PART VII

ADDITIONAL CONTRIBUTIONS

PAPER 50

MAGNETO-HYDROSTATIC EQUILIBRIUM IN AN EXTERNAL MAGNETIC FIELD

P. A. SWEET

University of London Observatory, London, England

ABSTRACT

The expression $\iiint_{\text{all space}} \Delta \mathbf{H}^2 \, dv$ for a change in magnetic energy is shown to be incorrect when applied to a body carrying an electric current and situated in an external magnetic field. A modified expression is derived.

Chandrasekhar's form of the virial theorem in a magnetic field is extended to the case where an external magnetic field is present.

I. INTRODUCTION

Gjellestad [1] has determined the elongation of a gravitating, homogeneous, incompressible, perfectly conducting fluid sphere with no interior magnetic field, but situated in a uniform external magnetic field. Problems of this nature have a bearing on the behaviour of interstellar gas clouds in a galactic magnetic field. In the above work the equilibrium configuration was found by a method, involving the principle of virtual work, along lines developed by Chandrasekhar and Fermi [2]. The expression adopted for the change $\Delta\mathfrak{M}$ in the magnetic energy of the body, consequent on a deformation, was given by

$$\Delta\mathfrak{M} = \frac{1}{8\pi} \iiint_{\text{all space}} \Delta \mathbf{H}^2 \, dv, \tag{1}$$

where **H** is the magnetic intensity. This formula is not correct, and, in view of possible future work along these lines, it seems worth while to reinvestigate it *ab initio*.

The stability of an interstellar gas cloud in a galactic magnetic field has been examined in a recent paper by Mestel and Spitzer [3] using a form of the virial theorem extended by Chandrasekhar and Fermi [2] to include forces of magnetic origin. The expression derived by Chandrasekhar and Fermi applies, however, only to a body carrying its own magnetic field, and it is therefore of interest to extend their results to the case where a magnetic field of external origin is present.

2. THE MAGNETIC ENERGY OF A CURRENT SYSTEM IN AN EXTERNAL FIELD

The rate of working of an electromagnetic field on a system of electric currents and charges is $\mathbf{E}.\mathbf{j}$ per unit volume, where $\mathbf{E}$ is the electric intensity and $\mathbf{j}$ is the electric current density. Gaussian units are used, and unit magnetic permeability and dielectric constants assumed. This expression is derived by considering a charge q moving with velocity $\mathbf{v}$; the Lorentz force on the charge is $q\mathbf{E}+q\mathbf{v}\times\mathbf{H}/c$, where c is the velocity of light. The rate of working on the charge is thus $q\mathbf{E}.\mathbf{v}$, and the above result follows on summing over all charges in a unit volume. The rate of working dW/dt on the whole body is therefore given by

$$dW/dt=\iiint_D \mathbf{E}.\mathbf{j}\,dv, \tag{2}$$

where D is any region containing the body. D, however, must not contain any part of the charge and current system responsible for the external field. This is because, when the body is deformed, the disturbances produced in the field may cause the external system to extract or supply electrostatic and magnetic energy, the amount depending on the mechanism maintaining the system.

It will be convenient to assume that the external field is produced by a fixed charge and current system, although the type of external system does not influence the final result. Now

$$\mathbf{H}=\mathbf{H}^{\text{ext}}+\mathbf{H}_1, \tag{3}$$

where $\mathbf{H}^{\text{ext}}$ is the magnetic field produced by the external current system, and $\mathbf{H}_1$ is the field arising from the current system in the body. In the region D curl $\mathbf{H}^{\text{ext}}=0$ since D contains only the currents in the body. Further, $\mathbf{H}^{\text{ext}}$ is constant in time, hence Maxwell's equations for this region can be written

$$\text{curl}\,\mathbf{H}_1=\frac{4\pi\mathbf{j}}{c}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t},\quad \text{curl}\,\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}_1}{\partial t}. \tag{4}$$

The right-hand side of (2) can now be transformed as in Poynting's theorem, and dW/dt thereby expressed in the form

$$\frac{dW}{dt}=-\frac{1}{8\pi}\iiint_D\frac{\partial}{\partial t}(\mathbf{E}^2+\mathbf{H}_1^2)\,dv-\frac{c}{4\pi}\iint_S(\mathbf{E}\times\mathbf{H}_1).d\mathbf{S}, \tag{5}$$

where S is the surface which bounds D.

