London Mathematical Society ISSN 1461–1570

RANGE OF THE FIRST THREE EIGENVALUES OF THE PLANAR DIRICHLET LAPLACIAN

MICHAEL LEVITIN AND RUSTEM YAGUDIN

Abstract

Extensive numerical experiments have been conducted by the authors, aimed at finding the admissible range of the ratios of the first three eigenvalues of a planar Dirichlet Laplacian. The results improve the previously known theoretical estimates of M. Ashbaugh and R. Benguria. Some properties of a maximizer of the ratio λ_3/λ_1 are also proved in the paper.

1. *Introduction*

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$. We consider the eigenvalue problem for the Dirichlet Laplacian,

$$-\Delta u = \lambda u \quad \text{in } \Omega; \tag{1.1}$$

$$u|_{\partial\Omega} = 0. \tag{1.2}$$

Let us denote the eigenvalues by $\lambda_1(\Omega), \lambda_2(\Omega), \ldots,$ where $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots$ (we will sometimes omit the explicit dependence on Ω when speaking about a generic domain). The corresponding orthonormal basis of real eigenfunctions will be denoted $\{u_j\}_{j=1}^{\infty}$.

For the last fifty years, the problem of obtaining *a priori* estimates of the eigenvalues and their ratios has attracted substantial attention. The existing results can be roughly divided into two groups – *universal* estimates, which are valid, as the name suggests, for all eigenvalues and all the domains in $\mathbb{R}^n$, and which do not take into account any geometric information, and *isoperimetric* estimates for low eigenvalues. We briefly survey some known results below; the reader is referred to the very detailed survey paper [**1**] and the references therein, for a full discussion.

1.1. *Universal estimates*

Probably the first, and best-known, estimate of this type is the Payne–Pólya–Weinberger inequality [**16**]:

$$\lambda_{m+1} \leqslant \lambda_m + \frac{4}{mn} \sum_{j=1}^{m} \lambda_j . \tag{PPW}$$

The research of the first author was partially supported by EPSRC grant GR/M20990 and by the EPSRC Spectral Theory Network. The research of the second author was partially supported by an ORS grant.

Received 11 March 2002, revised 10 October 2002; *published* 15 January 2003.
2000 Mathematics Subject Classification 35P15,65N25

This was subsequently improved by Hile and Protter [12], and, in the 1990s, by Hong Cang Yang [19], whose implicit estimate

$$\sum_{j=1}^{m}\left(\lambda_{m+1}-\lambda_j\right)\left(\lambda_{m+1}-\left(1+\frac{4}{n}\right)\lambda_j\right)\leqslant 0, \qquad \text{(HCY)}$$

remains the best universal estimate so far for the eigenvalues of the Dirichlet Laplacian.

The general method of obtaining (PPW) and (HCY), as well as similar estimates for a variety of other operators, has been the use of variational principles with some ingenious choices of trial functions; see [1]. Recently, an alternative abstract scheme, based on the so-called 'commutator trace identities' (which easily implies, in particular, (PPW) and (HCY)) has been developed in [14]; see also [11].

By their very nature, the universal estimates are generically non-sharp.

1.2. *Isoperimetric estimates*

Both (PPW) and (HCY) give, for $m=1$, the estimate

$$\frac{\lambda_2}{\lambda_1}\leqslant 1+\frac{4}{n}.$$

This upper bound cannot, in fact, be attained. Payne, Pólya, and Weinberger conjectured that the actual optimal upper bound on the ratio of the first two eigenvalues of the Dirichlet Laplacian is

$$\frac{\lambda_2}{\lambda_1}(\Omega)\leqslant\left.\frac{\lambda_2}{\lambda_1}\right|_{n\text{-dimensional ball}}=\frac{j_{n/2,1}^2}{j_{n/2-1,1}^2}=:K_n,\quad\text{for }\Omega\subset\mathbb{R}^n. \qquad (\text{AB}_0)$$

Note that $j_{p,q}$ stands here for the qth zero of the Bessel function $J_p(\rho)$; so, in the planar case $n=2$, $K_2\approx 2.5387$, compared with the (PPW) bound

$$\left.\frac{\lambda_2}{\lambda_1}(\Omega)\right|_{\Omega\subset\mathbb{R}^2}\leqslant 3.$$

Conjecture (AB_0) was eventually proved (only in the early 1990s) by Ashbaugh and Benguria [2, 4], using, in particular, symmetrization techniques going back to the Faber–Krahn inequality

$$\lambda_1(\Omega)\geqslant\lambda_1(\Omega^\star),$$

where $\Omega^\star$ is an n-dimensional ball of the same volume as Ω.

We would like to mention, in this context, extensive computational experiments designed by Haeberly [9, 10] to verify (AB_0).

1.3. *Statement of the problem*

As mentioned above, (AB_0) gives the full description of the range of the possible values of the ratio of the first two eigenvalues of the Dirichlet Laplacian, λ_2/λ_1, for domains in Euclidean space (the obvious lower bound is $\lambda_2/\lambda_1\geqslant 1$). In fact, similar results were also obtained for domains in $\mathbb{S}^n$ and $\mathbb{H}^n$. A natural extension would be to find optimal upper bounds on the range of the ratios of the first *three* eigenvalues of the Dirichlet Laplacian, $(\lambda_2/\lambda_1,\lambda_3/\lambda_1)$, in particular for planar domains. In other words, we would like to find, for $x := \lambda_2/\lambda_1$ and $y := \lambda_3/\lambda_1$, the function

$$y^*(x) := \max_{\Omega\subset\mathbb{R}^2 : (\lambda_2/\lambda_1)(\Omega)=x} \frac{\lambda_3}{\lambda_1}(\Omega) \tag{1.3}$$

and the number

$$Y^* := \max_{x\in[1,K_2]} y^*(x) = \max_{\Omega\subset\mathbb{R}^2} \frac{\lambda_3}{\lambda_1}(\Omega), \tag{1.4}$$

or their best possible estimates. We will use the notation of (1.3) and (1.4) when looking for maxima in particular classes of domains as well.

Despite the apparent simplicity of this problem, and the wide attention that it has attracted, it turns out to be rather difficult. In [**6, 7**], Ashbaugh and Benguria proved a complicated upper bound for $y^*(x)$, and also demonstrated that

$$3.1818 \approx \frac{35}{11} \leqslant Y^* \lessapprox 3.83103. \tag{1.5}$$

Their estimates improve upon previous results of their own, as well as results due to Payne, Pólya, and Weinberger, Brands, de Vries, Hile and Protter, Marcellini, Chiti and Hong Cang Yang; see [**6, 7**] and their earlier papers [**3, 5**] for an extensive bibliography and details of proofs. We present their estimates and other known facts in the next section; just note at the moment that the lower bound in (1.5) is attained when Ω is the rectangle $R_a := [0,1]\times[0,a]$ with $a = \sqrt{8/3}$.

