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ON THE STABILITY OF DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract

The local stability properties of non-linear differential-difference equations are investigated by considering the location of the roots of the eigen-equation derived from the linearised approximation of the original model. A general linear system incorporating one time delay is considered and local stability results are obtained for cases in which the coefficient matrices satisfy certain assumptions. The results have applications to recent Biological and Economic models incorporating time lags.

1. Introduction

In recent years considerable attention has been directed at differential-difference equations, and in obtaining the properties of their solution. Differential-difference equations arise quite naturally in modelling complex biological, economic and engineering systems when time lags are included. In modelling biological systems, delayed arguments reflect the inherent natality and regeneration lags; see, for example, May [6, chapter 4], Maynard-Smith [7, p. 43–5]. The behaviour of economic systems depends critically on the time lags between stimulus and response; see, for example, Kendall [4]. The equations arising from such deterministic models, with continuous time and discrete lags, take the form of systems of differential-difference equations, which are usually nonlinear.

The stability properties of these differential-difference equations are of particular interest, and it is found that the time lag frequently has a destabilising influence. Global stability in the phase plane can sometimes be obtained using Liapunov functions; see, for example, Elsgolts and Norkin [2, chapter 3]. Local stability in the phase plane is determined by linearization; the properties of the associated linear model being used to describe the range of parameters which imply local stability of the nonlinear system. Details of this approach are outlined in Bellman and Cooke [1, chapter 10].

This linearization approach frequently involves consideration of a quasipolynomial equation

$$pe^z + q - ze^z = 0,$$

where p and q may be either real or complex. When $\mathrm{Re}(z) < 0$, the system which gives rise to this quasipolynomial equation is locally stable. As described in the following section, some special cases of this equation have been studied previously. Here we complete the stability analysis by considering the remaining special cases for this equation. These are: p complex, q real (Section 3); p real, q complex (Section 4); p complex, q complex (Section 4). The various results are tabulated in Section 5 for ready reference.

2. Preliminaries

We consider the following general system of n linear differential-difference equations

$$\frac{dw_i(t)}{dt} = \sum_{j=1}^{n} a_{ij} w_j(t-T) + \sum_{j=1}^{n} b_{ij} w_j(t), \qquad i = 1, 2, \cdots, n \tag{2.1}$$

where $T > 0$ is a real constant time lag and $w(t)$ is an n-vector with components $w_1(t), w_2(t), \cdots, w_n(t)$. $A = (a_{ij})$ and $B = (b_{ij})$ are the matrices of coefficients appearing in equation (2.1). The parameters a_{ij} and b_{ij} are obtained from the particular biological or economic model which is under consideration, and these parameters are generally considered as being real. However, this assumption is immaterial to our subsequent discussion. Generally models of biological and economic systems give rise to quite complicated nonlinear equations. The linear approximations to these complex systems can often be cast in the general form of equation (2.1), and the local stability properties of the nonlinear model then studied in terms of the geometric distribution of the roots of that equation.

Being linear, equation (2.1) is amenable to standard techniques such as the Laplace Transform. This is equivalent to setting

$$w(t) = w(0)e^{\lambda t},$$

where the eigenvalue λ is complex, and $w(0)$ is some constant initial value of $w(t)$. On substituting into (2.1) we have

$$w(0)(\lambda I - e^{-\lambda T}A - B) = 0.$$

Discarding the trivial solution $w(0) = 0$, we obtain the eigenvalue equation

$$\det(\lambda I - e^{-\lambda T}A - B) = 0, \tag{2.2}$$

which gives rise to several special cases. One important case corresponds to $A = 0$ for which equation (2.1) is a system of ordinary differential equations. Then the eigenvalues λ coincide with the eigenvalues of the matrix B. Note that for $T = 0$, we obtain λ equal to the eigenvalues of the matrix $(A + B)$. Two other special cases of (2.2) are important to this discussion.

(a) $A = cI$ where c is a real constant, I is the identity matrix, and B is a general matrix.

(b) $B = dI$, where d is a real constant, and A is a general matrix.

In case (a), equation (2.2) reduces to

$$\det[(\lambda - ce^{-\lambda T})I - B] = 0, \tag{2.3}$$

which is really the equation from which the eigenvalues of B are determined. Thus

$$\lambda - ce^{-\lambda T} = \xi_B,$$

where ξ_B is a complex eigenvalue of B. For a particular model, the coefficient matrix and hence its eigenvalues, are known so that this is an equation for the unknown λ. Let $z = \lambda T$, whence we have

$$pe^z + q - ze^z = 0, \tag{2.4}$$

where $p = T\xi_B$ is complex and $q = cT$ is real.