https://doi.org/10.1017/S0074180900238060 Published online by Cambridge University Press

The circumstances of a body in a uniform external field are realized when the typical distance L of variation of the external field is large compared with the linear dimensions of the body. The dimensions of D can then be made much larger than those of the body while remaining small compared with L. If the term 'all space' is used in a restricted sense to denote such a region D, then (5) can be written

$$\frac{dW}{dt} = -\frac{1}{8\pi}\iiint_{\text{all space}} \frac{\partial}{\partial t}(\mathbf{E}^2 + \mathbf{H}_1^2)\,dv. \tag{6}$$

The surface integral in (5) has vanished because $\mathbf{H}_1$ tends to zero at least as fast as r^{-3} at large distances r from the body and $\mathbf{E}$ is bounded. The change $\Delta\mathfrak{M}_1$ in the magnetic energy of the body in a virtual deformation is, by definition, minus the magnetic part of the work done on the body by the field. From (6) it therefore follows that

$$\Delta\mathfrak{M}_1 = \frac{1}{8\pi}\iiint_{\text{all space}} \Delta\mathbf{H}_1^2\,dv \tag{7}$$

in contrast to the expression in (1). In using the expression $\Delta\mathfrak{M}_1$ in an energy equation it must be remembered that this only represents the total change in field energy provided that there is no charge density in the body, and provided that the deformation is made sufficiently slowly to avoid electromagnetic radiation.

It is of interest to examine $\Delta\mathfrak{M} - \Delta\mathfrak{M}_1$. Since $\Delta\mathbf{H}^{\text{ext}} = 0$ by definition, then

$$\Delta\mathbf{H}^2 - \Delta\mathbf{H}_1^2 = 2\mathbf{H}^{\text{ext}}.\Delta\mathbf{H}_1. \tag{8}$$

Thus

$$\Delta\mathfrak{M} - \Delta\mathfrak{M}_1 = \frac{1}{4\pi}\mathbf{H}^{\text{ext}}.\Delta\iiint_{\text{all space}} \mathbf{H}_1\,dv. \tag{9}$$

But

$$\iiint_{\text{all space}} \mathbf{H}_1\,dv = \tfrac{8}{3}\pi\mathbf{M}, \tag{10}$$

where $\mathbf{M}$ is the magnetic dipole moment of the body. To demonstrate this consider the identity

$$\iiint_D \mathbf{H}_1\,dv = -\iint_S \mathbf{A}_1 \times d\mathbf{S}, \tag{11}$$

where $\mathbf{A}_1$ is a vector potential for $\mathbf{H}_1$, D is a region containing the body, and S is the surface bounding D. Then

$$\mathbf{A}_1 = \mathbf{M} \times \mathbf{r}/r^3 + o\left(\frac{1}{r^3}\right) \tag{12}$$

for large values of r where $\mathbf{r}$ is the radius vector from an origin in the neighbourhood of the body. Eq. (10) follows on substituting this expression into the identity and allowing the region D to expand to include 'all space'. Hence

$$\Delta\mathfrak{M} - \Delta\mathfrak{M}_1 = \tfrac{2}{3}\mathbf{H}^{\text{ext}}.\Delta\mathbf{M}. \tag{13}$$

The expression $\Delta\mathfrak{M}$ is therefore incorrect whenever the magnetic dipole moment of the body changes.

3. THE VIRIAL THEOREM

The equations of motion of an inviscid fluid in a magnetic field and in its own gravitational field are

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}.\text{grad})\,\mathbf{v} = -\text{grad}\,p + \rho\,\text{grad}\,V + \frac{1}{4\pi}\,\text{curl}\,\mathbf{H}\times\mathbf{H}, \tag{14}$$

where p is the hydrostatic pressure, ρ is the density, $\mathbf{v}$ is the fluid velocity and V is the gravitational potential. Electrostatic and electromagnetic forces are ignored in this application.