In the current paper, we describe extensive numerical experiments aimed at improving (1.5). We also show, using perturbation techniques, that the rectangle $R_{\sqrt{8/3}}$ does not maximize the ratio λ_3/λ_1, and we indicate a class of domains among which a possible maximizer could be found.

2. *Known results for the range of* $(\lambda_2/\lambda_1, \lambda_3/\lambda_1)$ *for planar domains*

2.1. *Explicit solutions*

The spectral problem (1.1), (1.2) admits a full solution by separation of variables when Ω is, for example, a disjoint union of a number of rectangles or circles. For reference, we collect below the results on the range of $(\lambda_2/\lambda_1, \lambda_3/\lambda_1)$ in these cases.

Rectangles. Let $R_a := [0,1]\times[0,a]$ be a rectangle with the side ratio a; without loss of generality, $a \geqslant 1$. Then

$$\frac{\lambda_2}{\lambda_1}(R_a) = \frac{a^2+4}{a^2+1}$$

and

$$\frac{\lambda_3}{\lambda_1}(R_a) = \begin{cases} \dfrac{a^2+9}{a^2+1}, & \text{for } a \geqslant \sqrt{\dfrac{8}{3}}, \\ \dfrac{4a^2+1}{a^2+1}, & \text{for } 1 \leqslant a \leqslant \sqrt{\dfrac{8}{3}}. \end{cases}$$

https://doi.org/10.1112/S1461157000000346 Published online by Cambridge University Press

Thus, for rectangles, in the notation of (1.3) and (1.4),

$$y(x) = y^*(x)\big|_{\text{rectangles}} = \begin{cases} \frac{8}{3}x - \frac{5}{3}, & \text{for } 1 \leqslant x \leqslant \frac{20}{11}, \\ 5 - x, & \text{for } \frac{20}{11} \leqslant x \leqslant \frac{5}{2}, \end{cases} \tag{2.1}$$

and the maximum value of λ_3/λ_1 is

$$Y^*\big|_{\text{rectangles}} = \frac{35}{11},$$

attained when $a = \sqrt{8/3}$. Note that for this particular rectangle, λ_3 is a degenerate eigenvalue: $\lambda_3(R_{\sqrt{8/3}}) = \lambda_4(R_{\sqrt{8/3}})$, and it is the only a for which $\lambda_3(R_a)$ is not simple.

In the (x, y)-plane, (2.1) corresponds to the two straight lines intersecting at the point $(20/11, 35/11)$.

Circles. For a single circle, $x = y = K_2$. As is easily checked, for a union of two or more disjoint circles of arbitrary radii,

$$y^*(x)\big|_{\text{circles}} \equiv K_2, \quad \text{for } 1 \leqslant x \leqslant K_2. \tag{2.2}$$

Its graph in the (x, y)-plane is a straight line parallel to the x-axis.

Disjoint unions. The following easily checked fact shows that one cannot obtain higher values of $y^*(x)$ by considering disjoint unions of sets from two different classes. In other words, let $\mathcal{C}_j$ be two arbitrary classes of domains, with corresponding functions $y^*(x)|_{\mathcal{C}_j}$ (not necessarily defined for all $x \in [1, K_2]$). Then, for any domain $\Omega = \Omega_1 \sqcup \Omega_2$ with $\Omega_j \in \mathcal{C}_j$, we have, for $x = (\lambda_2/\lambda_1)(\Omega)$ and $y = (\lambda_3/\lambda_1)(\Omega)$, the inequality $y \leqslant \max\left(y^*(x)|_{\mathcal{C}_1}, y^*(x)|_{\mathcal{C}_2}, K_2\right)$.

Other domains. There are other domains, such as sectors of annuli, ellipses, and so on, for which the problem of finding the eigenvalues is reduced by separation of variables to a problem of solving some transcendental equations. The latter problem is often no easier than the numerical solution of the original problem, however, so we do not treat these cases here.

The graphs of $y(x) = y^*(x)\big|_{\text{rectangles}}$ and $y^*(x)\big|_{\text{circles}}$ are shown in Figure 1.

2.2. *Ashbaugh–Benguria estimates*

In [7], Ashbaugh and Benguria proved, using a wide variety of methods, the following upper bounds for $y^*(x)$:

$$y^*(x) < K_2 x, \qquad \text{for } 1 < x \leqslant 1.396^-, \tag{AB$_1$}$$

$$y^*(x) \leqslant 1 + x + \sqrt{2x - (1 + x^2)/2}, \qquad \text{for } 1.396^- \leqslant x \leqslant 1.634^-, \tag{AB$_2$}$$

$$y^*(x) \leqslant F(x), \qquad \text{for } 1.634^- \leqslant x \leqslant 1.676^-, \tag{AB$_3$}$$

$$y^*(x) \leqslant H(x) - x, \qquad \text{for } 1.676^- \leqslant x \leqslant 2.198^+, \tag{AB$_4$}$$

and

$$y^*(x) \leqslant G(x), \qquad \text{for } 2.198^+ \leqslant x \leqslant 2.539^-, \tag{AB$_5$}$$

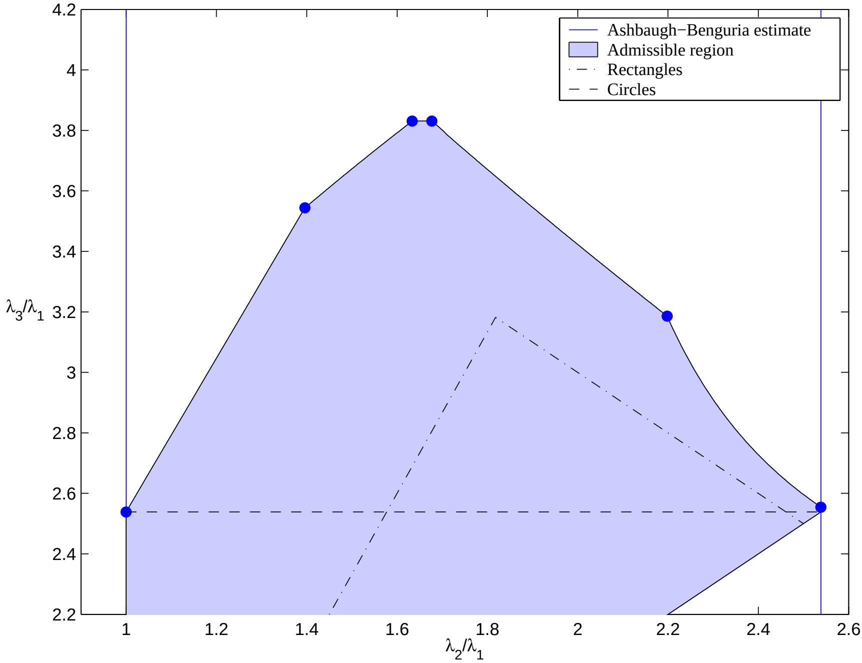

Figure 1: Admissible range (shaded) of $(\lambda_2/\lambda_1, \lambda_3/\lambda_1)$ according to [7]. Shown for comparison are the maximum values of λ_3/λ_1 as functions of λ_2/λ_1 for rectangles and disjoint unions of circles.

