

Self-affine sets with fibred tangents

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Abstract. We study tangent sets of strictly self-affine sets in the plane. If a set in this class satisfies the strong separation condition and projects to a line segment for sufficiently many directions, then for each generic point there exists a rotation \mathcal{O} such that all tangent sets at that point are either of the form $\mathcal{O}((\mathbb{R} \times C) \cap B(0, 1))$, where C is a closed porous set, or of the form $\mathcal{O}((\ell \times \{0\}) \cap B(0, 1))$, where ℓ is an interval.

1. Introduction

Taking tangents is a standard tool in analysis. Tangents are usually more regular than the original object and often they capture its local structure. Further, understanding how tangents behave at many points gives information about the global structure as well. For example, tangents of a differentiable function are affine maps, and they capture the full behaviour of the function. Similarly, tangents of measures and sets are useful in understanding the fine structure of the objects under study, as well as their global properties.

Tangent measures were introduced by Preiss [29], and they were a crucial ingredient in connecting densities to rectifiability. It should be noted that for general measures and sets tangents can be almost anything; see [6–8, 27, 28, 31]. However, under strong enough regularity assumptions, or when studying the tangents statistically, the tangent structure can describe the original object well. For example, recently Fraser [11] and Fraser *et al* [12] have used tangent sets to study dimensional properties of various fractal sets. This approach goes back to Mackay and Tyson [26] and Mackay [25]. Furthermore, the process of taking blow-ups of a measure or a set around a point induces a natural dynamical system. This makes it possible to apply ergodic-theoretical methods to understand the statistical behaviour of tangents. The general theory related to this was initiated by Furstenberg [14, 15]. It was greatly developed by Hochman [17] and recently enhanced

by Käenmäki *et al* [22]. This ‘zooming-in’ dynamics has been considered for specific sets and measures arising from dynamics; see, for example, [3–5, 9, 10, 16, 35]. The theory has found applications in arithmetics and geometric measure theory; see [18, 21]. Very recently, Kempton [24] studied the scenery flow of Bernoulli measures on self-affine sets associated to strictly positive matrices under the condition that projections of the self-affine measure in typical directions are absolutely continuous.

The main purpose of this paper is to investigate tangent sets of a self-affine set. A tangent set is a limit in the Hausdorff metric obtained from successive magnifications of the original set around a given point. These kinds of local structures of fractal sets are of course interesting in their own right, but on top of that, our effort is motivated by the presumption that understanding them could provide new methods in the study of self-affine sets.

Tangent measures of self-similar sets have been given a satisfactory description by Bandt [1]. He studied tangent measures (and their distribution) of self-similar sets satisfying the open set condition, showing that almost every point has the same collection of tangent sets. Intuitively it is plausible that all tangent sets are homothetic copies of the original self-similar set. For a self-affine set, the expected tangent behaviour is completely different, and not yet very well understood. Under iteration by affine maps, balls are often mapped to narrower and narrower ellipses. Thus it is intuitive that, when zooming into a small ball, the magnification will contain narrow fibres in different directions. The limit object should hence be contained in a set consisting of long line segments, even if the original self-affine set is totally disconnected. Bandt and Käenmäki [2] confirmed that this intuition is correct at least in the special case where all the mappings contract more in the vertical direction than in the horizontal direction. According to them, under a projection condition, tangent sets at generic points are product sets of a line and a perfect nowhere dense set.

In the present article, we generalize the result of Bandt and Käenmäki to a general class of self-affine sets satisfying the strong separation condition and a projection condition. We emphasize that we do not need to assume the matrices in the affine iterated function system to be strictly positive. Therefore, in this sense, our setting is also more general than that of Kempton. Some questions still remain open. It would be interesting to know when exactly the tangents of self-affine sets admit this kind of fibred tangent structure.

We begin the description of the set-up by defining self-affine sets. Fix $m \in \mathbb{N} \setminus \{1\}$ and for each $i \in \{1, \dots, m\}$ let $f_i: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a contractive map with

$$f_i(x) = A_i x + b_i,$$

where $A_i \in \mathbb{R}^{2 \times 2}$ is an invertible 2×2 matrix having operator norm strictly less than one, and $b_i \in \mathbb{R}^2$. The collection $\{f_1, \dots, f_m\}$ of affine mappings is an *affine iterated function system*, and it gives rise to a *self-affine set* E , which is the unique compact non-empty set satisfying

$$E = \bigcup_{i=1}^m f_i(E).$$

If the matrices A_i are diagonal and the sets $f_i(E)$ are sufficiently separated, then the set E is referred to as a *self-affine carpet*. For example, if the *strong separation condition* is

satisfied, that is, $f_i(E) \cap f_j(E) = \emptyset$ for $i \neq j$, then there is a one-to-one correspondence between a point $x \in E$ and its address: the *canonical projection* $\pi : \{1, \dots, m\}^{\mathbb{N}} \rightarrow E$ defined by

$$\pi(i_1, i_2, \dots) = \sum_{n=1}^{\infty} A_{i_1} \cdots A_{i_{n-1}} b_{i_n} \quad (1.1)$$

is bijective.

It is clear that general results concerning tangent sets cannot be obtained for all points. Therefore we restrict our analysis to points which are generic with respect to Bernoulli measures. This is a natural class of measures to consider, since often the measure with maximal dimension is Bernoulli. It should be noted that a *Bernoulli measure* ν_p , defined to be the product of a given probability vector $p = (p_1, \dots, p_m)$, is not a measure on E but on $\{1, \dots, m\}^{\mathbb{N}}$. Thus, precisely speaking, we consider generic points on E with respect to the pushforward measure $\pi \nu_p$. But when the canonical projection is bijective we interpret ν_p as a measure on E .

Here we shall only state an informal version of our main theorem, as some of the assumptions are too technical to be presented in the introduction. A precise formulation of the theorem can be found in Theorem 3.1. We refer the reader to Remarks 3.2 and 3.3 for sufficient and checkable conditions for the assumptions. Definitions of a tangent set and porosity are given in §2.3. Lyapunov exponents will be defined in §2.2; notice that their values depend on the Bernoulli measure. The closed unit ball is denoted by $B(0, 1)$.

INFORMAL VERSION OF THEOREM 3.1. *If a self-affine set E :*

- (1) *satisfies the strong separation condition;*
- (2) *projects to a line segment for sufficiently many directions;*
- (3) *has two distinct Lyapunov exponents,*

then for ν_p -almost every $x \in E$ there exists a rotation \mathcal{O} such that the tangent sets at x are either of the form $\mathcal{O}((\mathbb{R} \times C) \cap B(0, 1))$, where C is a closed porous set, or of the form $\mathcal{O}((\ell \times \{0\}) \cap B(0, 1))$, where ℓ is an interval containing at least one of the intervals $[-1, 0]$ and $[0, 1]$.

It is reasonable to assume the strong separation condition since the geometry of the limit set for an overlapping iterated function system can differ hugely from the non-overlapping case. Further, the projection condition (2) is necessary for the claim in this form to hold; see Example 4.5. The assumption on two distinct Lyapunov exponents guarantees that the system is ‘strictly affine’ – this kind of assumption does not seem too restrictive since the expected behaviour in the self-similar case is so different than the self-affine case. There could of course be a general theorem covering a wider class of iterated function systems, but even to guess at the statement of such a theorem seems very difficult.

Let us now compare the assumptions of this result to the setting of Bandt and Käenmäki [2]. They considered self-affine carpets on $[0, 1]^2$ for which all the maps of iterated function systems contract more vertically than horizontally. They required the rectangular strong separation condition and also assumed that the projection of the carpet onto the x -axis is the whole interval $[0, 1]$. Our assumptions (1)–(3) resemble their assumptions, but we can handle more general self-affine sets. In fact, our main result,

Theorem 3.1, contains the self-affine carpets as a special case, regardless of whether there is a geometrically dominant contraction direction or not. For a discussion on this and other examples, see §4. To finish with a concrete statement, we formulate a corollary which strictly generalizes the main result of [2]. The proof is given in §4.

COROLLARY 1.1. *Suppose that $\{f_1, \dots, f_m\}$ is an affine iterated function system defined on $[0, 1]^2$ such that the linear part of f_i is the diagonal matrix $\text{diag}(h_i, v_i)$ with $0 < h_i, v_i < 1$ for all $i \in \{1, \dots, m\}$ and*

- (1) $f_i([0, 1]^2) \cap f_j([0, 1]^2) = \emptyset$ for $i \neq j$;
- (2) $\#\{i \in \{1, \dots, m\} : \{c\} \times [0, 1] \cap f_i([0, 1]^2) \neq \emptyset\} \geq 2$ for all $c \in [0, 1]$;
- (3) $-\sum_{i=1}^m v_p(f_i([0, 1]^2)) \log h_i < -\sum_{i=1}^m v_p(f_i([0, 1]^2)) \log v_i$.

Then the tangent sets at v_p -almost every point of the associated self-affine carpet E are of the form $(\mathbb{R} \times C) \cap B(0, 1)$, where C is a perfect porous set.

2. Preliminaries

In this section and throughout the paper, the affine iterated function system $\{f_1, \dots, f_m\}$ and hence the self-affine set E remain fixed. Further, v_p is the Bernoulli measure corresponding to a probability vector $p = (p_1, \dots, p_m)$. We let $\bar{p} = \max_{i \in \{1, \dots, m\}} p_i$ and $\underline{p} = \min_{i \in \{1, \dots, m\}} p_i$.

