & c^{-1} \end{bmatrix}\begin{bmatrix} e & f \\ 0 & e^{-1} \end{bmatrix} = \begin{bmatrix} * & * \\ a^{-1}de & a^{-1}df + a^{-1}c^{-1}e^{-1} \end{bmatrix}$$

where $a > 0$, $c > 0$, $e > 0$; but $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ cannot be written in this form.

The order of generation of $SL(2, R)$ is 6, for

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix}\begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+bd & (1+bd)f+b \\ d & df+1 \end{bmatrix}$$

and $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in SL(2, R)$ can be so written if $C \neq 0$; conjugation by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

maps X to $-X$, Y to $-Z$, Z to $-Y$, and $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ to $\begin{bmatrix} D & C \\ B & A \end{bmatrix}$ and so $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can also be written as a product of six elements if $B \neq 0$. We saw earlier that conjugation by $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ permutes X, Y, Z up to sign; it takes $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ to $\begin{bmatrix} A & A-D \\ 0 & D \end{bmatrix}$, and so $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ can be written as a product of six elements unless $A = D$. Only the case of $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ remains. However

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and $\begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$ has the form $\begin{bmatrix} a+b & a \\ c & c+b \end{bmatrix}$, $b > 0$, $ac = (1-b^2)/b$.

No fixed expression of length six generates $SL(2, R)$. Indeed assume such an expression exists; apply a suitable automorphism to obtain an expression starting with XY. The expression must then be $XYZXYZ$, for otherwise two letters would occupy three successive places; the subgroup generated by any two letters has dimension two, and hence has order of generation two. Thus $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ could then be written as a product of length five. If $XYZXYZ$ gives all of $SL(2, R)$, then

$$\begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} c & 0 \\ d & c^{-1} \end{bmatrix}\begin{bmatrix} e+f & e \\ g & g+f \end{bmatrix}$$
$$= \begin{bmatrix} * & * \\ a^{-1}d(e+f) + a^{-1}c^{-1}g & a^{-1}de + a^{-1}c^{-1}(g+f) \end{bmatrix}$$

where $a > 0$, $c > 0$, $f > 0$, $e + g = (1-f^2)/f$; so $d = c^{-1}$ and $-1 = a^{-1}c^{-1}(e+f+g) = a^{-1}c^{-1}((1-f^2)/f + f) = a^{-1}c^{-1}f^{-1} > 0$.

An alternate geometric proof can be given using the ideas developed in [**5**].

5. All remaining groups are locally isomorphic to subgroups of $A(2)$. The initial calculations which follow hold for arbitrary semidirect products of R and R^n. Suppose, then, that A_0 is a fixed non-zero $n \times n$ matrix and $g = \{\langle rA_0|l\rangle | r \in R, l \in R^n\}$ and that g is the Lie algebra of

$$G = \{\langle e^{tA_0}, l\rangle \in A(n) | t \in R, l \in R^n\}.$$

For each fixed $l_1 \in R^n$, the map $\langle rA_0|l\rangle \to \langle rA_0|l - rl_1\rangle$ is a Lie algebra automorphism of g induced by the group automorphism

$$\psi_{l_1}: \langle e^{tA_0}, l\rangle \to \langle e^{tA_0}, l - e^{tA_0}l_1 + l_1\rangle.$$

This automorphism leaves the center of G, $\{\langle I, l\rangle | A_0 l = 0\}$, fixed and so induces an automorphism of all Lie groups locally isomorphic to G.

If $X = \langle A_0 | l\rangle \in g$, ψ_{l*} sends X to $\langle A_0 | 0\rangle$. Consequently, whenever $\{X_1, \ldots, X_k\}$ is a minimal generating set for g, we may assume $X_1 = \langle A_0 | 0\rangle$.

The following lemma requires no proof:

LEMMA. (1) *$X = \langle A_0|0\rangle$ and $Y = \langle rA_0|v\rangle$ generate g if and only if v is a cyclic vector for A_0 (i.e., R^n has no non-trivial invariant subspaces containing v).*

(2) *$X = \langle A_0|0\rangle$, $Y = \langle r_1 A_0|v_1\rangle$, and $Z = \langle r_2 A_0|v_2\rangle$ minimally generate g if and only if v_1, v_2, and $r_1 v_2 - r_2 v_1$ are not cyclic for A_0 and R^n has no non-trivial invariant subspace containing v_1 and v_2.*

6. Let $B \in GL(n, R)$, $BA_0B^{-1} = \lambda A_0$, $Bv = v$ whenever $A_0 v = 0$. The map $\langle A_0|l\rangle \rightarrow \langle BA_0B^{-1}|Bl\rangle$ is an automorphism of g induced by the automorphism $\langle e^{tA_0}, l\rangle \rightarrow \langle Be^{tA_0}B^{-1}, Bl\rangle$ of G; since this automorphism leaves the center of G fixed, it induces automorphisms of all Lie groups locally isomorphic to G.

We are interested in the special cases

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}, \quad |\alpha| \leqq 1, \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix}, \qquad \alpha \geqq 0.$$

If $A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, any B will work; since A_0 has no cyclic vectors, g has no generating sets of the form $\{X, Y\}$ and any $\{X, Y, Z\}$ is obviously equivalent to one listed in Theorem 4.