where the functions $H(x)$, $F(x)$ and $G(x)$ are defined as follows:

$$H(x) = \begin{cases} 6, & \text{for } x = 1, \\ \min\limits_{1 \leqslant \eta, \xi < x} \left(2\eta + \dfrac{4\beta(\beta+\gamma)^2(x-1)(x-\beta\gamma/(\beta+\gamma-1))^2}{(2\beta-1)(2\gamma-1)(x-\eta)(x-\xi)(4x-2-\eta-\xi)} \right), & \\ & \text{for } x > 1, \end{cases}$$

with $\beta = \eta + \sqrt{\eta^2 - \eta}$ and $\gamma = \xi + \sqrt{\xi^2 - \xi}$. $F(x)$ is the middle root of the cubic

$$2xy^3 - 2(5x^2 + 3x + 1)y^2 + (6x^3 + 39x^2 + 2x - 1)y - (24x^3 + 11x^2 - 4x - 1) = 0$$

and

$$G(x) = \inf_{\beta > 1/2} \left(\frac{\beta^2}{2\beta - 1} + \frac{x - \beta^2/(2\beta-1)}{C_2(\beta)(x - \beta^2/(2\beta-1)) - 1} \right),$$

with the infimum taken over values of β satisfying $x > \beta^2/(2\beta-1) + 1/C_2(\beta)$, and with

$$C_2(\beta) = \frac{2\beta - 1}{\beta} \frac{\int_0^{j_{0,1}} t^3 J_0^{2\beta}(t)\, dt}{\int_0^{j_{0,1}} t J_0^{2\beta}(t)\, dt}.$$

We recall that $J_0(t)$ denotes the standard Bessel function of order zero and $j_{0,1}$ is its first positive zero. For the derivation of the bound (AB_4) and further discussion of it, see [6]. The other bounds given above are due to Hong Cang Yang (see [19] for (AB_2)) and Ashbaugh and Benguria (see [3], [5] for (AB_1), [6] for (AB_5), and [7] for (AB_3)).

https://doi.org/10.1112/S1461157000000346 Published online by Cambridge University Press

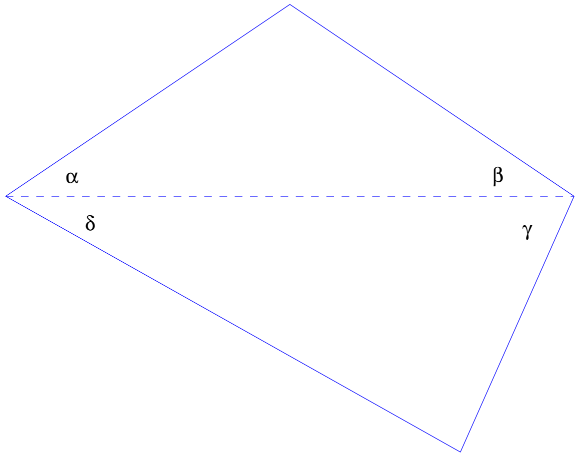

Figure 2: Parametrization of a quadrilateral.

The admissible region for $(\lambda_2/\lambda_1, \lambda_3/\lambda_1)$ defined jointly by (AB_0), the obvious bounds $\lambda_2/\lambda_1 \geqslant 1$ and $\lambda_3/\lambda_1 \geqslant \lambda_2/\lambda_1$, and the inequalities ($AB_1$) – ($AB_5$), is shown in Figure 1.

We make two remarks, following [7].

REMARK 2.1. The inequalities (AB_1) – (AB_5) all apply on broader intervals of x-values than the intervals specified explicitly with them; the given intervals indicate the range for which the corresponding inequality gives the best bound yet found.

REMARK 2.2. The absolute maximum of the right-hand sides of (AB_1) – (AB_5) occurs at the point where $F(x)$ has a maximum within the interval where it is the best bound. That happens at the point $(x, y) \approx (1.65728, 3.83103)$, and implies the best upper bound (1.5) yet proven for λ_3/λ_1.

3. *Numerical analysis of random domains*

To the best of our knowledge, there have been no large-scale numerical experiments on low eigenvalues of the Dirichlet Laplacian for the planar domain. In an attempt to improve the existing estimates on the range of $(\lambda_2/\lambda_1, \lambda_3/\lambda_1)$, we conducted such experiments for a variety of domain classes.

3.1. *General method*

For each particular domain, the calculation of the first three eigenvalues has been conducted using a standard finite element method implementation via PDEToolbox [15] and FEMLAB [8] in Matlab, with two or three mesh refinements. For simple domains with relatively 'high' values of the ratio λ_3/λ_1, optimization with respect to the parameters describing the domains of the particular class was performed in order to maximize this ratio. The results of the calculations for some classes of domains are described below, and are summarized at the end of this section.

For each class of domains, we represent the results in the following graphical form. The range $[1, K_2]$ of possible values of $x = \lambda_2/\lambda_1$ is split into subintervals of length δx (normally approximately equal to 0.05). In each subinterval we choose, if it exists, a domain

https://doi.org/10.1112/S1461157000000346 Published online by Cambridge University Press

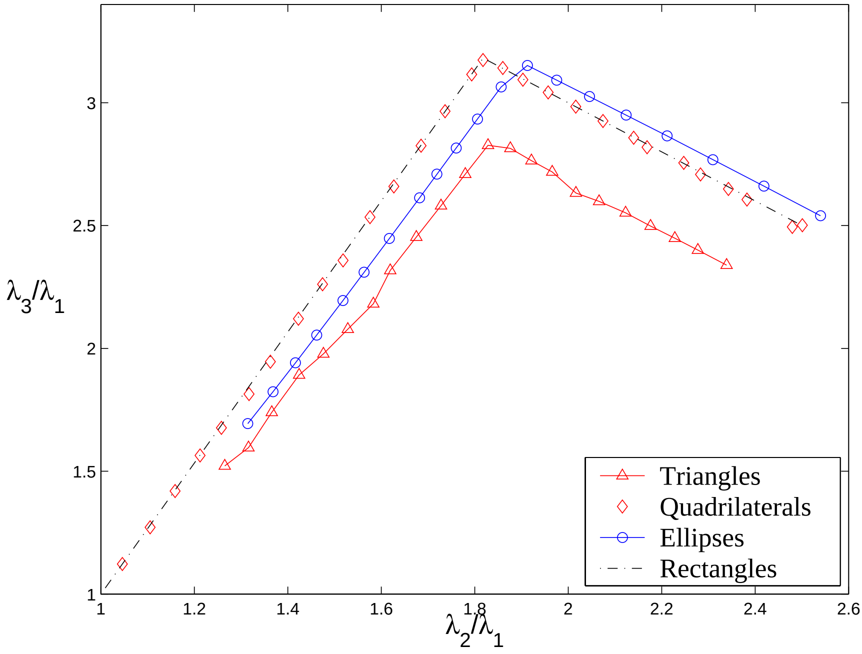

Figure 3: $y^*(x)$ for triangles, quadrilaterals and ellipses.

with maximal $y = \lambda_3/\lambda_1$, and plot the corresponding point (x, y). For comparison, the graphs of $y(x) = y^*(x)|_{\text{rectangles}}$ and/or $y^*(x)|_{\text{circles}}$ are shown.