2.1. *Symbolic space.* We begin by presenting the symbolic representation of points in E . Let $I = \{1, \dots, m\}$ be the collection of letters. Define $I^{\mathbb{N}}$ to be the collection of infinite words $\{1, \dots, m\}^{\mathbb{N}}$ and I^k to be the set of finite words $\{1, \dots, m\}^k$ for all k . The empty word is denoted by \emptyset . Let $I^* = \bigcup_{k=0}^{\infty} I^k$ and denote by proportional font letters $\dot{i}, \dot{j}, \dot{a}, \dot{u}$ and so on the finite or infinite words. For a finite word \dot{i} , define the *cylinder*

$$[\dot{i}] = \{\dot{j} \in I^{\mathbb{N}} : \dot{j}|_{|\dot{i}|} = \dot{i}\},$$

where $\dot{j}|_k$ are the first k letters of \dot{j} and $|\dot{i}|$ is the length of the word \dot{i} . We say that finite words \dot{i} and \dot{j} are *incomparable*, $\dot{i} \perp \dot{j}$, if $[\dot{i}] \cap [\dot{j}] = \emptyset$. Denote by $\dot{i}(j)$ the j th letter in \dot{i} . For any finite word \dot{i} , write $A_{\dot{i}} = A_{\dot{i}(1)} \cdots A_{\dot{i}(|\dot{i}|)}$ and $f_{\dot{i}} = f_{\dot{i}(1)} \circ \cdots \circ f_{\dot{i}(|\dot{i}|)}$. We refer to the images $f_{\dot{i}}(E)$ as level $|\dot{i}|$ *construction cylinders* and denote them by $E_{\dot{i}}$. Notice that $E_{\dot{i}} = \pi([\dot{i}])$, where π is the canonical projection defined in (1.1).

2.2. *Orientation of construction cylinders.* In this subsection, we study the orientation of the construction cylinders $E_{\dot{i}}$. In particular, in Lemma 2.1 we prove that they converge almost everywhere. This is a version of the Oseledets theorem; see [33, Theorem 10.2] or [30]. For the convenience of the reader, and since the statement does not seem to appear in the right form in the literature, we present a full proof.

Let us now make the above more precise. For each $\dot{i} \in I^*$ and $k \in \{1, 2\}$, let $\eta_k(\dot{i}) \in S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ be the eigenvectors of $A_{\dot{i}}^T A_{\dot{i}}$ with eigenvalues $\alpha_k^2(\dot{i})$ ordered so that $\alpha_1(\dot{i}) \geq \alpha_2(\dot{i})$. We call $\eta_k(\dot{i})$ the *singular vectors*, or *singular directions*, of $A_{\dot{i}}$ and $\alpha_k(\dot{i})$ the *singular values* of $A_{\dot{i}}$. From basic linear algebra it follows that

$$\alpha_1(\dot{i}) = \|A_{\dot{i}} \eta_1(\dot{i})\| = \max_{v \in S^1} \|A_{\dot{i}} v\| \quad \text{and} \quad \alpha_2(\dot{i}) = \|A_{\dot{i}} \eta_2(\dot{i})\| = \min_{v \in S^1} \|A_{\dot{i}} v\|.$$

Furthermore, if $\alpha_1(\dot{i}) > \alpha_2(\dot{i})$ we see that $\eta_1(\dot{i})$ is orthogonal to $\eta_2(\dot{i})$ and $A_{\dot{i}}\eta_1(\dot{i})$ is orthogonal to $A_{\dot{i}}\eta_2(\dot{i})$. The numbers $\alpha_1(\dot{i})$ and $\alpha_2(\dot{i})$ are the lengths of the principal semiaxes of the image of the unit ball $f_{\dot{i}}(B(0, 1))$. We write $\underline{\alpha} = \min_{i \in \{1, \dots, m\}} \alpha_2(i)$ and $\bar{\alpha} = \max_{i \in \{1, \dots, m\}} \alpha_1(i)$. Since the matrices A_i are invertible contractions we have $0 < \underline{\alpha} \leq \bar{\alpha} < 1$.

Note that when $\alpha_1(\dot{i}) = \alpha_2(\dot{i})$, the eigenspace of $A_{\dot{i}}^T A_{\dot{i}}$ is the whole of \mathbb{R}^2 . In this case, we can choose $\eta_1(\dot{i})$ and $\eta_2(\dot{i})$ to be any orthogonal vectors. Also when $\alpha_1(\dot{i}) > \alpha_2(\dot{i})$, the sign of $\eta_k(\dot{i})$ can be chosen freely. This will not make a significant difference in what follows. For each $k \in \{1, 2\}$ and all finite words \dot{i} , let

$$\vartheta_k(\dot{i}) = \frac{A_{\dot{i}}\eta_k(\dot{i})}{\|A_{\dot{i}}\eta_k(\dot{i})\|}.$$

The pair $(\vartheta_1(\dot{i}), \vartheta_2(\dot{i}))$ gives the directions of the principal semiaxes of the ellipse $f_{\dot{i}}(B(0, 1))$, and hence the ‘orientation’ of the construction cylinders $E_{\dot{i}}$.

For each $k \in \{1, 2\}$ and $\dot{i} \in I^{\mathbb{N}}$, we define a sequence by setting

$$\lambda_k(\dot{i}|_n) = -\frac{1}{n} \log \alpha_k(\dot{i}|_n). \tag{2.1}$$

Due to the subadditive ergodic theorem (see [33, Theorem 10.1] or [30]), the *Lyapunov exponents* $\lambda_k(\dot{i}) = \lim_{n \rightarrow \infty} \lambda_k(\dot{i}|_n)$ exist for ν_p -almost all $\dot{i} \in I^{\mathbb{N}}$ and for all $k \in \{1, 2\}$. Moreover, since Bernoulli measures are ergodic (see, for example, [20, Theorem 3.7]) there exists a constant λ_k such that $\lambda_k(\dot{i}) = \lambda_k$ for ν_p -almost all $\dot{i} \in I^{\mathbb{N}}$ and for all $k \in \{1, 2\}$. Note that we always have $\lambda_2 \geq \lambda_1$.

LEMMA 2.1. *If $\lambda_2 > \lambda_1$, then for each $k \in \{1, 2\}$ and for ν_p -almost every $\dot{i} \in I^{\mathbb{N}}$ there exists $\bar{\vartheta}_k(\dot{i}) \in S^1$ such that $\vartheta_k(\dot{i}|_n) \rightarrow \bar{\vartheta}_k(\dot{i})$ as $n \rightarrow \infty$.*

Proof. First we notice that $\vartheta_k(\dot{i}|_n)$ are the singular directions of $A_{\dot{i}|_n}^T$. This is true, since

$$A_{\dot{i}|_n} A_{\dot{i}|_n}^T \vartheta_k(\dot{i}|_n) = A_{\dot{i}|_n} \frac{A_{\dot{i}|_n}^T A_{\dot{i}|_n} \eta_k(\dot{i}|_n)}{\|A_{\dot{i}|_n} \eta_k(\dot{i}|_n)\|} = \alpha_k(\dot{i}|_n)^2 \frac{A_{\dot{i}|_n} \eta_k(\dot{i}|_n)}{\|A_{\dot{i}|_n} \eta_k(\dot{i}|_n)\|} = \alpha_k(\dot{i}|_n)^2 \vartheta_k(\dot{i}|_n).$$

By the orthogonality of $\vartheta_1(\dot{i}|_n)$ and $\vartheta_2(\dot{i}|_n)$, it suffices to show the convergence in the case $k = 2$. For each $n \in \mathbb{N}$, let θ_n denote the angle between $\vartheta_2(\dot{i}|_n)$ and $\vartheta_2(\dot{i}|_{n+1})$. We prove that (θ_n) is a Cauchy sequence. This yields the existence of the limit. Since $\vartheta_1(\dot{i}|_{n+1})$ and $\vartheta_2(\dot{i}|_{n+1})$ are orthogonal for all n , we can write

$$\vartheta_2(\dot{i}|_n) = \sin(\theta_n)\vartheta_1(\dot{i}|_{n+1}) + \cos(\theta_n)\vartheta_2(\dot{i}|_{n+1}).$$

Note that, by changing the sign if necessary, we can always choose $\vartheta_2(\dot{i}|_n)$ and $\vartheta_2(\dot{i}|_{n+1})$ so that $-\pi/2 \leq \theta_n \leq \pi/2$. Now we have

$$\begin{aligned} \|A_{\dot{i}|_{n+1}}^T \vartheta_2(\dot{i}|_n)\| &= \|A_{\dot{i}|_{n+1}}^T \sin(\theta_n)\vartheta_1(\dot{i}|_{n+1}) + A_{\dot{i}|_{n+1}}^T \cos(\theta_n)\vartheta_2(\dot{i}|_{n+1})\| \\ &\geq \|A_{\dot{i}|_{n+1}}^T \sin(\theta_n)\vartheta_1(\dot{i}|_{n+1})\| = |\sin(\theta_n)|\alpha_1(\dot{i}|_{n+1}) \\ &\geq |\sin(\theta_n)|\underline{\alpha}\alpha_1(\dot{i}|_n) \end{aligned}$$

and

$$\|A_{\dot{i}|_{n+1}}^T \vartheta_2(\dot{i}|_n)\| = \|A_{\dot{i}|_{n+1}}^T A_{\dot{i}|_n}^T \vartheta_2(\dot{i}|_n)\| \leq \alpha_1(\dot{i}(n+1))\|A_{\dot{i}|_n}^T \vartheta_2(\dot{i}|_n)\| \leq \bar{\alpha}\alpha_2(\dot{i}|_n).$$

To prove the claim, we show that $\sum_{n=1}^{\infty} |\theta_n|$ converges by applying the root test. Since we have $|\sin(\theta_n)| \leq (\bar{\alpha}/\underline{\alpha})\alpha_2(\dot{i}|_n)/\alpha_1(\dot{i}|_n)$, it suffices to show that

$$\limsup_{n \rightarrow \infty} \sqrt[n]{\alpha_2(\dot{i}|_n)/\alpha_1(\dot{i}|_n)} < 1.$$

This is equivalent to $\liminf_{n \rightarrow \infty} -(1/n) \log \alpha_2(\dot{i}|_n)/\alpha_1(\dot{i}|_n) > 0$, which is true by our assumption $\lambda_2 > \lambda_1$. □

Remark 2.2. It should be remarked that in the Oseledets theorem, compared to the setting of the iterated function systems, the matrices are iterated in the reverse order. This is why Lemma 2.1 gives the convergence of the images of the singular directions, $\vartheta_k(\dot{i}|_n)$, while the Oseledets theorem concerns the singular directions $\eta_k(\dot{i}|_n)$.