If $A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$, $0 < |\alpha| \leqq 1$, $\alpha \neq 1$, any diagonal B will work; $v = \begin{bmatrix} a \\ b \end{bmatrix}$ is cyclic if and only if $a \neq 0$, $b \neq 0$. It is then trivial to obtain the generators listed in Theorem 4.

If $A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & 0 \\ 0 & 1 \end{bmatrix}$; $v = \begin{bmatrix} a \\ b \end{bmatrix}$ is cyclic if and only if $a \neq 0$, $b \neq 0$. Again we easily obtain the generators listed. In the case of $(ax + b) \times R$, B need not preserve the center and can be taken to be $\begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix}$; a smaller list of generators then suffices.

If $A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{11} \end{bmatrix}$ then cyclic vectors have the form $\begin{bmatrix} a \\ b \end{bmatrix}$, $b \neq 0$. It is now easy to obtain the generators listed. There are no minimal triple systems since whenever v_1 and v_2 are noncyclic, v_1 and v_2 generate an invariant subspace smaller than R^2.

The same arguments work for $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; in this case we need not worry about the center since $G \Big/ \left\{\left\langle I, \begin{bmatrix} n \\ 0 \end{bmatrix}\right\rangle \Big| n \in Z\right\}$ is not a matrix group. The generating pair $\left\{\langle A_0|0\rangle, \left\langle A_0 \Big| \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\rangle\right\}$ is equivalent to the pair $\left\{\langle A_0|0\rangle, \left\langle 0 \Big| \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\rangle\right\}$

via the isomorphism

$$\left\langle tA_0 \middle| \begin{bmatrix} a \\ b \end{bmatrix} \right\rangle \to \left\langle (t+b)A_0, \begin{bmatrix} a \\ b \end{bmatrix} \right\rangle .$$

If

$$A_0 = \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix}, \qquad B = \lambda \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

then any non-zero vector is cyclic; the results listed follow easily.

7. The groups

$$\left\{ \langle A, l \rangle \in A(2) | A = e^{\alpha t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \right\}, \qquad \alpha > 0,$$

were considered in [**4**]. The group $\tilde{E}(2)/Z = E(2)$ was considered in [**2**]. When the generating pair is $\left\{ \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \middle| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \middle| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle \right\}$, the order of generation was shown to be infinite; this implies that it is also infinite in $\tilde{E}(2)/kZ$ for all k. When the generating pair is

$$\{X, Y\} = \left\{ \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \middle| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle, \left\langle 0 \middle| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle \right\}$$

it was shown that some elements in $E(2)$ cannot be written in the form YXY; the same conclusion then follows for $\tilde{E}(2)/kZ$. Finally it was shown that all elements in $E(2)$ can be written in the form XYX. If $\xi \in \tilde{E}(2)$, the element in $E(2)$ induced by ξ can be written $\exp(tX) \circ \exp(uY) \circ \exp(vX)$, so $\xi = \eta \exp(tX) \circ \exp(uY) \circ \exp(vX)$ for some

$$\eta \in \operatorname{Ker}\, [\tilde{E}(2) \xrightarrow{\pi} E(2)] .$$

(Here we have used exp to denote both the exponential map on $E(2)$ and the exponential map on $\tilde{E}(2)$ and trust that no confusion will result!) But $\exp(tX)$ generates the center of $\tilde{E}(2)$ and the kernel of π lies in this center, so

$$\begin{aligned} \xi &= \exp(t_1X) \circ \exp(tX) \circ \exp(uY) \circ \exp(vZ) \\ &= \exp([t_1 + t]X) \circ \exp(uY) \circ \exp(vZ). \end{aligned}$$

Thus all elements in $\tilde{E}(2)$ can be written in the form XYX; the same result then follows for $\tilde{E}(2)/kZ$.

Only cases 2_α, $|\alpha| \leqq 1$, 3, and 4 remain.

8. Consider $G = \{ \langle e^{tA_0}, l \rangle \in A(n) | t \in R, l \in R^n \}$. If v is cyclic for A_0 and $X = \langle A_0 | 0 \rangle$, $Y = \langle 0 | v \rangle$, the order of generation of G with respect to $\{X, Y\}$ is

https://doi.org/10.4153/CJM-1975-048-0 Published online by Cambridge University Press

$\leqq 2n$. Indeed

$$\exp(t_1 Y) \circ \exp(u_1 X) \circ \ldots \circ \exp(t_n Y) \circ \exp(u_n X)$$
$$= \langle e^{(u_1 + \ldots + u_n)A_0}, t_1 v + t_2 e^{u_1 A_0} v + \ldots + t_n e^{(u_1 + \ldots + u_{n-1})A_0} v \rangle.$$

Let $r_1 = u_1, r_2 = u_1 + u_2, \ldots, r_{n-1} = u_1 + \ldots + u_{n-1}$; it suffices to choose these r's so that $v, e^{r_1 A_0}v, \ldots, e^{r_{n-1}A_0}v$ are linearly independent. This is certainly possible, for otherwise $\det(v, e^{r_1 A_0}v, \ldots, e^{r_{n-1}A_0}v) = R(r_1, \ldots, r_{n-1}) \equiv 0$, in which case

$$\frac{\partial^{\frac{1}{2}n(n-1)} R}{\partial r_1 \partial r_2^{\,2} \cdot \ldots \cdot \partial r_{n-1}^{\,n-1}} (0, \ldots, 0) = \det(v, A_0 v, \ldots, A_0^{\,n-1} v) = 0.$$

But v is cyclic and so $\{v, A_0 v, \ldots, A_0^{\,n-1} v\}$ is a basis for R^n.