3.2. *Triangles, quadrilaterals and ellipses*

We start with cyclic calculations for all triangles, with angle step 2.5°. For the triangles with relatively high ratios of λ_3/λ_1 we repeat the procedure in the local neighbourhood with angle step 0.5°. The results are shown in Figure 3.

The computational procedure for quadrilaterals is essentially the same as that for triangles, with parameters α, β, γ and δ in the region $(0, \pi)$; see Figure 2.

We choose an angle step of 2.5°. Note that quadrilaterals with negative α and β, or γ and δ, do not have to be considered separately – they fit into the scheme above if we choose another diagonal as a starting point, and re-scale. In cases of relatively high ratio λ_3/λ_1 ($\geqslant 3$), we repeated the calculation with an angle step 0.5° in the local neighbourhood of that quadrilateral. The results are shown in Figure 3.

The results of cyclic calculations for the ellipses, with axis ratio varying between 1 and 5 with step 0.1 (0.001 in the vicinity of the ellipse with highest λ_3/λ_1), are also shown in Figure 3.

Remark 3.1. Rather surprisingly, Figure 3 suggests that

$$y^*(x)\Big|_{\text{quadrilaterals}} \approx y^*(x)\Big|_{\text{rectangles}} . \tag{3.1}$$

We give a partial explanation of this fact in the next section; see Remark 4.6.

https://doi.org/10.1112/S1461157000000346 Published online by Cambridge University Press

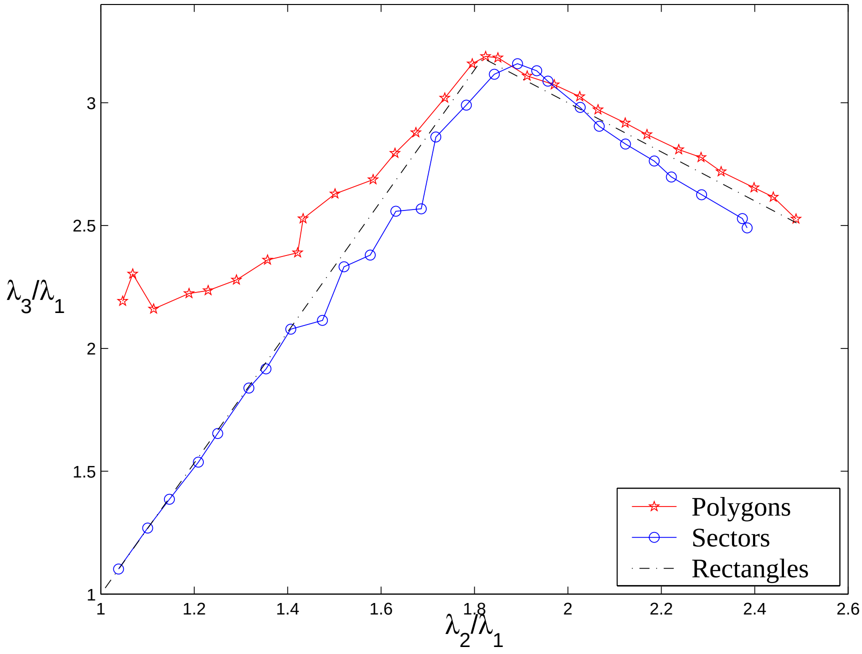

Figure 4: $y^*(x)$ for random sectors of annuli and pseudo-random polygons.

3.3. *Annuli and random sectors of annuli*

The calculations for annuli with inner radius 1 and outer radius r demonstrate that the corresponding value $(\lambda_3/\lambda_1)(r)$ is monotonically increasing from 1 to K_2 as r changes from 1 to ∞ (although convergence, for large r, is very slow – just logarithmic). These results are not very informative, and we do not include them in the graphs or the summary table below.

In calculations for sectors of the annuli of angle θ, we choose r randomly in the interval $(1, 20)$ and θ randomly in the interval $(0.01\pi, 1.99\pi)$. The results of the calculations are shown in Figure 4

3.4. *Pseudo-random polygons*

For polygons with more than four vertices, cyclic calculations through all the possible values of the geometric parameters with some reasonable step become impractical, due to the time constraints. Instead, we choose to perform calculations for randomly generated polygons. We employ the following simple procedure for generating a pseudo-random polygon with N vertices, $\mathbf{v}_1, \dots, \mathbf{v}_N$, lying inside a square $[0, 1]^2$.

Vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. These are chosen randomly, using any pseudo-random generator.

Vertices $\mathbf{v}_j$, $j = 4, \dots, N-1$. We choose a possible vertex at random. If the interval $[\mathbf{v}_{j_1}, \mathbf{v}_j]$ intersects any of the previously constructed sides $[\mathbf{v}_{k-1}, \mathbf{v}_k]$, $k = 1, \dots, j-2$, then we make another random choice.

Vertex $\mathbf{v}_N$ This is constructed in the same manner, but we additionally check that the interval $[\mathbf{v}_N, \mathbf{v}_1]$ does not intersect any of the existing sides.

https://doi.org/10.1112/S1461157000000346 Published online by Cambridge University Press

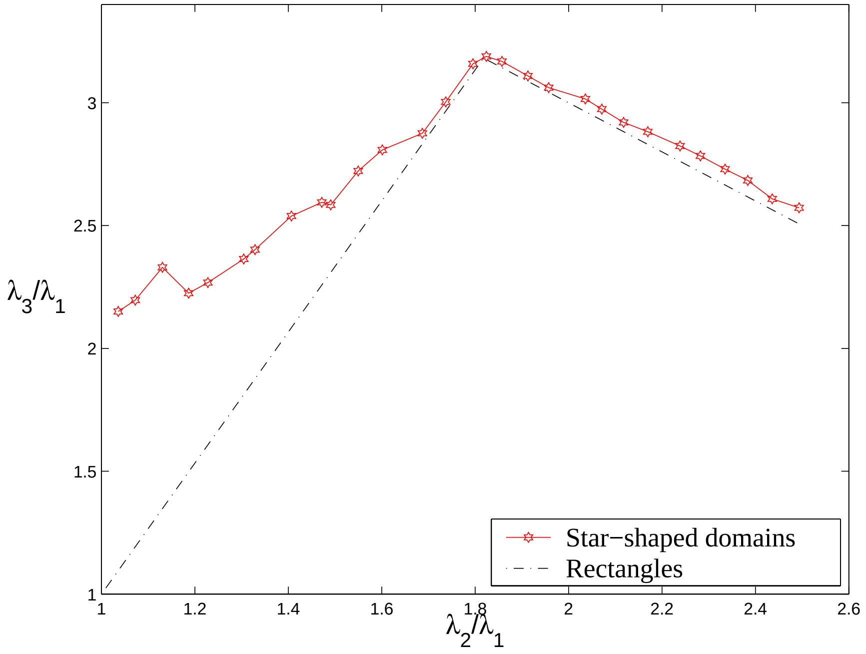

Figure 5: $y^*(x)$ for pseudo-random star-shaped domains.