The following example shows that in our setting, we cannot expect the convergence of the singular directions in a set of positive measure. Let $\{A_i x + b_i\}_{i \in I}$ be an affine iterated function system. Suppose that for some $j \in I^*$ there exists $0 < \alpha < 1$ such that $A_j = \alpha R$, where R is a rotation of angle $\theta \neq \pi$. Let μ be an ergodic measure on $I^{\mathbb{N}}$ so that $\lambda_2 > \lambda_1$ and $\mu[j] > 0$. For any $\dot{i} \in I^*$ the direction $\eta_1(\dot{i})$ differs from $\eta_1(\dot{i}|_j)$ exactly by angle θ . From [23, Lemma 2.3] we see that the set where j occurs infinitely often is of full measure, so the set where $\eta_1(\dot{i}|_n)$ converges is of measure zero.

2.3. Zooming and patterns. For $\dot{i} \in I^{\mathbb{N}}$ and $0 < t < 1$, let $n = n(\pi(\dot{i}), t)$ be the largest integer for which the closed ball $B(\pi(\dot{i}), t)$ only intersects one level n construction cylinder $E_{\dot{i}|_n}$. The existence of such n is guaranteed by the strong separation condition. For reasons that will soon become apparent, this is called the *construction level of the zoom*. It is easy to see that $n(\pi(\dot{i}), t)$ increases as t decreases to zero. We also abbreviate $n(\pi(\dot{i}), t)$ by $n(\dot{i}, t)$ and $\dot{i}|_{n(\dot{i}, t)}$ by $\dot{i}|_t$.

For any vector $w \in \mathbb{R}^2 \setminus \{0\}$, let proj_w be the orthogonal projection onto the line $\{tw : t \in \mathbb{R}\}$. Let $j \in I^*$. We use $h_{\dot{i}|_t}(j)$ to denote the diameter of the projection $\text{proj}_{\vartheta_1(\dot{i}|_t)}(E_{\dot{i}|_t|_j})$. Similarly, $v_{\dot{i}|_t}(j)$ is used to denote the diameter of $\text{proj}_{\vartheta_2(\dot{i}|_t)}(E_{\dot{i}|_t|_j})$. In the course of the proofs the construction cylinders will typically be written in the *singular basis* $(\vartheta_1(\dot{i}|_t), \vartheta_2(\dot{i}|_t))$, turning the directions $\vartheta_1(\dot{i}|_t)$ and $\vartheta_2(\dot{i}|_t)$ horizontal and vertical, respectively, which is what the symbols h and v stand for.

Notice that the construction cylinder $E_{\dot{i}|_t|_j}$ is included in a closed rectangle with sides parallel to $\vartheta_1(\dot{i}|_t)$ and $\vartheta_2(\dot{i}|_t)$ and of side lengths $h_{\dot{i}|_t}(j)$ and $v_{\dot{i}|_t}(j)$; see Figure 1. This *construction rectangle* is denoted by $R_{\dot{i}|_t}(j)$. For the empty word \emptyset we use the notation $R_{\dot{i}|_t}$. The next lemma highlights the relationship between the side lengths of $R_{\dot{i}|_t}(j)$ and the singular values of $A_{\dot{i}|_t}$.

LEMMA 2.3. *If the self-affine set E is not contained in a line, then there is a constant $L \in \mathbb{N}$ such that*

$$L^{-1} \alpha_1(\dot{i}|_t) \underline{\alpha}^{|\dot{j}|} \leq h_{\dot{i}|_t}(j) \quad \text{and} \quad v_{\dot{i}|_t}(j) \leq \alpha_2(\dot{i}|_t) \bar{\alpha}^{|\dot{j}|} L$$

for all $\dot{i} \in I^{\mathbb{N}}$, $j \in I^*$, and $t > 0$.

Proof. Since E is compact and is not contained in any line, there exist a number $L \in \mathbb{N}$ and balls $B(x, L)$ and $B(y, L^{-1})$ such that E is included in $B(x, L)$ and $B(y, L^{-1})$ is included

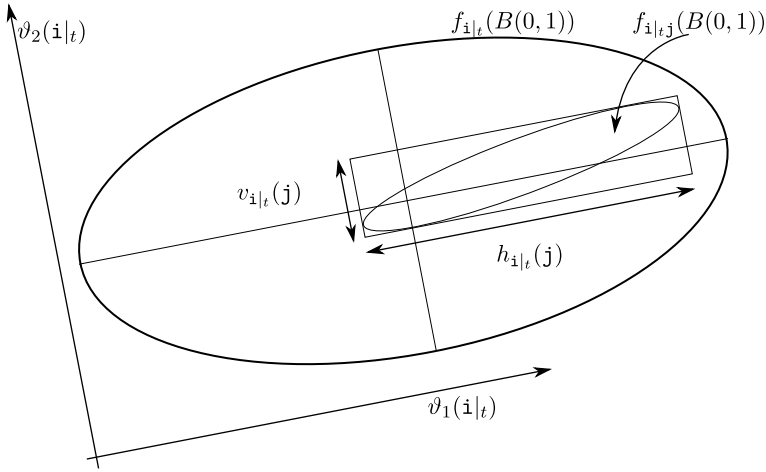


FIGURE 1. The construction rectangle $R_{i|t}(j)$. picture. For illustrative purposes, the construction cylinders have been replaced by ellipses.

in the convex hull of E . Therefore we have $h_{i|t}(j) \geq L^{-1}\alpha_1(i|t)j \geq L^{-1}\alpha_1(i|t)\underline{\alpha}^{|j|}$ and $v_{i|t}(j) \leq L\alpha_2(i|t)j \leq L\alpha_2(i|t)\bar{\alpha}^{|j|}$ as claimed. \square

Fix $x \in E$ and $t > 0$. We define the t -screen at x to be the closed ball $B(x, t)$ and the zoom function $Z_{x,t}: B(x, t) \rightarrow B(0, 1)$ by setting

$$Z_{x,t}(y) = \frac{y - x}{t}.$$

Then we define the scenery $N_{x,t}$ around x at scale t by setting $N_{x,t} = Z_{x,t}(E \cap B(x, t))$. We consider distances of compact subsets of \mathbb{R}^2 in terms of the Hausdorff metric d_H . The notation dist is used for distances between points, or points and compact sets, in the Euclidean metric. Any limit of the sequence (N_{x,t_n}) , where $t_n \downarrow 0$, is called a tangent set of E at a point x . We call a subset $S \subset B(0, 1)$ an ε -pattern, if

$$S = \left(\mathbb{R} \times \bigcup_{i=1}^l I_i \right) \cap B(0, 1),$$

where I_i are intervals of length less than ε for all $i \in \{1, \dots, l\}$. We emphasize that these intervals are not assumed to be disjoint. Our goal is to study ε -patterns coming from the construction rectangles – even though the cylinders E_i are disjoint, the construction rectangles might overlap.

Fix $i \in I^{\mathbb{N}}$ and let $B(i, t) = B(\pi(i), t)$. Since the directions $\bar{\vartheta}_k(i)$ can differ for different i , we zoom into the set by applying, at each step, an appropriate rotation. To make this precise, consider the set in the singular basis, and define the rotated screen $M_{i,t}$ by setting

$$M_{i,t} = \mathcal{O}_{i,t}(\mathcal{Z}_{\pi(i),t}(B(i, t) \cap E)) = \mathcal{O}_{i,t}(N_{\pi(i),t}).$$

Here $\mathcal{O}_{i,t}$ is the rotation that takes the singular basis $(\vartheta_1(i|t), \vartheta_2(i|t))$ to the standard basis of \mathbb{R}^2 . Thus $M_{i,t}$ is the intersection of the set E with a small ball around the point

$\pi(\mathfrak{i})$, scaled up to fill the whole unit ball, and turned around until $\vartheta_1(\mathfrak{i}|_t)$ is horizontal. We also define the *approximative sceneries* $P_{\mathfrak{i},t}^K$ for all $K \in \mathbb{N}$ as

$$P_{\mathfrak{i},t}^K = \mathcal{O}_{\mathfrak{i},t} \left(\mathcal{Z}_{\pi(\mathfrak{i}),t} \left(B(\mathfrak{i}, t) \cap \left(\bigcup_{j \in R(\mathfrak{i},t,K)} R_{\mathfrak{i}|_t}(j) \right) \right) \right),$$

where $R(\mathfrak{i}, t, K) = \{j \in I^K : B(\mathfrak{i}, t) \cap E_{\mathfrak{i}|_t,j} \neq \emptyset\}$. Thus $P_{\mathfrak{i},t}^K$ tells us the level K approximation of the set around $\pi(\mathfrak{i})$ in the singular basis, after we have first zoomed into the scale t . Note that both $M_{\mathfrak{i},t}$ and $P_{\mathfrak{i},t}^K$ are subsets of $B(0, 1)$. We say that $T \subset B(0, 1)$ is a *modified tangent* of E at $\pi(\mathfrak{i})$ if there exists a sequence $t_n \downarrow 0$ such that $M_{\mathfrak{i},t_n} \rightarrow T$.

A set $A \subset \mathbb{R}^2$ is *porous* if there exists $0 < \alpha \leq 1$ such that for every $x \in A$ and $0 < r < \text{diam}(A)$ there is a point $y \in A$ for which $B(y, \alpha r) \subset B(x, r) \setminus A$. We remark that a more precise name for this porosity condition is uniform lower porosity. It is well known that porous sets have zero Lebesgue measure; see, for example, [19, Proposition 3.4]. Therefore a closed porous set is nowhere dense. Any ‘fat’ Cantor set serves as an example of a closed nowhere dense set which is not porous.