If $g \in G$, $g^{-1} = \exp(t_1 Y) \circ \exp(u_2 X) \circ \ldots \circ \exp(t_n Y) \circ \exp(u_n X)$, then

$$g = \exp(-u_n X) \circ \exp(-t_n Y) \circ \ldots \circ \exp(-u_1 X) \circ \exp(-t_1 Y).$$

In short, every element can be written in the form $YX \ldots YX$ and also in the form $XY \ldots XY$.

If $A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}, \alpha \neq 1, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, or $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, the actual order of generation is exactly $2n = 4$. Indeed $\left\langle I, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$ cannot be written as

$$\exp(u_1 X) \circ \exp(t_2 Y) \circ \exp(u_2 X) = \langle e^{(u_1 + u_2)A_0}, t_2 e^{u_1 A_0} v \rangle$$

or

$$\exp(t_1 Y) \circ \exp(u_1 X) \circ \exp(t_2 Y) = \langle e^{u_1 A_0}, t_1 v + t_2 e^{u_1 A_0} v \rangle.$$

(More generally, it can be shown that whenever A_0 has only real eigenvalues, the order of generation of G with respect to $\{\langle A_0 | 0 \rangle, \langle 0 | v \rangle\}$ is exactly $2n$. The key lemma needed to establish this result appears in a paper by Polya [**6**].)

9. Let $G = (ax + b) \times R$ or $(ax + b) \times S^1$. Since the order of generation of $(ax + b) \times R$ with respect to $\{\langle 1|0 \rangle \times 0, \langle 0|1 \rangle \times 1\}$ is four, the order of generation of $(ax + b) \times S^1$ with respect to these generators is three or four. It cannot be three, for $\exp tX \circ \exp uY \circ \exp vX = \langle e^{t+v}, e^t u \rangle \times u$ and $\exp tY \circ \exp uX \circ \exp vY = \langle e^u, t + e^u v \rangle \times (t + v)$; therefore $\langle 1, 0 \rangle \times r$ can be written as a product of three elements only if $r \sim 0$.

Consider $\{X, Y\} = \{\langle 1|0 \rangle \times 0, \langle 1|1 \rangle \times \lambda\}$ where $\lambda \neq 0$. Then

$$\exp tX \circ \exp uY \circ \exp vX \circ \exp wY = \langle e^{t+u+v+w}, e^{t+u+v+w} - e^{t+u+v} + e^{t+u} - e^t \rangle X \lambda(u + w).$$

But every $\langle a, b\rangle \times c$ can be so written. Indeed chose u so large that

$$d = \frac{b - a + e^u(ae^{-c/\lambda})}{e^u - 1}$$

is positive, choose t so that $e^t = d$ and let $v = \ln a - c/\lambda - t, w = c/\lambda - u$. Then every element in $(ax + b) \times R$ and $(ax + b) \times S^1$ can be written as a product of 4 terms ending in $\exp wY$ and by the argument in VIII, every element is also a product of 4 terms ending in $\exp wX$.

Finally, $\exp tX \circ \exp uY \circ \exp vX = \langle e^{t+u+v}, e^{t+u} - e^t\rangle \times \lambda u$ and $\exp uY \circ \exp vX \circ \exp wY = \langle e^{u+v+w}, e^{u+v+w} - e^{u+v} + e^u - 1\rangle \times \lambda(u + w)$; elements of the form $\langle 1, 0\rangle \times r$ can be written in the required form only if $r \sim 0$.

Consider $\{X, Y, Z\} = \{\langle 1|0\rangle \times 0, \langle 0|1\rangle \times 0, \langle 0|0\rangle \times 1\}$; then

$$\begin{aligned} \exp tX \circ \exp uY \circ \exp vZ &= \langle e^t, e^t u\rangle \times v \\ \exp uY \circ \exp tX \circ \exp vZ &= \langle e^t, u\rangle \times v \\ \exp uY \circ \exp vZ \circ \exp tX &= \langle e^t, u\rangle \times v \end{aligned}$$

and each of these forms gives all of G. If $g \in G$,

$$g^{-1} = \exp(tX) \circ \exp(uY) \circ \exp(vZ)$$

and so $g = \exp(-vZ) \circ \exp(-uY) \circ \exp(-tX)$; thus every element can be written in the form ZYX. Similarly any permutation of XYZ produces all elements of G.

Consider $\{X, Y, Z\{ = \{\langle 1|0\rangle \times 0, \langle 0|1\rangle \times 0, \langle 1|0\rangle \times \lambda\}$ for $\lambda \neq 0$; then

$$\begin{aligned} \exp tX \circ \exp uY \circ \exp vZ &= \langle e^{t+v}, e^t u\rangle \times \lambda v \\ \exp uY \circ \exp tX \circ \exp vZ &= \langle e^{t+v}, u\rangle \times \lambda v \\ \exp uY \circ \exp vZ \circ \exp tX &= \langle e^{t+v}, u\rangle \times \lambda v \end{aligned}$$

and the preceding argument is again valid.