To avoid infinite loops, we abort the construction if the number of attempts at some stage exceeds some sufficiently big number (say, 200). We also put in place a restriction forbidding very small angles (which require special efforts in mesh generation).

The collated results of the calculations for pseudo-random pentagons, hexagons and decagons are shown in Figure 4. These results also include experiments on random perturbations of the rectangles constructed in the following way: N points were randomly chosen on the sides of the rectangle R_a, with $a \in (1, 5)$, and $1 \leqslant N \leqslant 8$, and these points and the four vertices of the original rectangle were randomly moved by a distance not exceeding $0.1a$ to form an $(N + 4)$-gon.

3.5. *Star-shaped domains* (*simply and non-simply connected*)

The procedure described above for random polygons does not work very effectively for polygons with large numbers of vertices – it often takes a long time to generate a suitable vertex $\mathbf{v}_j$ with $j \gtrsim 10$. Thus, in these cases we restricted ourselves to star-shaped polygonal domains, which are much easier to construct. That is, for the vertices $\mathbf{v} = re^{i\theta}$, we chose the angles θ randomly between 0 and 2π, and the radii r randomly between the given numbers r_1 and r_2. We conducted a series of experiments with a fixed number of vertices (thirteen, seventeen and twenty-three), as well as a series of runs where the number of vertices was chosen randomly between four and thirty.

Additionally, we conducted a series of experiments of non-simply connected domains of the types $R_{\sqrt{8/3}} \setminus S_1$, $S_2 \setminus R_{\sqrt{8/3}}$, and $S_1 \setminus S_2$, where S_j are random star-shaped polygons such that $S_1 \subset R_{\sqrt{8/3}} \subset S_2$, and $R_{\sqrt{8/3}}$ is the rectangle with the maximum λ_3/λ_1.

The results for pseudo-random star-shaped domains are collated in Figure 5.

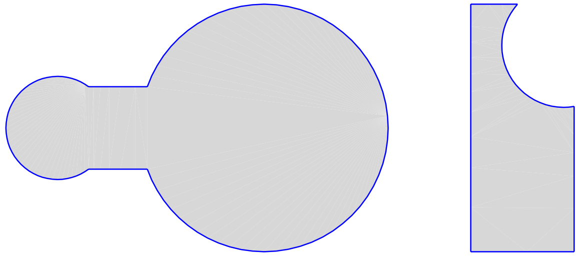

Figure 6: Typical dumbbell and jigsaw-piece domains.

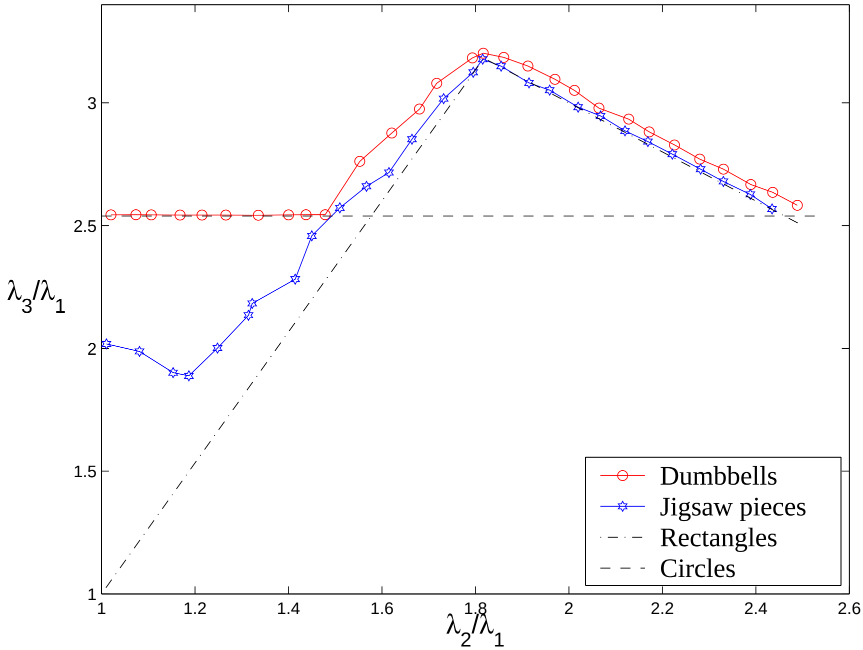

Figure 7: $y^*(x)$ for pseudo-random dumbbells and jigsaw pieces.

3.6. *Dumbbells and jigsaw pieces*

By a *dumbbell*, we understand a domain of the type

$$([0, l] \times [-h, h]) \cup C((0, 0), r_1) \cup C((l, 0), r_2), \tag{3.2}$$

where l, h, r_1 and r_2 are positive parameters, and $C(\mathbf{v}, r)$ denotes a circle with radius r centred at $\mathbf{v}$. By a *jigsaw piece*, we understand a domain of the type $R \setminus C$, where R is a rectangle, and C is a circle with a centre 'near' the boundary of the rectangle. Typical dumbbell and jigsaw-piece domains are shown in Figure 6.

The results of numerical experiments on dumbbells and jigsaw pieces, with cyclical/random choice of the parameters, are shown in Figure 7. For dumbbells, we also optimized over the parameters for domains with $\lambda_3/\lambda_1 \approx 3.2$, additionally allowing the centres of the circles to move in the vertical direction along the sides of rectangles. However, this did not lead to any improvement in the results.

https://doi.org/10.1112/S1461157000000346 Published online by Cambridge University Press

Table 1: Summary statistics for numerical experiments. The value in the fourth column is $\delta_4 := (\lambda_4 - \lambda_3)/(\lambda_3(\Omega^*))$, where Ω^* is the domain that maximizes the ratio $\lambda_3/\lambda_1 = Y^*$ in the corresponding class of domains.

Type of domain	No. of experiments	Y^*	δ_4
Triangles	2145	2.827	0.016
Quadrilaterals	13222	3.183	$1.6 \cdot 10^{-5}$
Sectors	360	3.149	$4.6 \cdot 10^{-5}$
Ellipses	142	3.167	$9.1 \cdot 10^{-5}$
Random polygons	26867	3.189	$6.1 \cdot 10^{-4}$
Random star-shaped domains	18320	3.159	0.022
Dumbbells	2871	3.202	$1.1 \cdot 10^{-4}$
Jigsaw pieces	1420	3.178	0.003
Total	65337	3.202	$1.1 \cdot 10^{-4}$

3.7. *Summary of the numerical experiments*

We summarize the results of our numerical experiments in Table 1.