LEMMA 2.4. *If T is a modified tangent of the self-affine set E , then T is closed and porous.*

Proof. By [34], the set E is porous and thus, by [8, Proposition 5.6], any tangent set is also porous. Since this remains true also for modified tangent sets, the proof is complete. \square

Lemma 2.1 says that the rotations $\mathcal{O}_{\mathfrak{i},t}$ converge for almost every $\mathfrak{i} \in I^{\mathbb{N}}$. Therefore there is a correspondence between tangents and modified tangents in our setting. Thus all results obtained for modified tangent sets are valid for tangent sets and vice versa. In what follows, we mostly consider modified tangents because it is convenient to have a fixed orientation of the construction rectangles.

2.4. *Idea of the proof.* The idea of the proof is to show that at almost every point of E , at all small scales, the approximative sceneries are ε -patterns; see Lemma 3.6. In this we are following Bandt and Käenmäki [2]. In their situation all of the construction rectangles are uniformly flat in the vertical direction, but in our case it is not immediately clear what ‘vertical’ even means, and flatness of the construction rectangles is not guaranteed in any direction. To deal with this difficulty, we first let the screen rotate according to the singular basis with the construction level of the zoom, turning the construction rectangles so that they are flat in a controllable way in the rotated basis.

On the other hand, to make sure that the construction rectangles are flat enough in the vertical direction of the singular basis, we prove in Lemmas 2.5 and 3.5 that there is a set of large measure so that the construction rectangles for points in this set are long in the horizontal direction and narrow in the vertical direction. Lemma 2.5 is proved below but Lemma 3.5 is postponed to the next section, as it requires some more definitions.

To make use of Lemma 3.6, we show in Lemma 3.7 that the approximative sceneries get close to the rotated screens and thus can be used to approximate the modified tangent sets. Here we do not know whether the rectangles in the approximative sceneries overlap or not, but this is not a problem, since by recalling Lemma 2.4, we can deduce that the

horizontal projection C of the modified tangent set is porous. Finally, we use Lemma 2.1 to transfer the result to the original tangent sets of E .

LEMMA 2.5. *If $\lambda_1 < \theta < \gamma < \lambda_2$, then for all $\varrho > 0$ there is a set E_ϱ with $v_p(E_\varrho) \geq 1 - \varrho$ such that the following two conditions are satisfied.*

- (1) *There are numbers $a = a(\gamma, \theta) > 1$ and $N(\varrho) \in \mathbb{N}$ such that, for all $i \in E_\varrho$ and $n \geq N(\varrho)$, we have $\alpha_1(i|_n) > a^n \alpha_2(i|_n)$.*
- (2) *For all $D \in \mathbb{N}$ there is $N(D) \in \mathbb{N}$ such that, for all $i \in E_\varrho$, and for all t with $n(i, t) = n$, and for the $k \in \mathbb{N}$ satisfying $N(D)k \leq n < N(D)(k + 1)$, we have $h_{i|_t}(j) > 2D\alpha_2(i|_n)$ for all $j \in I^k$.*

Proof.

(1) Fix $\varrho > 0$. By Egorov’s theorem, we find a set $E_\varrho \subset I^\mathbb{N}$ with $v_p(I^\mathbb{N} \setminus E_\varrho) \leq \varrho$ where $\lambda_1(i|_n)$ and $\lambda_2(i|_n)$ from (2.1) converge uniformly. Thus we find $N = N(\varrho) \in \mathbb{N}$ so that

$$\alpha_1(i|_n) \geq e^{-\theta n} \quad \text{and} \quad \alpha_2(i|_n) \leq e^{-\gamma n}$$

for all $n \geq N$. Letting $a = e^{\gamma - \theta}$, we have $\alpha_1(i|_n)/\alpha_2(i|_n) \geq e^{(\gamma - \theta)n} = a^n$, and the first claim is proved.

(2) Let $L \in \mathbb{N}$ be as in Lemma 2.3. Fix $D \in \mathbb{N}$ and choose $N(D) \geq N(\varrho)$ so that

$$a^{N(D)} \underline{\alpha} > 2DL. \tag{2.2}$$

Let $i \in E_\varrho$ and t be so small that $n := n(i, t) \geq N(D)$. Fix $k \in \mathbb{N}$ so that $N(D)k \leq n < N(D)(k + 1)$, and let $j \in I^k$. Now, by Lemma 2.3 and (1), we have

$$\begin{aligned} h_{i|_t}(j) &\geq \alpha_1(i|_t) \underline{\alpha}^k L^{-1} > a^n \alpha_2(i|_t) \underline{\alpha}^k L^{-1} \\ &\geq \alpha_2(i|_t) (a^{N(D)} \underline{\alpha})^k L^{-1} > 2^k D^k L^{k-1} \alpha_2(i|_t) \geq 2D\alpha_2(i|_t) \end{aligned}$$

as claimed. □

3. Main result

Let us begin by formulating the main theorem of the paper.

THEOREM 3.1. *Suppose that $\{f_1, \dots, f_m\}$ is an affine iterated function system and E the associated self-affine set. If*

- (1) *there exists $\delta > 0$ is such that $\min\{\text{dist}(f_i(E), f_j(E)) : i \neq j\} > \delta$; that is, E satisfies the strong separation condition;*
- (2) *for v_p -almost all $i \in I^\mathbb{N}$ there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $j \in I^*$, the projection $\text{proj}_{\overline{\vartheta}(i)}(E_{i|_n j})$ is a line segment;*
- (3) *the probability vector $p = (p_1, \dots, p_m)$ is such that the Lyapunov exponents satisfy $\lambda_1 < \lambda_2$,*

then for v_p -almost all $x \in E$ the tangent sets at x are either of the form $\mathcal{O}((\mathbb{R} \times C) \cap B(0, 1))$, where C is a closed porous set, or of the form $\mathcal{O}((\ell \times \{0\}) \cap B(0, 1))$, where ℓ is an interval containing at least one of the intervals $[-1, 0]$ and $[0, 1]$. Here \mathcal{O} is the rotation that takes $(1, 0)$ to $\overline{\vartheta}_1(\pi^{-1}(x))$.

Remark 3.2. To verify assumption (3) in Theorem 3.1, it suffices to check that the iterated function system is pinching and twisting since then assumption (3) follows immediately from [32, Theorem 1.2]; see also [13]. An affine iterated function system is *pinching* if for any constant $C > 1$ there is a finite word \dot{i} such that $\alpha_1(\dot{i}) > C\alpha_2(\dot{i})$. It is *twisting* if for any finite set of vectors $\{v, w_1, \dots, w_n\} \subset \mathbb{R}^2$, there exists a finite word \dot{j} so that $A_{\dot{j}}v$ is not parallel to w_j for any $j \in \{1, \dots, n\}$. It is worthwhile to remark that, in particular, [32, Theorem 1.2] applies to any Bernoulli measure obtained from a positive probability vector.

In the carpet case, where the linear part of f_i is the diagonal matrix $\text{diag}(h_i, v_i)$, assumption (3) is equivalent to

$$-\sum_{i=1}^m v_p(f_i([0, 1]^2)) \log h_i \neq -\sum_{i=1}^m v_p(f_i([0, 1]^2)) \log v_i. \tag{3.1}$$

Indeed, by the ergodic theorem, the left-hand side of (3.1) equals $\lim_{n \rightarrow \infty} (1/n) \log h_{\dot{i}|_n}$ and the right-hand side equals $\lim_{n \rightarrow \infty} (1/n) \log v_{\dot{i}|_n}$ for v_p -almost all $\dot{i} \in I^{\mathbb{N}}$. Fix \dot{i} so that these limits and $\lambda_1(\dot{i})$, $\lambda_2(\dot{i})$, and $\bar{\vartheta}_1(\dot{i})$ exist. By Lemma 2.1, $\vartheta_k(\dot{i}|_n)$ converges to $\bar{\vartheta}_k(\dot{i})$ for both $k \in \{1, 2\}$. Therefore, for some $k \in \{1, 2\}$, we have $\alpha_k(\dot{i}|_n) = h_{\dot{i}|_n}$ and $\alpha_{3-k}(\dot{i}|_n) = v_{\dot{i}|_n}$ for all large enough n . This clearly implies that $\lambda_k = -\sum_{i=1}^m v_p(f_i([0, 1]^2)) \log h_i$ and $\lambda_{3-k} = -\sum_{i=1}^m v_p(f_i([0, 1]^2)) \log v_i$.

From the point of view of our theorem, it would be interesting to know if for any self-affine set there exists a positive probability vector p that gives rise to distinct Lyapunov exponents. Pinching is definitely a necessary condition for this: having distinct Lyapunov exponents implies that $\alpha_1(\dot{i}|_n)$ gets exponentially larger than $\alpha_2(\dot{i}|_n)$ v_p -almost everywhere. Since it is easy to define Bernoulli measures having distinct Lyapunov exponents on self-affine carpets we see that twisting is not a necessary condition. We also remark that there exist affine iterated function systems where the mappings are not similitudes but the Lyapunov exponents coincide for all ergodic measures. For example, choose mappings that have the same linear part which is a composition of a diagonal contraction having distinct eigenvalues and a rotation of 90 degrees. Since the second-level compositions of the mappings are similitudes it is impossible to have $\lambda_1 < \lambda_2$ for any ergodic measure.