Consider $\{X, Y, Z\} = \{\langle 1|0\rangle \times 0, \langle 1|1\rangle \times 0, \langle 0|0\rangle \times 1\}$:

$$\begin{aligned} \exp tX \circ \exp uY \circ \exp vZ &= \langle e^{t+u}, e^{t+u} - e^t\rangle \times v \\ \exp uY \circ \exp tX \circ \exp vZ &= \langle e^{t+u}, e^u - 1\rangle \times v \\ \exp uY \circ \exp vZ \circ \exp tX &= \langle e^{t+u}, e^u - 1\rangle \times v. \end{aligned}$$

An element of the form $\langle a, b\rangle \times c$ can be written in the first way if and only if $a - b > 0$. It can be written in the other two ways if and only if $b > -1$. Hence every element can be written in one of these forms, but no fixed order produces all elements of G.

10. Let

$$G = \left\{\langle A, l\rangle \in A(2) | A = \begin{bmatrix} e^t & 0 \\ 0 & e^{\alpha t}\end{bmatrix}\right\}, \qquad 0 < |\alpha| \leqq 1.$$

https://doi.org/10.4153/CJM-1975-048-0 Published online by Cambridge University Press

If $\{X, Y, Z\} = \left\{\left\langle\begin{bmatrix}1 & 0\\0 & \alpha\end{bmatrix}\middle|\begin{bmatrix}0\\0\end{bmatrix}\right\rangle, \left\langle 0\middle|\begin{bmatrix}1\\0\end{bmatrix}\right\rangle, \left\langle 0\middle|\begin{bmatrix}0\\1\end{bmatrix}\right\rangle\right\}$, then

$$\exp tX \circ \exp uY \circ \exp vZ = \left\langle\begin{bmatrix}e^t & 0\\0 & e^{\alpha t}\end{bmatrix}, \begin{bmatrix}e^t u\\e^{\alpha t}v\end{bmatrix}\right\rangle$$

$$\exp uY \circ \exp tX \circ \exp vZ = \left\langle\begin{bmatrix}e^t & 0\\0 & e^{\alpha t}\end{bmatrix}, \begin{bmatrix}u\\e^{\alpha t}v\end{bmatrix}\right\rangle$$

$$\exp uY \circ \exp vZ \circ \exp tX = \left\langle\begin{bmatrix}e^t & 0\\0 & e^{\alpha t}\end{bmatrix}, \begin{bmatrix}u\\v\end{bmatrix}\right\rangle.$$

Thus, each form suffices to give all of G.

If $\{X, Y, Z\} = \left\{\left\langle\begin{bmatrix}1 & 0\\0 & \alpha\end{bmatrix}\middle|\begin{bmatrix}0\\0\end{bmatrix}\right\rangle, \left\langle 0\middle|\begin{bmatrix}1\\0\end{bmatrix}\right\rangle, \left\langle\begin{bmatrix}1 & 0\\0 & \alpha\end{bmatrix}\middle|\begin{bmatrix}0\\1\end{bmatrix}\right\rangle\right\}$, then

$$\exp tX \circ \exp uY \circ \exp vZ = \left\langle\begin{bmatrix}e^{t+v} & 0\\0 & e^{\alpha(t+v)}\end{bmatrix}, \begin{bmatrix}e^t u\\\frac{1}{\alpha}(e^{\alpha(t+v)} - e^{\alpha t})\end{bmatrix}\right\rangle$$

$$\exp uY \circ \exp tX \circ \exp vZ = \left\langle\begin{bmatrix}e^{t+v} & 0\\0 & e^{\alpha(t+v)}\end{bmatrix}, \begin{bmatrix}u\\\frac{1}{\alpha}(e^{\alpha(t+v)} - e^{\alpha t})\end{bmatrix}\right\rangle$$

$$\exp uY \circ \exp vZ \circ \exp tX = \left\langle\begin{bmatrix}e^{t+v} & 0\\0 & e^{\alpha(t+v)}\end{bmatrix}, \begin{bmatrix}u\\\frac{1}{\alpha}(e^{\alpha v} - 1)\end{bmatrix}\right\rangle.$$

An element of the form $\left\langle\begin{bmatrix}a & 0\\0 & a^\alpha\end{bmatrix}, \begin{bmatrix}b\\c\end{bmatrix}\right\rangle$ can be written in one of the first two ways if and only if $a^\alpha - \alpha c > 0$. It can be written in the third form if and only if $\alpha c > -1$. The claimed result now follows.

Exactly the same analysis applies when

$$\{X, Y, Z\} = \left\{\left\langle\begin{bmatrix}1 & 0\\0 & \alpha\end{bmatrix}\middle|\begin{bmatrix}0\\0\end{bmatrix}\right\rangle, \left\langle\begin{bmatrix}1 & 0\\0 & \alpha\end{bmatrix}\middle|\begin{bmatrix}1\\0\end{bmatrix}\right\rangle, \left\langle 0\middle|\begin{bmatrix}0\\1\end{bmatrix}\right\rangle\right\}.$$

Suppose $\alpha = 1$ and

$$\{X, Y, Z\} = \left\{\left\langle\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\middle|\begin{bmatrix}0\\0\end{bmatrix}\right\rangle, \left\langle\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\middle|\begin{bmatrix}1\\0\end{bmatrix}\right\rangle, \left\langle\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\middle|\begin{bmatrix}0\\1\end{bmatrix}\right\rangle\right\}.$$