As seen in the last column of Table 1, in each class of domains the maximum of the ratio λ_3/λ_1 is attained, within the accuracy of computations, on a domain with *degenerate* eigenvalue $\lambda_3 \approx \lambda_4$. The same, of course, holds for rectangles; see (2.1). This allows us to conjecture that the absolute maximum and any local maxima of $\lambda_3 \approx \lambda_4$ are also attained on domains with degenerate λ_3. We give a partial proof of this conjecture in the next section.

The computed absolute maximum ratio $Y^* \approx 3.202$ is attained on the dumbbell-shaped domain (3.2) with $l = 1$, $h = 1.4510$, $r_1 = 0.7814$, and $r_2 = 0.7818$; see Figure 9. Note that the maximum value Y^* is only slightly higher than the corresponding value $Y^*|_{\text{rectangles}} \approx 3.1818$.

Remark 3.2. Additional experiments were conducted in order to check whether a maximizer is likely to be a simply connected domain. That is, for the dumbbell-shaped domain Ω described above, we computed the eigenvalues for a number of domains obtained by removing a small hole from Ω. In all the cases, the ratio of the third and the first eigenvalue for a perturbed problem was quite significantly less than that for Ω.

The graph of the function $y^*(x)$, built on the basis of all the numerical experiments, is shown in Figure 8.

4. *Asymptotic results*

In this section, using standard perturbation techniques, we establish several results which, although they do not give the full answer to the question of maximizing the ratio $(\lambda_3/\lambda_1)(\Omega)$ among all planar domains Ω, give some indication of which domains may or may not be a maximizer. We first prove the following theorem; we note here that Niculae Mandrache has informed us that he has independently obtained a similar result.

Theorem 4.1. *The rectangle $R_{\sqrt{8/3}}$ does not maximize the λ_3/λ_1 among all planar domains.*

This should be compared, however, with Remark 4.6 below.

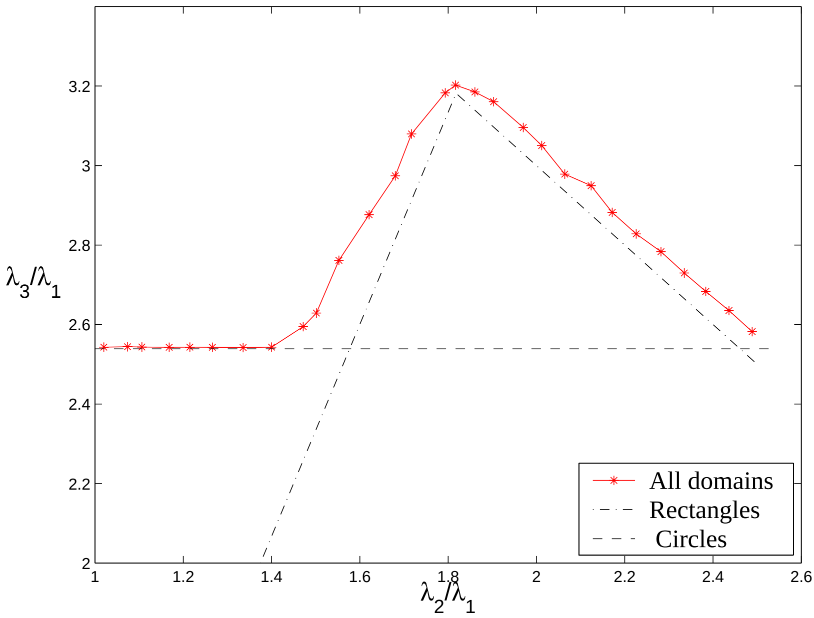

Figure 8: $y^*(x)$ for all computed domains.

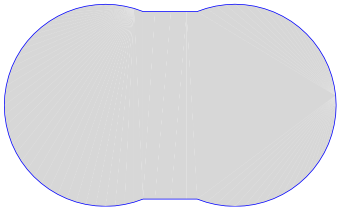

Figure 9: Domain maximizing λ_3/λ_1 on the basis of computations.

We also give a proof of the following, more general, result, which justifies the remark made at the end of the previous section.

THEOREM 4.2. *Suppose that* $\Omega \subseteq \mathbb{R}^2$ *is a local maximizer of* $(\lambda_3/\lambda_1)(\Omega)$ *among planar domains with sufficiently smooth boundaries. Then* $\lambda_3(\Omega) = \lambda_4(\Omega)$.

We should emphasize here that neither the statement nor the proof (found below) of Theorem 4.2 is fully rigorous. In the former, we do not discuss the requirements on the smoothness of the boundary, or the concept of a local maximizer; nor do we prove that maximizers actually exist. In the latter, we rely on the following unproven, although very plausible, conjecture.

CONJECTURE 4.3. *Let* λ_3 *be a simple eigenvalue of the Dirichlet Laplacian on a planar connected domain. Then not all nodal lines of the corresponding eigenfunction are closed.*

https://doi.org/10.1112/S1461157000000346 Published online by Cambridge University Press

Such a conjecture is not unreasonable since it is, in general, quite difficult to construct domains for which even one nodal line of a low eigenfunction is closed; see [13].

Before giving the proofs of Theorems 4.1 and 4.2, we recall, without proof, some classical results from domain perturbation theory. The details can be found in, for example, [17, 18].

4.1. *Domain perturbations*

For simplicity, we restrict ourselves to domains in $\mathbb{R}^2$; all the results stated here hold in any dimension.

Consider, for small absolute values of the real parameter ε, a family of bounded domains Ω^ε in $\mathbb{R}^2$ of a variable $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2)$, which are transformed by the change of coordinates

$$\mathbf{x} = \tilde{\mathbf{x}} + \varepsilon \mathbf{S}(\mathbf{x}) \tag{4.1}$$

into the domain $\Omega = \Omega^0$ in $\mathbb{R}^2$ of variable $\mathbf{x}$. We assume that the boundary $\partial\Omega$ and the vector-function $\mathbf{S}$ are sufficiently smooth.

Let $\mathbf{n}$ be the outer unit normal to $\partial\Omega$, and denote

$$f = \mathbf{S} \cdot \mathbf{n}$$

(in fact, εf is a smooth function on $\partial\Omega$ which gives, up to the leading order for small ε, the normal distance between $\partial\Omega$ and $\partial\Omega^\varepsilon$).

Denote by $\lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \ldots$ the eigenvalues of the Dirichlet Laplacian on Ω, and by $\{u_j\}$ the corresponding basis of normalized orthogonal eigenfunctions (which are chosen real). Also, denote by λ_j^ε the eigenvalues of the Dirichlet Laplacian on Ω^ε. For sufficiently small $|\varepsilon|$, the λ_j^ε are continuous functions of ε, and tend to λ_j as $\varepsilon \to 0$.