Remark 3.3. Assumption (2) in Theorem 3.1 is referred to as the *projection condition*. It is satisfied, for example, if the projection of the set $\bigcup_{i=1}^m f_i(X)$ in any direction is a line segment, where X is the convex hull of E . To see this, fix a line ℓ that intersects $\bigcup_{i=1}^m f_i(X)$. Let j_1 be such that ℓ intersects $f_{j_1}(X)$. The crucial observation now is that the line $f_{j_1}^{-1}(\ell)$ intersects $\bigcup_{i=1}^m f_i(X)$. If this were not the case, then the line $f_{j_1}^{-1}(\ell)$ would divide the convex hull X into two parts, both of which contain sets $f_i(X)$. This contradicts the assumption. Therefore, if j_2 is such that $f_{j_1}^{-1}(\ell)$ intersects $f_{j_2}(X)$, then we see that ℓ intersects $f_{j_1} \circ f_{j_2}(X)$. Continuing in this manner, we find $\dot{j} = (j_1, j_2, \dots) \in I^{\mathbb{N}}$ such that $\pi(\dot{j}) \in \ell$, which is what we wanted to show.

It is worth noticing that in the carpet case, if (3.1) holds, then it suffices to consider only one projection: there are exactly two singular directions which are invariant under all the maps, so that in any case, in order for assumption (2) to hold, it suffices to consider at most two directions. We may assume that the right-hand side of (3.1) is

greater than the left-hand side. This guarantees that the horizontal direction is the direction $\overline{\vartheta}_1(\dot{i})$ for ν_p -almost all $\dot{i} \in I^{\mathbb{N}}$. Indeed, notice that for almost all $\dot{i} \in I^{\mathbb{N}}$, eventually $\text{diam}(\text{proj}_x(R_{\dot{i}|_n})) > \text{diam}(\text{proj}_y(R_{\dot{i}|_n}))$, where proj_x and proj_y denote the projections onto the x -axis and y -axis, respectively. This means that for almost all $\dot{i} \in I^{\mathbb{N}}$, eventually $\text{diam}(\text{proj}_x(R_{\dot{i}|_n})) = \alpha_1(\dot{i}|_n)$. Hence the vector $\vartheta_1(\dot{i}|_n)$ eventually becomes horizontal. Observe that it is essential that the projection condition is defined pointwise.

Remark 3.4. Assumptions (1) and (2) of Theorem 3.1 imply that E is not contained in a line: assume, to the contrary, that E is contained in a line ℓ . Then, for any n , all the rectangles $R_{\dot{i}|_n}$ intersect this line. Condition (1) guarantees that E itself cannot be a line segment. Thus $E_{\dot{i}|_n}$ is not a line segment. Also, the direction $\overline{\vartheta}_2(\dot{i})$ is not the direction of the line ℓ for any \dot{i} in a set of full measure, since then the projection onto $\overline{\vartheta}_1(\dot{i})$ would be a single point and not a line segment. Thus, for almost all \dot{i} , the angle between $\overline{\vartheta}_2(\dot{i})$ and ℓ is positive, implying that $\text{proj}_{\overline{\vartheta}_1(\dot{i})}(E_{\dot{i}|_n \dot{j}})$ is not a line segment, thus giving a contradiction with assumption (2).

Therefore, since E is not contained in any line, there is a construction level k_0 such that when $|\dot{i}| = k_0$, no line in any direction intersects all ellipses $f_{\dot{i}}(B)$, where B is a ball containing E . Without loss of generality we may assume that $k_0 = 1$, since otherwise we can consider the iterated function system $\{f_{\dot{i}} : \dot{i} \in I^{k_0}\}$. This is not a restriction, since tangent sets of E only depend on the set E itself and not on the iterated function system that generates E .

We will now start preparations for the proof of the main result. Working in the setting of Lemma 2.5, we introduce a sequence of lemmas gradually converging to the proof. Fix $\varrho > 0$ and let E_ϱ be as in Lemma 2.5. Furthermore, let $K \in \mathbb{N}$ and denote

$$\varepsilon(K) = L\delta^{-1}\overline{\alpha}^K,$$

where L is as in Lemma 2.3 and $\delta > 0$ as in assumption (1) of Theorem 3.1.

LEMMA 3.5. *Under the assumptions of Theorem 3.1, for every $K \in \mathbb{N}$ there exists $t_0 > 0$ such that $v_{\dot{i}|_t}(\dot{j}) \leq t\varepsilon(K)$ for all $t < t_0$, $\dot{i} \in E_\varrho$, and $\dot{j} \in I^K$.*

Proof. The t -screen $B(\pi(\dot{i}), t)$ intersects at least two level $n(\dot{i}, t) + 1$ construction cylinders, and by assumption (1) of Theorem 3.1, we have the estimate $t \geq \alpha_2(\dot{i}|_t)\delta \geq \underline{\alpha}^{n(\dot{i}, t)}\delta$, where δ is as in that assumption. This shows that we have a uniform lower bound for $n(\dot{i}, t)$ that increases as t decreases. Take $t_0 > 0$ so small that, for all $t < t_0$, it is the case that $n(\dot{i}, t) > K$ for all $\dot{i} \in I^{\mathbb{N}}$. By Lemma 2.3, we then have

$$v_{\dot{i}|_t}(\dot{j}) \leq \alpha_2(\dot{i}|_t)\overline{\alpha}^K L \leq t\delta^{-1}\overline{\alpha}^K L = t\varepsilon(K)$$

as claimed. □

As described in §2.4, we want to investigate the size of the set of points for which the approximative scenery is not an $\varepsilon(K)$ -pattern. For technical reasons we consider the following sets. For every $D > 0$ and $K, n \in \mathbb{N}$ we define

$$A_{n,K}(D) = \{\dot{i} \in E_\varrho : \text{there is } t \text{ such that } n(\dot{i}, t) = n, \\ D\alpha_2(\dot{i}|_n) > t, \text{ and } P_{\dot{i},t}^K \text{ is not an } \varepsilon(K)\text{-pattern}\},$$

and let $B_K(D) = \limsup_{n \rightarrow \infty} A_{n,K}(D)$.

In the following two lemmas, the reader should bear in mind that while the strong separation condition guarantees the construction cylinders to be disjoint, the corresponding construction rectangles may overlap. This is not a problem since, for example, the proof of the following lemma concerns only the number of certain vertical edges. Recall also that the choice of $n(\mathfrak{i}, t)$ guarantees that the screen $B(\pi(\mathfrak{i}), t)$ contains points only from one level n construction cylinder.

LEMMA 3.6. *Under the assumptions of Theorem 3.1, we have $v_p(B_K(D)) = 0$ for all $D > 0$ and $K \in \mathbb{N}$.*

Proof. Fix D and K , and let $N(D)$ be as in Lemma 2.5. Our plan is to prove that

$$\sum_{n=1}^{\infty} v_p(A_{n,K}(D)) < \infty$$

since then the claim follows from the Borel–Cantelli lemma. To that end, we estimate the measure of the sets $A_{n,K}(D)$.

We will cover the set $A_{n,K}(D)$ by construction cylinders corresponding to forbidden words which will be defined shortly. Throughout, we are considering the situation in the singular basis and thus, we shall refer to the directions $\vartheta_1(\mathfrak{i}|_n)$ and $\vartheta_2(\mathfrak{i}|_n)$ as horizontal and vertical, respectively. This should not be a cause of confusion, as the basis in use is clear from the context. The *forbidden words* are defined in the following way: for any $k \in \mathbb{N}$ and $|\mathfrak{i}| = n$ with $N(D)k \leq n < N(D)(k + 1)$ the word $\mathfrak{i}j$ is n -forbidden, if $|j| = k$ and $R_{\mathfrak{i}}(j)$ intersects any of the vertical line segments $\{c\} \times \mathbb{R}$ where

$$c \in \{e \pm D\alpha_2(\mathfrak{i}) : e \text{ is an } x \text{ coordinate of any of the } 2m^K$$

$$\text{vertical edges of the rectangles } R_{\mathfrak{i}}(a) \text{ with } a \in I^K\}.$$

Now fix a point $x = \pi(\mathfrak{u}) \in \pi(A_{n,K}(D))$. Then there is t such that $n(x, t) = n$ and $P_{\mathfrak{u},t}^K$ is not an $\varepsilon(K)$ -pattern. By Lemma 3.5, for each $|\mathfrak{a}| = K$, the rectangles $R_{\mathfrak{u}|_n}(\mathfrak{a})$ have height at most $\varepsilon(K)t$. Thus the only way the approximative scenery $P_{\mathfrak{u},t}^K$ around x is not an $\varepsilon(K)$ -pattern is if some endpoint of a rectangle is in the t -screen $B(x, t)$. More precisely, this means that within distance t from x in the horizontal direction, there is an endpoint of a rectangle from level $n + K$. By Lemma 2.5(2) and the definition of $A_{n,K}(D)$,

$$h_{\mathfrak{u}|_n}(\mathfrak{u}(n + 1), \dots, \mathfrak{u}(n + k)) > 2D\alpha_2(\mathfrak{u}|_n) > D\alpha_2(\mathfrak{u}|_n) > t,$$

so that in this case the rectangle $R_{\mathfrak{u}|_n}(\mathfrak{u}(n + 1), \dots, \mathfrak{u}(n + k))$ necessarily intersects one of the forbidden line segments and hence $\mathfrak{u}|_{n+k}$ is an n -forbidden word. (See Figure 2.) By this argument, we see that $\pi(A_{n,K}(D))$ is covered by n -forbidden rectangles, as claimed.