Then,

$$\exp tX \circ \exp uY \circ \exp vZ = \left\langle\begin{bmatrix}e^{t+u+v} & 0\\0 & e^{t+u+v}\end{bmatrix}, \begin{bmatrix}e^{t+u} - e^t\\e^{t+u+v} - e^{t+u}\end{bmatrix}\right\rangle$$

$$\exp uY \circ \exp tX \circ \exp vZ = \left\langle\begin{bmatrix}e^{t+u+v} & 0\\0 & e^{t+u+v}\end{bmatrix}, \begin{bmatrix}e^u - 1\\e^{t+u+v} - e^t\end{bmatrix}\right\rangle$$

$$\exp uY \circ \exp vZ \circ \exp tX = \left\langle\begin{bmatrix}e^{t+u+v} & 0\\0 & e^{t+u+v}\end{bmatrix}, \begin{bmatrix}e^u - 1\\e^{u+v} - e^u\end{bmatrix}\right\rangle.$$

An element $\left\langle\begin{bmatrix}a & 0\\0 & a\end{bmatrix}, \begin{bmatrix}b\\c\end{bmatrix}\right\rangle$ can be written in the first form if and only if

$a - c > 0$ and $a - c - b > 0$. It can be written in the second form if and only if $a - c > 0$ and $b > -1$. It can be written in the third form if and only if $b > -1$ and $b + c > -1$. None of these forms, and so no form, suffices to write all elements of G. However, the first two forms suffice to write $\left\langle \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \begin{bmatrix} b \\ c \end{bmatrix} \right\rangle$ whenever $c \leqq 0$. If $c > 0$,

$$\left\langle \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \begin{bmatrix} b \\ c \end{bmatrix} \right\rangle^{-1} = \left\langle \begin{bmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{bmatrix}, \begin{bmatrix} -a^{-1} & b \\ -a^{-1} & c \end{bmatrix} \right\rangle$$

and $-a^{-1}c < 0$, and so $\left\langle \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \begin{bmatrix} b \\ c \end{bmatrix} \right\rangle^{-1}$ can be generated by three elements. It follows that $\left\langle \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \begin{bmatrix} b \\ c \end{bmatrix} \right\rangle$ itself can be generated by three elements.

Finally, let $\alpha \neq 1$, $\{X, Y\} = \left\{ \left\langle \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \middle\| \begin{bmatrix} -1 \\ -\alpha \end{bmatrix} \right\rangle \right\}$.

Then

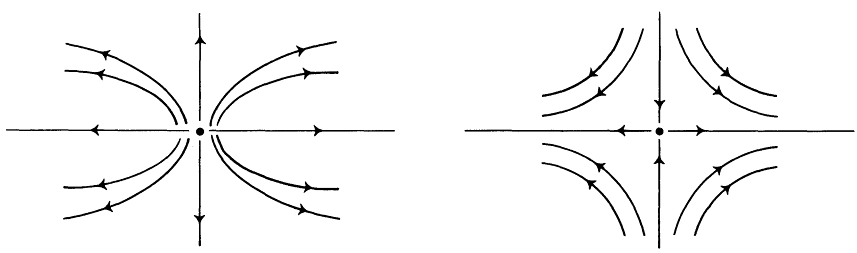
$$\exp tX = \left\langle \begin{bmatrix} e^t & 0 \\ 0 & e^{\alpha t} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle \quad \text{and} \quad \exp tY = \left\langle \begin{bmatrix} e^t & 0 \\ 0 & e^{\alpha t} \end{bmatrix}, \begin{bmatrix} -e^t + 1 \\ -e^{\alpha t} + 1 \end{bmatrix} \right\rangle.$$

Notice that the equation $\exp tY \circ \exp uX \circ \exp vY \circ \exp wX = \langle A, l \rangle$ can be solved for t, u, v, and w precisely when t, u, and v can be chosen so that $\exp tY \circ \exp uX \circ \exp vY$ applied to the zero vector gives l. Indeed $\exp wX$ leaves the zero vector fixed; once t, u, and v are known, w can be chosen to make the first part of the expression equal to A.

The orbits

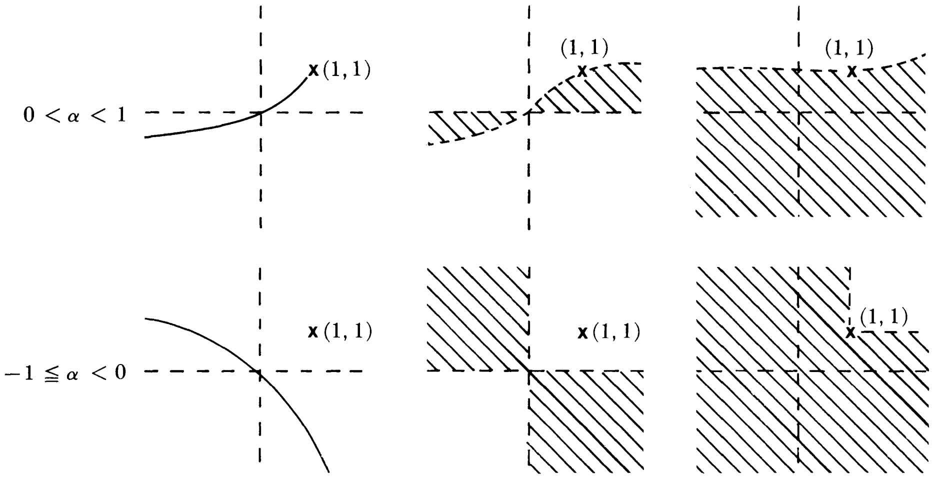
$$\begin{bmatrix} e^t & 0 \\ 0 & e^{\alpha t} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} e^t v_1 \\ e^{\alpha t} v_2 \end{bmatrix}$$

of $exp(tX)$ acting on the plane are as sketched below.