The following two results go back to Rellich [17].

Proposition 4.4. *Let λ_j, $j \geqslant 1$, be a* simple *eigenvalue of the Dirichlet Laplacian on Ω. Then λ_j^ε has the asymptotic expansion*

$$\lambda_j^\varepsilon = \lambda_j + \varepsilon\tilde{\lambda}_{j,1} + \varepsilon^2\tilde{\lambda}_{j,2} + \ldots \tag{4.2}$$

as $\varepsilon \to 0$, where

$$\tilde{\lambda}_{j,1} = -\int_{\partial\Omega} f \left|\frac{\partial u_j}{\partial n}\right|^2 d\sigma. \tag{4.3}$$

The situation is slightly more complicated when $\lambda_j = \cdots = \lambda_{j+m}$ is an eigenvalue of multiplicity $m+1$. For simplicity, we consider just the case $m = 1$.

Proposition 4.5. *Let $\lambda_k = \lambda_{k+1}$ be a* double *eigenvalue of the Dirichlet Laplacian on Ω. Then, as $\varepsilon \to 0$, λ_k^ε and $\lambda_{k+1}^\varepsilon$ still have the asymptotic expansions* (4.2) *(where $j = k, k+1$) with*

$$\tilde{\lambda}_{k,1} = \frac{1}{\varepsilon}\min(\varepsilon\mu_1, \varepsilon\mu_2), \qquad \tilde{\lambda}_{k+1,1} = \frac{1}{\varepsilon}\max(\varepsilon\mu_1, \varepsilon\mu_2), \tag{4.4}$$

where μ_1, μ_2 are two real roots of the quadratic equation

$$(F_{k,k} + \mu)(F_{k+1,k+1} + \mu) - F_{k,k+1}^2 = 0 \tag{4.5}$$

and

$$F_{p,q} = \int_{\partial\Omega} f \frac{\partial u_p}{\partial n}\frac{\partial u_q}{\partial n} d\sigma. \tag{4.6}$$

We will, in fact, be interested in the asymptotic expansion of $\lambda_j^\varepsilon/\lambda_1^\varepsilon$, which follows from (4.2):

$$\frac{\lambda_j^\varepsilon}{\lambda_1^\varepsilon} = \frac{\lambda_j}{\lambda_1} + \frac{\varepsilon}{(\lambda_1)^2}(\tilde{\lambda}_{j,1}\lambda_1 - \tilde{\lambda}_{1,1}\lambda_j) + O(\varepsilon^2). \tag{4.7}$$

4.2. *Proof of Theorem 4.1*

Let $\Omega = R_{\sqrt{8/3}}$ be the rectangle $\{(x_1, x_2) : 0 < x_1 < 1,\ 0 < x_2 < \sqrt{8/3}\}$. We shall construct an explicit perturbation Ω^ε using (4.1) such that the first correction term in the asymptotic formula (4.7) is positive for $\varepsilon > 0$, and therefore $\lambda_3^\varepsilon/\lambda_1^\varepsilon > \lambda_3/\lambda_1$ for sufficiently small positive ε.

Let

$$\Omega^\varepsilon = \left\{(x_1, x_2) : 0 < x_1 < 1,\ 0 < x_2 < \sqrt{\frac{8}{3}} + \varepsilon g(x_1)\right\},$$

where

$$g(x_1) = c_0 + \sum_{l=0}^{\infty} \sqrt{2} c_l \cos(\pi l x_1).$$

We will choose the coefficients c_l later.

The corresponding function f appearing in the asymptotic formulae above is

$$f(x_1, x_2) = \begin{cases} g(x_1), & \text{if } x_2 = \sqrt{\frac{8}{3}},\ 0 \leqslant x_1 \leqslant 1, \\ 0, & \text{if } (x_1, x_2) \in \partial\Omega,\ x_2 \neq \sqrt{\frac{8}{3}}. \end{cases}$$

Note that we shall use (4.3) for computing $\tilde{\lambda}_{1,1}$, and (4.4) for computing $\tilde{\lambda}_{3,1}$ and $\tilde{\lambda}_{4,1}$, since $\lambda_3 = \lambda_4$ is a double eigenvalue of the unperturbed problem. Elementary but tedious calculations show that the correction terms $\tilde{\lambda}_{k,1}$, $k = 1, 2, 3, 4$, depend only upon the parameters c_j with $j = 0, \ldots, 4$. For brevity, we omit the explicit expressions.

Let us choose the parameters $c_0, \ldots, c_4$ in such a way that $\tilde{\lambda}_{3,1} = \tilde{\lambda}_{4,1}$ (that is, λ_3^ε remains a double eigenvalue up to the linear terms in ε). This, by Proposition 4.5, happens when $F_{3,3} = F_{4,4}$ and $F_{3,4} = 0$, which in turn leads to the following conditions on coefficients c_j:

$$c_3 = c_1, \qquad c_4 = 9c_2 - 8\sqrt{2}c_0. \tag{4.8}$$

Under conditions (4.8), the asymptotic formula (4.7) simplifies dramatically, and becomes

$$\frac{\lambda_3^\varepsilon}{\lambda_1^\varepsilon} - \frac{\lambda_3}{\lambda_1} = \frac{96\sqrt{3}}{121}\left(c_2 - \sqrt{2}c_0\right)\varepsilon + O(\varepsilon^2),$$

and we can choose c_0 and c_2 in such a way that its right-hand side is positive for sufficiently small positive ε. This proves Theorem 4.1.

REMARK 4.6. Let $\Omega = R_a$ be any rectangle, and consider the perturbations Ω^ε as above, but with function f *linear* in x_1, x_2 (and, naturally, a replacing $\sqrt{8/3}$ throughout). Thus, we are considering *quadrilaterals* Ω^ε which are small perturbations of the rectangle R_a. The same elementary calculations then imply that, for $x^\varepsilon := (\lambda_2/\lambda_1)(\Omega^\varepsilon)$ and $y^\varepsilon := (\lambda_3/\lambda_1)(\Omega^\varepsilon)$, we obtain, up to and inclusive of the terms of order ε,

$$y^\varepsilon = y^*(x^\varepsilon)\big|_{\text{rectangles}},$$

where $y^*(x)|_{\text{rectangles}}$ is given by the right-hand side of (2.1). In other words, up to the terms of order ε, the rectangles are local maximizers among all quadrilaterals that are sufficently 'close' to them; see Remark 3.1.