Now, let $\mathfrak{i} \in I^n$. By Remark 3.4, no line in any direction in the rectangle $R_{\mathfrak{i}}$ intersects all the sub-rectangles $R_{\mathfrak{i}}(j)$ of level $n + 1$. Therefore, the relative total mass of the sub-rectangles $R_{\mathfrak{i}}(j)$ that the line intersects in the vertical direction is at most $1 - \underline{p}$. By the self-affinity and properties of the Bernoulli measures, we see that the relative total mass of the level $n + 2$ sub-rectangles, which the vertical line intersects, is at most $(1 - \underline{p})^2$. Iterating this, and noticing that there are $4m^K$ forbidden line segments, we get that

$$v_p(A_{n,K}(D)) \leq \sum_{\mathfrak{i}j \text{ is } n\text{-forbidden}} v_p(E_{\mathfrak{i}j}) \leq \sum_{\mathfrak{i} \in I^n} v_p(E_{\mathfrak{i}})(1 - \underline{p})^k 4m^K = (1 - \underline{p})^k 4m^K \tag{3.2}$$

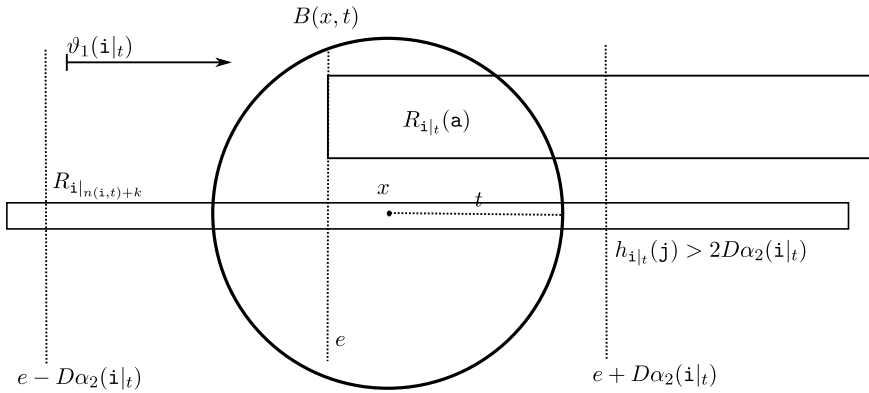


FIGURE 2. The point x belongs to a forbidden rectangle because within distance t away in the horizontal direction an endpoint a of level $n + K$ construction rectangle $R_{i|_t}(a)$ appears.

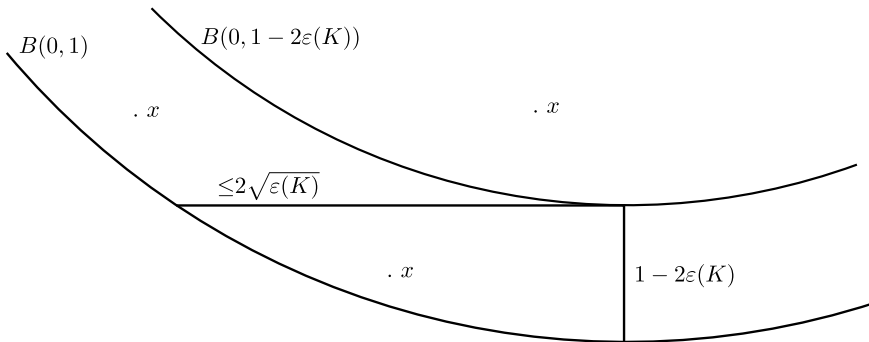


FIGURE 3. The possible positions of x in the proof of Lemma 3.7.

for $N(D)k \leq n < N(D)(k + 1)$. Thus

$$\sum_{n=1}^{\infty} v_p(A_{n,K}(D)) \leq \sum_{n=1}^{N(D)-1} v_p(A_{n,K}(D)) + N(D) \frac{(1-p)4m^K}{1-(1-p)} < \infty,$$

finishing the proof. □

LEMMA 3.7. *Under the assumptions of Theorem 3.1, for every $K \in \mathbb{N}$ and for almost every $i \in E_Q$ there exists $t_0 > 0$ such that $d_H(P_{i,t}^K, M_{i,t}) < 5\sqrt{\varepsilon(K)}$ for all $0 < t < t_0$.*

Proof. Let $i \in I^{\mathbb{N}}$ be generic in the sense that $\bar{\vartheta}_1(i)$ from Lemma 2.1 exists. Now $t_0 > 0$ is determined from n_0 of assumption (2) in Theorem 3.1 and from Lemmas 2.1 and 3.5 so that, for all $0 < t < t_0$ and $j \in I^*$, the projection $\text{proj}_{\bar{\vartheta}_1(i)}(E_{i|_t}(j))$ is a line segment, $\vartheta_1(i|_t)$ is close to $\bar{\vartheta}_1(i)$, and the height of $R_{i|_t}(j)$ is at most $\varepsilon(K)$ for all $j \in I^K$. Notice that $M_{i,t} \subset P_{i,t}^K$. Fix a point $x \in P_{i,t}^K$ and let $j \in I^K$ be such that $x \in \mathcal{O}_{i,t} \mathcal{Z}_{\pi(i),t} R_{i|_t}(j)$. We now wish to prove that within $5\sqrt{\varepsilon(K)}$ away from x there is a point $y \in M_{i,t}$. The study divides into three separate cases; see Figure 3.

If x is far from the boundary of the screen, we can find the point y on the line ℓ crossing through $\mathcal{Z}_{\pi(i),t}^{-1}(x)$ in direction $\bar{\vartheta}_2(i)$. Assume that $x \in B(0, 1 - 2\varepsilon(K))$, say,

and denote by θ the angle between $\bar{\vartheta}_1(\dot{i})$ and $\vartheta_2(\dot{i}|_t)$. By Lemma 2.1, we may assume that $\sin(\theta) \geq 1/2$. By assumption (2) of Theorem 3.1, there exists $z \in \ell \cap E_{\dot{i}|_t, j}$, as the projection $\text{proj}_{\bar{\vartheta}_1(\dot{i})}^{-1}(E_{\dot{i}|_t, j})$ is connected. By Lemma 3.5, we get that $v_{\dot{i}|_t}(j) \leq t\varepsilon(K)$. Thus $\text{dist}(\mathcal{Z}_{\pi(\dot{i}), t}^{-1}(x), z) \leq 2t\varepsilon(K)$, and so $z \in B(\pi(\dot{i}), t)$ by the triangle inequality. When zooming out, we get $\text{dist}(x, y) \leq 2\varepsilon(K)$ for the point $y = \mathcal{Z}_{\pi(\dot{i}), t}(z) \in M_{\dot{i}, t}$.

If x is very close to the boundary of the screen, it is possible that the above reasoning gives a point y which is not inside the screen. These points need to be dealt with differently. Consider first the case $\mathcal{O}_{\dot{i}, t} \mathcal{Z}_{\pi(\dot{i}), t} R_{\dot{i}|_t}(j) \cap B(0, 1 - 2\varepsilon(K)) = \emptyset$. Let $y \in M_{\dot{i}, t} \cap \mathcal{O}_{\dot{i}, t} \mathcal{Z}_{\pi(\dot{i}), t} R_{\dot{i}|_t}(j)$ be arbitrary – such a point exists by the definition of $P_{\dot{i}, t}^K$. By estimating the length of line segments contained in the annulus $B(0, 1) \setminus B(0, 1 - 2\varepsilon(K))$, we can bound the horizontal distance between x and y by $4\sqrt{\varepsilon(K)}$. Thus $\text{dist}(x, y) \leq 4\sqrt{\varepsilon(K)} + \varepsilon(K)$.

To conclude, let $x \notin B(0, 1 - 2\varepsilon(K))$ but assume that $\mathcal{O}_{\dot{i}, t} \mathcal{Z}_{\pi(\dot{i}), t} R_{\dot{i}|_t}(j) \cap B(0, 1 - 2\varepsilon(K)) \neq \emptyset$. Then there is a point $x' \in B(0, 1 - 2\varepsilon(K))$ such that, as for x and y above, $\text{dist}(x, x') \leq 2\sqrt{\varepsilon(K)}$. For this x' we find $y \in M_{\dot{i}, t}$ as in the first case, with $\text{dist}(x', y) \leq 2\varepsilon(K)$. Thus $\text{dist}(x, y) \leq 2\sqrt{\varepsilon(K)} + 2\varepsilon(K)$. Since $\varepsilon(K) \leq \sqrt{\varepsilon(K)}$, the proof is complete. □

We now combine the above lemmas to prove Lemma 3.8. We will then be in a position to prove the main theorem.

LEMMA 3.8. *Under the assumptions of Theorem 3.1, for v_p -almost all $\dot{i} \in E_Q$, the modified tangent sets at $\pi(\dot{i})$ are either of the form $(\mathbb{R} \times C) \cap B(0, 1)$, where C is a closed porous set, or of the form $(\ell \times \{0\}) \cap B(0, 1)$, where ℓ is an interval containing at least one of the intervals $[-1, 0]$ and $[0, 1]$.*

Proof. Recall that by Lemma 3.6, the set $B(M) = \bigcup_{K=0}^\infty B_K(M)$ has zero measure for all $M \in \mathbb{N}$. Fix a point $\dot{i} \in E_Q \setminus \bigcup_{M=1}^\infty B(M)$, and a modified tangent set $T = \lim_{k \rightarrow \infty} M_{\dot{i}, t_k}$ at $x = \pi(\dot{i})$. We assume that \dot{i} is generic, in the sense that $\bar{\vartheta}_1(\dot{i})$ from Lemma 2.1 exists.

Now, there are two options. Either

$$\lim_{k \rightarrow \infty} \frac{\alpha_2(\dot{i}|_{n(\dot{i}, t_k)})}{t_k} = 0, \tag{3.3}$$

or there exist $D > 0$ and infinitely many k such that

$$\frac{\alpha_2(\dot{i}|_{n(\dot{i}, t_k)})}{t_k} > \frac{1}{D}. \tag{3.4}$$

We will first consider the situation where (3.3) holds. For the time being, fix $t > 0$ and the corresponding $n = n(\dot{i}, t)$ so that $n - 1 \geq n_0$, where n_0 is as in assumption (2) of Theorem 3.1, and assume that both endings of the construction rectangle $R_{\dot{i}|_n}$ can be seen in the screen. It might be that one of the endings of $\text{proj}_{\bar{\vartheta}_1(\dot{i})}^{-1}(E_{\dot{i}|_n})$ is at the end of the line segment $\text{proj}_{\bar{\vartheta}_1(\dot{i})}^{-1}(E_{\dot{i}|_{n-1}})$, but not both of them. Thus, by assumption (2) of Theorem 3.1, for at least one of the endpoints of $R_{\dot{i}|_n}$, a line ℓ in the direction $\bar{\vartheta}_2(\dot{i})$ through the endpoint necessarily also intersects another cylinder $E_{\dot{i}|_{n-1}j}$ for some $j \in \{1, \dots, m\}$. This is the case, since otherwise there is a hole in the projection of $E_{\dot{i}|_{n-1}}$ onto the line in direction $\bar{\vartheta}_1(\dot{i})$.