$0 < \alpha < 1$ $\qquad\qquad$ $-1 \leqq \alpha < 0$

Since $\exp tY = \left\langle I, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle \circ \exp tX \circ \left\langle I, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\rangle$, the orbit picture of $\exp tY$ is just that of $\exp tX$ translated by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The successive images of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by $\exp vY$, $\exp uX \circ \exp vY$, $\exp tY \circ \exp uX \circ \exp vY$ are as below.



For our purpose it suffices to observe that the final pictures omit some points in the plane and that they contain all (x, y) with $y < 1$. We now offer an analytic justification of these facts. First of all, $(1, 1)$ is not in either final picture, otherwise it would equal $\exp uX \circ \exp vY\begin{bmatrix}0\\0\end{bmatrix}$ since it is left fixed by $\exp tY$; thus $(\exp vY)\begin{bmatrix}0\\0\end{bmatrix} = \exp(-u)X\begin{bmatrix}1\\1\end{bmatrix}$ or $(-e^t + 1, -e^{\alpha t} + 1) = (e^{-u}, e^{-\alpha u})$ or $1 = e^t + e^{-u} = e^{\alpha t} + e^{-\alpha u}$; it follows that $t < 0$, $u > 0$, $\alpha t < 0$, $\alpha u > 0$, and so $\alpha > 0$. Then $[1 - e^{-u}]^\alpha = 1 - [e^{-u}]^\alpha$; however if $0 < \alpha < 1$ and $0 < r < 1$, the equation $[1 - r]^\alpha = 1 - r^\alpha$ cannot hold.

Let (x, y) be a point with $y < 1$. Then $\exp tY \circ \exp uX \circ \exp vY\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$ precisely when $\exp uX \circ \exp vY\begin{bmatrix}0\\0\end{bmatrix} = \exp(-tY)\begin{bmatrix}x\\y\end{bmatrix}$, i.e. when

$$\begin{bmatrix} e^u(1 - e^v) \\ e^{\alpha u}(1 - e^{\alpha v}) \end{bmatrix} = \begin{bmatrix} e^{-t}(x - 1) + 1 \\ e^{-\alpha t}(y - 1) + 1 \end{bmatrix}.$$

Let $r = e^{-t}(x - 1) + 1$, $s = e^{-\alpha t}(y - 1) + 1$. We want to find u and v so $e^u(1 - e^v) = r$, $e^{\alpha u}(1 - e^{\alpha v}) = s$.

Suppose for the moment that $-1 \leqq \alpha < 0$. Choose t so large that $r > 0$, $s < 0$. For each $v < 0$ there is a unique u such that $e^u = r/(1 - e^v)$. Choose v so that

$$\left(\frac{r}{1 - e^v}\right)^\alpha = \frac{-s}{e^{\alpha v} - 1}$$

or $r^\alpha/-s = (1 - e^v)^\alpha/(e^{\alpha v} - 1)$; this is possible because

$$\lim_{v \to 0} \frac{(1 - e^v)^\alpha}{e^{\alpha v} - 1} = \infty \quad \text{and} \quad \lim_{v \to -\infty} \frac{(1 - e^v)^\alpha}{e^{\alpha v} - 1} = 0.$$

https://doi.org/10.4153/CJM-1975-048-0 Published online by Cambridge University Press

Now suppose $0 < \alpha < 1$. There is no hope of solving for u and v unless r and s have the same sign. If $r > 0$ and $s > 0$, there is a unique u for each $v < 0$ with $e^u = r/(1 - e^v)$; we want $r^\alpha/(1 - e^v)^\alpha = s/1 - e^{\alpha v}$ or $r^\alpha/s = (1 - e^v)^\alpha/(1 - e^{\alpha v})$. Since

$$\lim_{v\to 0} \frac{(1 - e^v)^\alpha}{1 - e^{\alpha v}} = \infty \quad \text{and} \quad \lim_{v\to -\infty} \frac{(1 - e^v)^\alpha}{1 - e^{\alpha v}} = 1,$$

the equations can be solved for u and v provided $r^\alpha/s > 1$. If $r < 0$ and $s < 0$, there is a unique u for each $v > 0$ with $e^u = -r/(e^v - 1)$; we want

$$\frac{(-r)^\alpha}{(e^v - 1)^\alpha} = \frac{-s}{e^{\alpha v} - 1} \quad \text{or} \quad \frac{(-r)^\alpha}{-s} = \frac{(e^v - 1)^\alpha}{e^{\alpha v} - 1}.$$