4.3. *Proof of Theorem 4.2*

Suppose that Ω is a planar domain with sufficiently smooth boundary, which locally maximizes the ratio λ_3/λ_1 in the following sense: for any sufficiently smooth perturbation Ω^ε determined by (4.1), we have

$$\frac{\lambda_3^\varepsilon}{\lambda_1^\varepsilon} \leqslant \frac{\lambda_3}{\lambda_1}. \tag{4.9}$$

Assume additionally that λ_3 is a simple eigenvalue of the Dirichlet Laplacian in the unperturbed domain Ω. We shall show that this assumption leads to the contradiction to Conjecture 4.3.

Since both λ_1 and λ_3 are simple eigenvalues, the asymptotic formula (4.7) becomes, in accordance with Proposition 4.4,

$$\frac{\lambda_3^\varepsilon}{\lambda_1^\varepsilon} - \frac{\lambda_3}{\lambda_1} = \frac{\varepsilon}{(\lambda_1)^2} \int_{\partial\Omega} f \left(\lambda_3 \left| \frac{\partial u_1}{\partial n} \right|^2 - \lambda_1 \left| \frac{\partial u_3}{\partial n} \right|^2 \right) d\sigma + O(\varepsilon^2).$$

Now, as ε can be chosen both positive and negative, (4.9) can hold only if

$$\int_{\partial\Omega} f \left(\lambda_3 \left| \frac{\partial u_1}{\partial n} \right|^2 - \lambda_1 \left| \frac{\partial u_3}{\partial n} \right|^2 \right) d\sigma = 0,$$

and since f is an arbitrary smooth function, this requires

$$\lambda_3 \left| \frac{\partial u_1}{\partial n} \right|^2 = \lambda_1 \left| \frac{\partial u_3}{\partial n} \right|^2$$

everywhere on $\partial\Omega$. But the normal derivative of the first eigenfunction of the Dirichlet Laplacian is non-zero everywhere on the boundary, so the last formula implies that the third eigenfunction has the same property, and therefore all its nodal lines are closed, in contradiction with Conjecture 4.3.

5. *Final remarks*

On the basis of the numerical computations and the results proven above, we make the following conjecture, most of which has still to be rigorously established (or disproved).

CONJECTURE. *The domain maximizing the ratio λ_3/λ_1 for planar domains is close in shape to the optimal computed dumbbell-shaped domain shown in Figure 9, is simply connected, and has a smooth boundary. The maximal admissible value Y^* is approximately equal to, or is slightly greater than,* 3.202.

Acknowledgements. The original suggestion to conduct numerical experiments on the range of $(\lambda_2/\lambda_1, \lambda_3/\lambda_1)$ for planar domains came from Brian Davies; we would like to thank him, as well as Mark Ashbaugh and Leonid Parnovski, for valuable discussions and advice. It is the authors, nevertheless, who take full responsibility for the realization of this idea, and any criticism for possible shortcomings of this realization should be addressed to them.

References

1. M. S. Ashbaugh, 'Isoperimetric and universal inequalities for eigenvalues', *Spectral theory and geometry* (*Edinburgh*, 1998), London Math. Soc. Lecture Notes 273 (ed. E. B. Davies and Yu. Safarov, Cambridge Univ. Press, Cambridge, 1999) 95–139. 1, 2

2. M. S. Ashbaugh and R. D. Benguria, 'Proof of the Payne–Pólya–Weinberger conjecture', *Bull. Amer. Math. Soc.* 25 (1991) 19–29. 2

3. M. S. Ashbaugh and R. D. Benguria, 'Isoperimetric bound for λ_3/λ_2 for the membrane problem', *Duke Math. J.* 63 (1991) 333–341. 3, 5

4. M. S. Ashbaugh and R. D. Benguria, 'A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacian and extensions', *Ann. of Math.* 135 (1992) 601–628. 2

5. M. S. Ashbaugh and R. D. Benguria, 'Isoperimetric bounds for higher eigenvalue ratios for the n-dimensional fixed membrane problem', *Proc. Royal Soc. Edinburgh* 123A (1993) 977–985. 3, 5

6. M. S. Ashbaugh and R. D. Benguria, 'The range of values of λ_2/λ_1 and λ_3/λ_1 for the fixed membrane problem', *Rev. Math. Phys.* 6 (1994) 999–1009. 3, 3, 5, 5

7. M. S. Ashbaugh and R. D. Benguria, 'Bounds for ratios of first, second and third membrane eigenvalues', *Nonlinear problems in applied mathematics: in honor of Ivor Stakgold on his seventieth birthday* (ed. T. S. Angell, L. P. Cook, R. E. Kleinmann, and W. E. Olmstead, Society for Industrial and Applied Mathematics, Philadelphia, 1996) 30–42. 3, 3, 4, 5, 5, 6

8. Comsol AB, 'FEMLAB 2.0 User's Guide and Introduction', Comsol AB, 2000; see also `http://www.femlab.com`. 6

9. J.-P. Haeberly, 'On shape optimizing the ratio of the first two eigenvalues of the Laplacian', Computer Science Technical Report 586, Courant Institute, New York University, 1991. 2

10. J.-P. Haeberly and M. L. Overton, 'A hybrid algorithm for optimizing eigenvalues of symmetric definite pencils', *SIAM J. Matrix Anal. Appl.* 15 (1994) 1141–1156. 2

11. E. M. Harrell, II and J. Stubbe, 'On trace identities and universal eigenvalue estimates for some partial differential operators', *Trans. Amer. Math. Soc.* 349 (1997) 2037–2055. 2

12. G. N. Hile and M. H. Protter, 'Inequalities for eigenvalues of the Laplacian', *Indiana Univ. Math. J.* 29 (1980) 523–538. 2

13. M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili, 'The nodal line of the second eigenfunction of the Laplacian in $\mathbb{R}^2$ can be closed', *Duke Math. J.* 90 (1997) 631–640. 13

14. M. Levitin and L. Parnovski, 'Commutators, spectral trace identities, and universal estimates for eigenvalues', *J. Funct. Anal.* 192 (2002) 425–445. 2

15. MathWorks, Inc., 'Partial Differential Equation Toolbox user's guide', The MathWorks, Inc., 1995; see also `http://www.mathworks.com`. 6

16. L. E. Payne, G. Pólya and H. F. Weinberger, 'On the ratio of consecutive eigenvalues', *J. Math. Phys.* 35 (1956) 289–298. 1

17. F. Rellich, *Perturbation theory of eigenvalue problems* (Gordon and Breach, New York, 1969). 13, 13

18. J. Sanchez-Hubert and E. Sanchez-Palencia, *Vibration and coupling of continuous systems – asymptotics methods* (Springer, Berlin, 1989). 13

19. Hong Cang Yang, 'An estimate of the difference between consecutive eigenvalues', preprint IC/91/60 of the Intl. Centre for Theoretical Physics, Trieste, 1991; revised preprint, Academia Sinica, 1995. 2, 5

Michael Levitin M.Levitin@ma.hw.ac.uk
http://www.ma.hw.ac.uk/∼levitin
Rustem Yagudin

Department of Mathematics
Heriot-Watt University
Riccarton, Edinburgh EH14 4AS
U. K.