Let d denote the distance from x to that ending of the rectangle $R_{\dot{i}|_n}$ and let e denote the distance between $E_{\dot{i}|_n}$ and $E_{\dot{i}|_{n-1}j}$ along the line ℓ . Notice that e is bounded from above by $v_{\dot{i}|_{n-1}}/\cos\theta_n$, where θ_n is the angle between $\vartheta_2(\dot{i}|_{n-1})$ and $\bar{\vartheta}_2(\dot{i})$. By Lemma 2.1, taking t small enough, we may assume that $\cos\theta_n \geq 1/2$, and hence $e \leq 2v_{\dot{i}|_{n-1}}$. We want to show that $d/t \rightarrow 1$ as $t \rightarrow 0$. Assume that this is not the case, that is, $d < ct$ for some $c < 1$. Then, since the screen $B(x, t)$ intersects only the cylinder $E_{\dot{i}|_n}$, using Lemma 2.3,

$$t \leq \text{dist}(x, E_{\dot{i}|_{n-1}j}) \leq d + v_{\dot{i}|_n} + e \leq ct + v_{\dot{i}|_n} + 2v_{\dot{i}|_{n-1}} \leq ct + L(1 + 2\bar{\alpha})\alpha_2(\dot{i}|_n).$$

This gives $t \leq C\alpha_2(\dot{i}|_n)$ for some absolute constant $C > 0$. Thus, for all small t ,

$$\frac{\alpha_2(\dot{i}|_n(\dot{i}, t))}{t} > 0.$$

These observations mean that if (3.3) holds, then either both endings of the construction rectangle $R_{\dot{i}|_{t_k}}$ can be seen in the screen for only finitely many k , or at least one of the endpoints of the rectangles reaches the boundary of the screen $B(x, t_k)$ in the limit.

In the latter case, using Lemma 2.3, $P_{x,t}^0$ consists of a strip with height converging to zero. By the argument of Lemma 3.7 and compactness, the modified tangent set T is a horizontal line segment, containing at least one of the line segments $[1, 0] \times \{0\}$ and $[-1, 0] \times \{0\}$.

Assume now that at most one of the endings is seen in the screen, apart from perhaps finitely many (t_k) . Since there are only two endings, there is a sub-sequence of (t_k) such that, along that sub-sequence, either the left or the right endpoint is never in the screen, and we can use the argument from the previous paragraph to deduce the same claim. Notice that because the limit set T exists, it is unique, and thus it suffices to prove the convergence along a sub-sequence.

Let us then assume that (3.4) holds, that is, $D\alpha_2(\dot{i}|_{t_k}) > t_k$ along a sub-sequence which we continue to denote by (t_k) . Let M be an integer with $M \geq D$ and notice that $\dot{i} \notin B(M)$. Fix K , and notice that also $\dot{i} \notin B_K(M)$, so that $\dot{i} \in A_{n,K}(M)$ for at most finitely many $n \in \mathbb{N}$.

For all k it is the case that $M\alpha_2(\dot{i}|_{t_k}) \geq D\alpha_2(\dot{i}|_{t_k}) > t_k$, so that if there are only finitely many k such that $P_{\dot{i},t_k}^K$ would be an $\varepsilon(K)$ -pattern, then $x \in A_{n(x,t_k),K}(M)$ for infinitely many k . Since $t_k \rightarrow 0$ implies $n(x, t_k) \rightarrow \infty$, this gives infinitely many $A_{n,K}$, which cannot be the case. Hence for all K there is a t_K such that $P_{\dot{i},t_K}^K$ is an $\varepsilon(K)$ -pattern and that Lemma 3.7 holds. Since there is a sequence of ε -patterns converging to the modified tangent set T , it must be of the form $(\mathbb{R} \times C) \cap B(0, 1)$ for some set $C \subset [-1, 1]$. Since, by Lemma 2.4, T is closed and porous in $B(0, 1)$, the same must hold for C in $[-1, 1]$. \square

We can now prove Theorem 3.1.

Proof of Theorem 3.1. So far, in the previous lemmas, we have verified the claims for almost all points in the set E_ϱ having measure at least $1 - \varrho$. Since the claim of Lemma 3.8 does not depend on ϱ , it actually holds for almost all points in E : if there is an exceptional set of positive measure, then we can repeat the argument for some ϱ smaller than half the size of the exceptional set to get a contradiction.

Let us now assume that T is a tangent set of E at $x = \pi(\dot{i})$, along a sequence (t_n) . By the above discussion, we may assume that Lemma 3.8 holds at x . By compactness, we find

a sub-sequence, also denoted by (t_n) , such that $M_{\dot{i},t_n} \rightarrow F$ along this sub-sequence. By Lemma 3.8, we know that F is either of the form $(\mathbb{R} \times C) \cap B(0, 1)$, where C is a closed porous set, or of the form $(\ell \times \{0\}) \cap B(0, 1)$, where ℓ is an interval with $[-1, 0] \subset \ell$ or $[0, 1] \subset \ell$. Let \mathcal{O} be the rotation taking $\bar{\vartheta}_1(\dot{i})$ to $(1, 0)$, where $\bar{\vartheta}_1(\dot{i})$ is from Lemma 2.1.

It suffices to show that $T = \mathcal{O}^{-1}F$. Since $\vartheta_1(\dot{i}|_{t_n}) \rightarrow \bar{\vartheta}_1(\dot{i})$, the rotations $\mathcal{O}_{\dot{i},t_n}$ converge to \mathcal{O} . Let $\varepsilon > 0$ and choose n_0 so that $d_H(\mathcal{O}T, \mathcal{O}_{\dot{i},t_n}T) \leq \varepsilon$, $d_H(T, N_{\dot{i},t_n}) \leq \varepsilon$, and $d_H(M_{\dot{i},t_n}, F) \leq \varepsilon$ for all $n \geq n_0$. By the triangle inequality, we then have

$$d_H(\mathcal{O}T, F) \leq d_H(\mathcal{O}T, \mathcal{O}_{\dot{i},t_n}T) + d_H(\mathcal{O}_{\dot{i},t_n}T, M_{\dot{i},t_n}) + d_H(M_{\dot{i},t_n}, F) \leq 3\varepsilon$$

which completes the proof.

4. Discussion

We formulated our main theorem by using assumptions as general as possible. In Remarks 3.2 and 3.3, we provided the reader with sufficient and checkable conditions. In this section, we continue this discussion by examining the effect of some of the assumptions and exhibiting examples. We also prove Corollary 1.1.

We say that a subset of the self-affine set $A \subset E$ satisfies the *line condition*, if there exists $N \in \mathbb{N}$ such that, for all $\pi(\dot{i}) \in A$ and for all large n , any line in direction $\vartheta_2(\dot{i}|_{n-N})$ that intersects $E_{\dot{i}|_n}$ also intersects $E_{\dot{i}|_{n-N+j}}$ for some $j \in I^*$ that satisfies $\dot{i}(n - N + 1) \neq j(1)$.

PROPOSITION 4.1. *Under the assumptions of Theorem 3.1, if for all $\varrho > 0$ there is a subset $E_\varrho \subset E$ with $v_p(E_\varrho) \geq 1 - \varrho$ satisfying the line condition for some $N_\varrho \in \mathbb{N}$, then for v_p -almost all $x \in E$ the tangent sets are of the form $\mathcal{O}((\mathbb{R} \times C) \cap B(0, 1))$, where C is a closed porous set and \mathcal{O} is the rotation taking $(0, 1)$ to $\bar{\vartheta}_1(\pi^{-1}(x))$.*

Proof. Fix $\varrho > 0$. By the line condition, there exist a set $E_\varrho \subset E$ with $v_p(E_\varrho) \geq 1 - \varrho$ and an integer $N_\varrho \in \mathbb{N}$, where

$$\alpha_2(\dot{i}|_t) \geq \min\{\alpha_2(\dot{i}) : \dot{i} \in I^{N_\varrho}\}t$$

for all small $t > 0$. This means that option (3.3) in the proof of Lemma 3.8 is impossible in this set. Hence, a closed porous set is the only possible outcome. Letting $\varrho \downarrow 0$ proves the claim. □

Remark 4.2. Without the line condition, it is certainly possible that the tangent sets are of the form $\mathcal{O}((\ell \times \{0\}) \cap B(0, 1))$ for a suitable rotation \mathcal{O} . For example, consider a self-affine carpet for which the first-level construction rectangles are horizontally aligned and disjoint, and their projection onto the x -axis is a line segment but, except for the vertical edges, no construction rectangle is above another. It is evident that at a generic point the tangent sets are of the form $(\ell \times \{0\}) \cap B(0, 1)$, where ℓ is an interval containing at least one of the intervals $[-1, 0]$ and $[0, 1]$.

With Proposition 4.1, we are now in a position to prove Corollary 1.1.

Proof of Corollary 1.1. To apply Proposition 4.1, we have to verify the assumptions of Theorem 3.1 and check that the line condition holds. Assumption (1) is trivially satisfied and, by recalling Remark 3.2, condition (3) of Corollary 1.1 implies assumption (3).