The virial theorem is derived by taking the scalar product of both sides of (14) with $\mathbf{r}$ the position vector and integrating over all space. The term 'all space' here is used in the restricted sense described in the previous section. After making the usual transformations of the integrals involving the mechanical terms the virial theorem is given by

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2T + 3(\gamma-1)\,U + \Omega + \frac{1}{4\pi}\iiint_{\text{all space}} \mathbf{r}.(\text{curl}\,\mathbf{H}\times\mathbf{H})\,dv, \tag{15}$$

where $I = \iiint_{\text{all space}} \rho r^2\,dv$, T is the kinetic energy of mass motion, U is the internal energy and Ω is the gravitational energy of the body. The ratio of specific heats is taken as uniform. By substituting $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}^{\text{ext}}$ and noting that $\mathbf{H}^{\text{ext}}$ is constant the magnetic term in (15) can be written

$$\mathscr{I} = \frac{1}{4\pi}\iiint_{\text{all space}} \mathbf{r}.(\text{curl}\,\mathbf{H}_1\times\mathbf{H}_1)\,dv + \frac{1}{4\pi}\iiint_{\text{all space}} \mathbf{r}.(\text{curl}\,\mathbf{H}_1\times\mathbf{H}^{\text{ext}})\,dv. \tag{16}$$

Since $\mathbf{H}_1$ is solenoidal the first integrand in (16) may be transformed as follows:

$$\mathbf{r}.(\text{curl}\,\mathbf{H}_1\times\mathbf{H}_1) = \tfrac{1}{2}\mathbf{H}_1^2 + \text{div}\,(\mathbf{r}.\mathbf{H}_1\mathbf{H}_1 - \tfrac{1}{2}\mathbf{r}\mathbf{H}_1^2). \tag{17}$$

The second integral in (16) can be expressed in the form

$$\mathbf{H}^{\text{ext}}.\left[\frac{1}{4\pi}\iiint_{\text{all space}} \mathbf{r}\times\text{curl}\,\mathbf{H}_1\,dv\right].$$

https://doi.org/10.1017/S0074180900238060 Published online by Cambridge University Press

By a well-known formula (Stratton [4])

$$\frac{1}{4\pi}\iiint_{\text{all space}} \mathbf{r}\times\text{curl}\,\mathbf{H}_1\,dv = 2\mathbf{M}, \tag{18}$$

hence, on substituting the expressions given by (17) and (18) into (16), and using Gauss's theorem,

$$\mathscr{I} = \frac{1}{8\pi}\iiint_{\text{all space}} \mathbf{H}_1^2\,dv + \frac{1}{4\pi}\,\mathscr{L}t_{R=\infty}\iint_{\Sigma_R} (\mathbf{r}\,.\,\mathbf{H}_1\mathbf{H}_1 - \tfrac{1}{2}\mathbf{H}_1^2\mathbf{r})\,.\,d\mathbf{S} + 2\mathbf{M}\,.\,\mathbf{H}^{\text{ext}}, \tag{19}$$

where Σ_R is a sphere of radius R centred at the origin. At large distances from the origin $\mathbf{H}_1 = o(r^{-3})$. The second integral on the right-hand side of (19) therefore vanishes.

The generalized virial theorem is therefore

$$\frac{1}{2}\frac{d^2}{dt^2}T = 2T + 3(\gamma-1)\,U + \Omega + \frac{1}{8\pi}\iiint_{\text{all space}} \mathbf{H}_1^2\,dv + 2\mathbf{M}\,.\,\mathbf{H}^{\text{ext}}. \tag{20}$$

The first member of the magnetic part agrees with the expression derived by Chandrasekhar [2] in the absence of an external magnetic field. The second term gives the contribution when an external magnetic field is present.

REFERENCES

[1] Gjellestad, G. *Astrophys. J.* **120**, 172, 1954.
[2] Chandrasekhar, S. and Fermi, E. *Astrophys. J.* **118**, 116, 1953.
[3] Mestel, L. and Spitzer, L. *Mon. Not. R. Astr. Soc.* **116**, 503, 1956.
[4] Stratton, J. A. *Electromagnetic Theory* (McGraw Hill, 1941), p. 235.