Since

$$\lim_{v\to 0} \frac{(e^v - 1)^\alpha}{e^{\alpha v} - 1} = \infty \quad \text{and} \quad \lim_{v\to \infty} \frac{(e^v - 1)^\alpha}{e^{\alpha v} - 1} = 1,$$

the equations can be solved for u and v provided $(-r)^\alpha/-s > 1$. If $r = s = 0$, let $v = 0$. Choose t_0 so that $s(t_0) = e^{-\alpha t_0}(y - 1) + 1 = 0$. If $r(t_0) = 0$, the results follows. If $r(t_0) > 0$, $r(t_0 + \Delta t) > 0$ and $s(t_0 + \Delta t) > 0$ for all sufficiently small positive Δt, then

$$\frac{r(t_0 + \Delta t)^\alpha}{s(t_0 + \Delta t)} > 1$$

if Δt is small enough. Finally if $r(t_0) < 0$, $r(t_0 + \Delta t) < 0$ and $s(t_0 + \Delta t) < 0$ for all sufficiently small negative Δt, then

$$\frac{[-r(t_0 + \Delta t)]^\alpha}{-s(t_0 + \Delta t)} > 1$$

if Δt is small enough.

Since the final pictures omit points in the plane, some elements of G cannot be written in the form $\exp tY \circ \exp uX \circ \exp vY \circ \exp wX$ whence the inverses of these elements cannot be written in the form $\exp tX \circ \exp uY \circ \exp vX \circ \exp wY$. However, every element in G can be written in one of these two forms. Indeed, notice that conjugation by $\left\langle -I, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$ interchanges $\exp tX$ and $\exp tY$ and converts

$$\left\langle \begin{bmatrix} e^t & 0 \\ 0 & e^{\alpha t} \end{bmatrix}, \begin{bmatrix} l_x \\ l_y \end{bmatrix} \right\rangle \text{ to } \left\langle \begin{bmatrix} e^t & 0 \\ 0 & e^{\alpha t} \end{bmatrix}, \begin{bmatrix} -l_x - e^t + 1 \\ -l_y - e^{\alpha t} + 1 \end{bmatrix} \right\rangle;$$

if $l_y < 1$ then

$$\left\langle \begin{bmatrix} e^t & 0 \\ 0 & e^{\alpha t} \end{bmatrix}, \begin{bmatrix} l_x \\ l_y \end{bmatrix} \right\rangle = \exp tY \circ \exp uX \circ \exp vY \circ \exp wX$$

by the preceding orbital analysis. If $l_y \geqq 1$, $-l_y - e^{\alpha t} + 1 < 0$ and

$$\left\langle -I, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle \circ \left\langle \begin{bmatrix} e^t & 0 \\ 0 & e^{\alpha t} \end{bmatrix}, \begin{bmatrix} l_x \\ l_y \end{bmatrix} \right\rangle \circ \left\langle -I, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle^{-1}$$

$$= \exp tY \circ \exp uX \circ \exp vY \circ \exp wX,$$

then

$$\left\langle \begin{bmatrix} e^t & 0 \\ 0 & e^{\alpha t} \end{bmatrix}, \begin{bmatrix} l_x \\ l_y \end{bmatrix} \right\rangle = \exp tX \circ \exp uY \circ \exp vX \circ \exp wY$$

11. Let

$$G = \left\{ \langle A, l \rangle \in A(2) | A = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \right\},$$

$$\{X, Y\} = \left\{ \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \middle| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \middle| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle \right\}.$$

Then

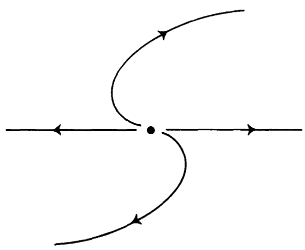
$$\exp(tX) = \left\langle \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}, 0 \right\rangle, \qquad \exp(tY) = \left\langle \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}, \begin{bmatrix} te^t - e^t + 1 \\ e^t - 1 \end{bmatrix} \right\rangle.$$

As in §10, we can write $\langle A, l \rangle$ in the form $\exp tY \circ \exp uX \circ \exp vY \circ \exp wX$ precisely when t, u, and v can be chosen so that $(\exp tY \circ \exp uX \circ \exp vY) \begin{bmatrix} 0 \\ 0 \end{bmatrix} = l$.

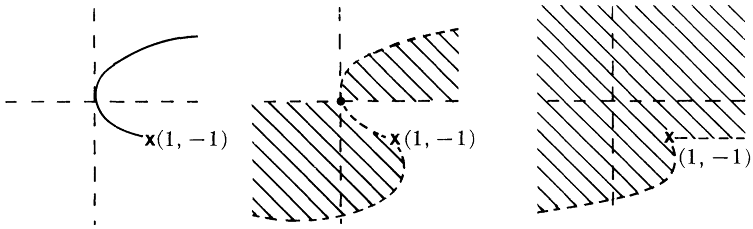
Consider a typical orbit

$$\begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = e^t \begin{bmatrix} v_1 - tv_2 \\ v_2 \end{bmatrix}$$

of $\exp(tX)$ acting on the plane. If $v_2 = 0$, this orbit is exactly the left or right half of the x-axis. Otherwise it passes through a non-zero point $\begin{bmatrix} 0 \\ w \end{bmatrix}$ on the y-axis and so consists of all $e^t \begin{bmatrix} tw \\ w \end{bmatrix}$. Thus the orbits of $\begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$ are as sketched below; all orbits approach the origin horizontally.