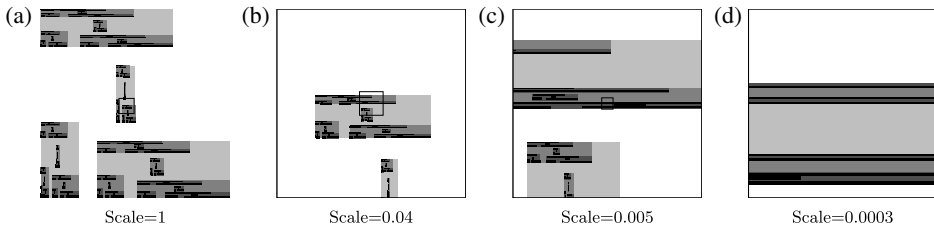


FIGURE 4. The tangent structure arises as one magnifies deeper and deeper in Example 4.3. Here the point we zoom upon is $(0.453\ 846, 0.486\ 659)$. For illustrative purposes we have used a square screen instead of a circular one. (a) Scale = 1. (b) Scale = 0.04. (c) Scale = 0.005. (d) Scale = 0.0003.

Moreover, Remark 3.3 guarantees that it suffices to check the projection condition only onto the horizontal direction. But this is guaranteed by condition (2) of Corollary 1.1 since every line in the vertical direction meets at least two of the first-level construction rectangles. This also means that the whole set E satisfies the line condition (with $N = 1$). Proposition 4.1 thus applies and shows that the tangent sets at almost all points are of the form $(\mathbb{R} \times C) \cap B(0, 1)$, where C is a closed porous set. It remains to prove that C does not contain any isolated points. Here one can argue in exactly the same way as in the proof of [2, Theorem 1].

Example 4.3. We exhibit an example for which Proposition 4.1 applies. Let

$$\begin{aligned}
 f_1(x) &= \begin{pmatrix} 0.2 & 0 \\ 0 & 0.4 \end{pmatrix} x, & f_2(x) &= \begin{pmatrix} 0.7 & 0 \\ 0 & 0.3 \end{pmatrix} x + \begin{pmatrix} 0.3 \\ 0 \end{pmatrix}, \\
 f_3(x) &= \begin{pmatrix} 0.7 & 0 \\ 0 & 0.2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0.8 \end{pmatrix}, & f_4(x) &= \begin{pmatrix} 0.1 & 0 \\ 0 & 0.3 \end{pmatrix} x + \begin{pmatrix} 0.4 \\ 0.4 \end{pmatrix},
 \end{aligned}$$

and $p = (1/6, 1/3, 1/3, 1/6)$. See Figure 4(a) for an illustration.

Let us verify the assumptions of Theorem 3.1 and check that the line condition is satisfied. Looking at Figure 4(a), it is apparent that assumption (1) is satisfied. Verifying assumption (3) just requires checking that (3.1) holds. For assumption (2) to hold, since the right-hand side of (3.1) is greater than the left-hand side in this example, we only need to check that the projection onto the x -axis is a line-segment. This is again clear from Figure 4(a). It is also clear from the picture that the set $A_\lambda = ([0, 1 - \lambda] \times [0, 1]) \cap E$ satisfies the line condition for all $\lambda > 0$. Since the Bernoulli measure ν_p has no atoms, we see that for each $\varrho > 0$ there is $\lambda(\varrho) > 0$ such that $\nu_p(A_{\lambda(\varrho)}) \geq 1 - \varrho$.

Therefore, according to Proposition 4.1, for ν_p -almost every $x \in E$ the tangent sets are of the form $(\mathbb{R} \times C) \cap B(0, 1)$, where C is a closed porous set. To perceive this result in Figure 4, observe how the tangent structure begins to ‘converge’ as we magnify deeper and deeper into the set E . In Figure 4(a–c), one can see the screen of the next magnification. Each picture shows four subsequent levels of the construction so that the level $n(x, t)$ is white and the color gets darker as n increases.

The next example shows that Theorem 3.1 applies also outside the class of self-affine carpets.

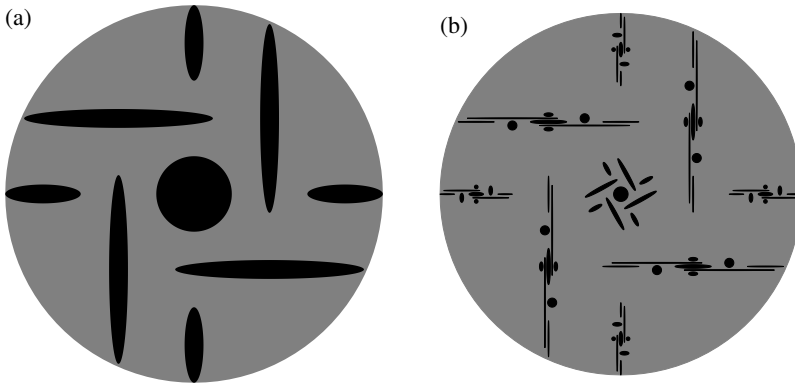


FIGURE 5. The first- and second-level images of $B(0, 1)$ inside $B(0, 1)$ for the iterated function system of Example 4.4.

Example 4.4. Let R_1 be the rotation of angle $\pi/2$ (counter-clockwise) and set

$$f_1(x) = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.05 \end{pmatrix} x + \begin{pmatrix} 0.4 \\ -0.4 \end{pmatrix}, \quad f_5(x) = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.05 \end{pmatrix} x + \begin{pmatrix} 0.8 \\ 0 \end{pmatrix},$$

and $f_{k+i}(x) = R_1^i f_k(x)$ for $k \in \{1, 5\}$ and $i \in \{1, 2, 3\}$. Finally, let $f_9(x) = 0.2R_2(x)$, where R_2 is a rotation of angle θ so that π/θ is irrational.

From Figure 5(a) and (b) one sees that the strong separation condition and the condition given in Remark 3.3, to ensure assumption (2) of Theorem 3.1, are satisfied. When considering the projection condition, it is helpful to notice that the images of $[-1, 1] \times \{0\}$ and $\{0\} \times [-1, 1]$ under f_i are contained in $f_i(X)$, where X is the convex hull of E . Examining how these images are located in $f_i(B(0, 1))$ helps to verify the condition given in Remark 3.3. In this reasoning, it is essential that $f_5, f_6, f_7,$ and f_8 fix points in the unit circle. Observe that it is not necessary for the linear parts of f_1, \dots, f_8 to be diagonal.

To verify assumption (3) of Theorem 3.1, we rely on Remark 3.2 and check that the iterated function system is pinching and twisting. Given a constant $C > 1$, consider the word \mathfrak{i} with $\mathfrak{i}(n) = 1$ for all $n \in \mathbb{N}$. Now $\alpha_1(\mathfrak{i}|_n) = 0.5^n$ and $\alpha_2(\mathfrak{i}|_n) = 0.05^n$, so it is clear that for some large n we have that $\alpha_1(\mathfrak{i}|_n) > C\alpha_2(\mathfrak{i}|_n)$. On the other hand, given a finite set of vectors $\{v, w_1, \dots, w_n\}$, consider the word \mathfrak{i} with $\mathfrak{i}(n) = 9$ for all $n \in \mathbb{N}$. Since π/θ is irrational, the set $\{f_{\mathfrak{i}|_n}(v)/\|f_{\mathfrak{i}|_n}(v)\|\}_{n=1}^\infty$ is dense in S^1 , and so there exists n such that $f_{\mathfrak{i}|_n}(v)$ is not parallel to any $w \in \{w_1, \dots, w_n\}$. Thus the Lyapunov exponents are different for any Bernoulli measure.

We finish the discussion by giving an example which shows that assumption (2) in Theorem 3.1 is necessary.

Example 4.5. Consider the set $F = C_{1/3} \times C_{1/4}$, where C_λ is the Cantor set constructed by using the contraction ratio λ . Then F satisfies assumptions (1) and (3) of Theorem 3.1, with $p = (1/4, 1/4, 1/4, 1/4)$.

Let T be the tangent set of F along a sequence (t_n) at a point $(x^{(1)}, x^{(2)}) \in F$. Denote by T_1 and T_2 the tangent sets of $C_{1/3}$ and $C_{1/4}$, at points $x^{(1)}$ and $x^{(2)}$ along the sequence (t_n) . (If necessary, take the sub-sequence of (t_n) along which these both exist.) Fix $\varepsilon > 0$.

Denote by A_ε the set of points within distance ε from a set A . For notational simplicity, we interpret $\mathcal{Z}_{x,t}$ to be a mapping defined on a square centred at x with side length t . We use the same symbol $\mathcal{Z}_{x,t}$ to denote the corresponding zoom on the real line. Then, for large enough n ,

$$\begin{aligned}\mathcal{Z}_{x,t_n}(Q(x, t_n) \cap F) &= \mathcal{Z}_{x,t_n}(Q(x^{(1)}, t_n) \cap C_{1/3}) \times \mathcal{Z}_{x,t_n}(Q(x^{(2)}, t_n) \cap C_{1/4}) \\ &\subset (T_1)_\varepsilon \times (T_2)_\varepsilon \subset (T_1 \times T_2)_{3\varepsilon},\end{aligned}$$

where the first inclusion uses the fact that F is a product set, and the second the definition of tangent sets. Similarly, for large n ,

$$\mathcal{Z}_{x,t_n}(Q(x, t_n) \cap F)_{3\varepsilon} \supset T_1 \times T_2.$$

This proves that a tangent set at a point of F is a product of tangent sets of $C_{1/3}$ and $C_{1/4}$.

By [3], tangent sets of a Cantor set are $C^{1+\gamma}$ -images of the set itself. This implies that the tangent sets at points of F are of the form $T_1 \times T_2$ where T_1 and T_2 are $C^{1+\gamma}$ -images of the Cantor sets. In particular, the claim of Theorem 3.1 does not hold.

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