Since $\exp(tY) = \left\langle I, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle \circ \exp(tX) \circ \left\langle I, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\rangle$, the orbit picture of $\exp(tY)$ is just that of $\exp tX$ translated by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The successive images of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by $\exp vY$, $\exp uX \circ \exp vY$, and $\exp tY \circ \exp uX \circ \exp vY$ are as follows:



As in § 10 we show that the final picture omits some points in the plane and contains all (x, y) with $y > -1$. First of all, $(1, -1)$ is not in the final picture, otherwise it would equal $(\exp uX \circ \exp vY)\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, since it is left fixed by $\exp tY$ thus $(\exp vY)\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \exp(-uX)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or

$$(ve^v - e^v + 1, e^v - 1) = (e^{-u} + ue^{-u}, -e^{-u})$$

or $e^v + e^{-u} = 1$ and $-ve^v + ue^{-u} = 0$; in order that $e^v + e^{-u} = 1$, $v < 0$ and $u > 0$, so $-ve^v + ue^{-u} > 0$.

Let (x, y) be a point such that $y > -1$. Then

$$\exp tY \circ \exp uX \circ \exp vY\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

precisely when $\exp uX \circ \exp vY\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \exp(-tY)\begin{bmatrix} x \\ y \end{bmatrix}$, i.e.,

$$\begin{bmatrix} (u-1)(e^{u+v} - e^u) + ve^{u+v} \\ e^{u+v} - e^u \end{bmatrix} = \begin{bmatrix} e^{-t}(x - 1 - t(y+1)) + 1 \\ e^{-t}(y+1) - 1 \end{bmatrix}.$$

Let $r = e^{-t}(x - 1 - t(y + 1)) + 1$, $s = e^{-t}(y + 1) - 1$. Notice that $(u - 1)(e^{u+v} - e^u) + ve^{u+v} = r$, $e^{u+v} - e^u = s$ precisely when $v = \ln(1 + s/e^u)$ and $(u - 1)s + (e^u + s)\ln(1 + s/e^u) = r$. Choose t so large that $r > 0$, $-1 < s < 0$. Consider $f(u) = r - (u - 1)s - (e^u + s)\ln(1 + s/e^u)$. If $u > 1$, this expression is positive. But

$$\lim_{u \to \ln(-s)} f(u) = r - [\ln(-s) - 1]s = e^{-t}(x + y - t(y + 1)) + (1 - e^{-t}(y + 1)) \ln (1 - e^{-t}(y + 1)) = e^{-t}[x + y - t(y + 1) + (e^t - (y + 1)) \ln (1 - e^{-t}(y + 1))];$$

this expression is negative for large t. Therefore the above equations can be solved for t, u, and v.

https://doi.org/10.4153/CJM-1975-048-0 Published online by Cambridge University Press

Since the final picture omits points in the plane, some elements of G cannot be written in the form $\exp tY \circ \exp uX \circ \exp vY \circ \exp wX$, and their inverses cannot be written in the form $\exp tX \circ \exp uY \circ \exp vX \circ \exp wY$. However every element of G can be written in one of these two forms, by the argument used in discussing the previous group; indeed conjugation by $\left\langle -I, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle$ interchanges $\exp tX$ and $\exp tY$ and converts

$$\left\langle \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}, \begin{bmatrix} l_x \\ l_y \end{bmatrix} \right\rangle \text{ to } \left\langle \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}, \begin{bmatrix} te^t - e^t - l_x \\ e^t - 1 - l_y \end{bmatrix} \right\rangle;$$

if $l_y \leqq -1$, $e^t - 1 - l_y > 0$.

12. Concluding remarks.

It is easy to extend our analysis to the non-linear group

$$\left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \right\} \Big/ \left\{ \begin{bmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| \, n \in Z \right\};$$

the order of generation is always 4. Since some automorphisms of $\tilde{G}$ do not induce automorphisms of G, a slight modification of our argument is required.

The groups $\widetilde{SL}(2, R)/kZ$ for $k \neq 1, 2$ require an intricate discussion to be given later. When neither generator is elliptic, the order of generation increases with increasing k.

In contrast to results of earlier papers, we have found many examples here where the minimal order of generation is greater than the dimension of the group.

References

1. N. Jacobson, *Lie algebras* (Interscience Publishers, New York, 1962).
2. F. Lowenthal, *Uniform finite generation of the isometry groups of Euclidean and non-Euclidean geometry*, Can. J. Math. *23* (1971), 364–373.
3. ——— *Uniform finite generation of the rotation group*, Rocky Mountain J. Math. *1* (1971), 575–586.
4. ——— *Uniform finite generation of the affine group*, Pacific J. Math. *40* (1972), 341–348.
5. ——— *Uniform finite generation of $SU(2)$ and $SL(2, R)$*, Can. J. Math. *24* (1972), 713–727.
6. G. Pólya, *On the mean-value theorem corresponding to a given linear homogeneous differential equation*, Trans. Amer. Math. Soc. *24* (1922), 312–324.
7. J. P. Serre, *Algèbres de Lie semi-simples complexes* (W. A. Benjamin, Inc., New York, 1966).
8. S. Sternberg, *Lectures on differential geometry* (Prentice Hall, Englewood Cliffs, N.J., 1964).
9. Hidehiko Yamabe, *On an arcwise connected subgroup of a Lie group*, Osaka J. Math. *2* (1950), 13–14.

University of Oregon,
Eugene, Oregon;
University of Wisconsin at Parkside,
Kenosha, Wisconsin