In case (b), equation (2.2) reduces to

$$\det[(\lambda - d)e^{\lambda T}I - A] = 0,$$

whence

$$(\lambda - d)e^{\lambda T} = \xi_A,$$

where ξ_A is a complex eigenvalue of the known matrix A. Setting $z = \lambda T$, $p = dT$, and $q = T\xi_A$, this equation reduces to (2.4), except that p is now real and q complex.

Equation (2.4) is studied in ecological contexts (Maynard-Smith [7, p. 43], May [6, p. 94]) particularly with a view to determining those regions of the (p, q) parameter space, which yield solutions $z = \alpha + i\beta$, with $\mathrm{Re}(z) < 0$. These studies have usually been directed at finding the local stability region of the parameter space for an associated nonlinear model. In particular, the generalized logistic equation with time lag, corresponds on linearization to (2.1) with $B = 0$ and $n = 1$, and yields an eigenvalue equation of the form of (2.4), with $p = 0$ and q real. This generalized logistic equation has been widely used in connection with one species modelling (May [6, chapter 4]; properties of its solution are outlined in Wright [8]).

A more general one species model is discussed by Maynard-Smith [7], and this corresponds to $n = 1$, $A = c$, $B = d$ with c and d real, in equation (2.1). This case leads to (2.4) with both p and q real, and this particular eigenvalue equation is discussed by Bellman and Cooke [1] (p. 444). Goel, Maitra and Montroll [3] consider the case $p = 0$ and q complex in their study of nonlinear models of interacting species. They prove the existence of a region in the q-plane bounded by Archimedean spirals, for which there is local stability in the phase plane. This region is called a 'teardrop' by May [6, p. 16]. This result is generalised in Section 4 to include the case $p \neq 0$.

3. p complex and q real

Consider equation (2.4) with q real and $p = x + iy$, complex. Let $z = \alpha + i\beta$, multiply by e^{-z} and extract real and imaginary parts whence

$$x - \alpha = -qe^{-\alpha}\cos\beta, \tag{3.1}$$

$$y = \beta + qe^{-\alpha}\sin\beta. \tag{3.2}$$

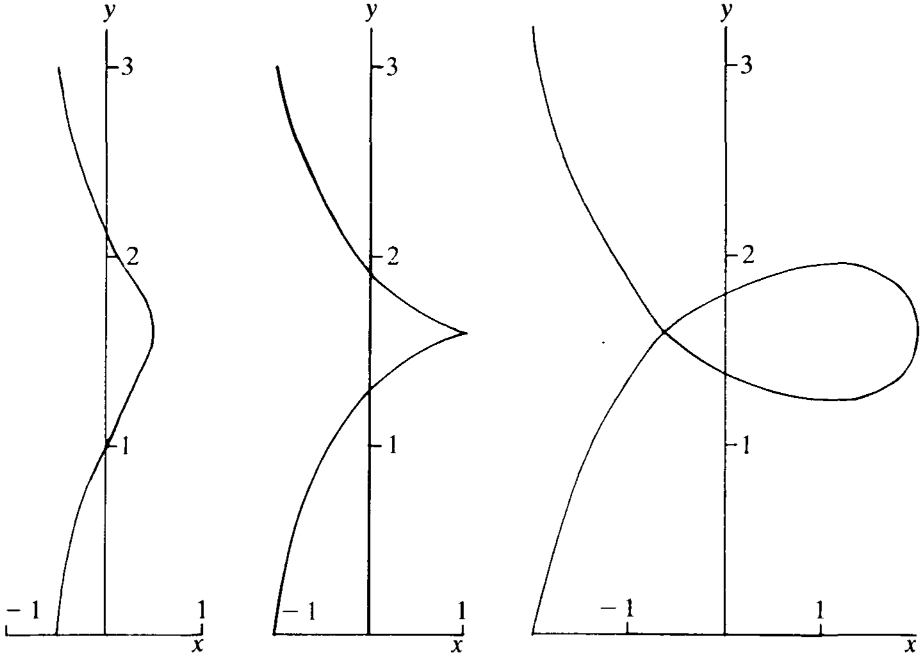
We now seek to determine the conditions on p and q such that $\alpha < 0$; i.e. (2.1) has decaying solutions. It is convenient to regard α and q as preassigned parameters and to treat x and y as functions of the independent parameter β.

Now if α and q are constant, equations (3.1) and (3.2) represent a trochoid (see Figure 1) in the (x, y) plane, in terms of the independent parameter β (see Lockwood [5, p. 146]). Consider first the degenerate case $q = 0$; then

$$\alpha = \operatorname{Re}(p),$$

and the linear system (2.1) is stable provided $\operatorname{Re}(p) < 0$. But if $q > 0$, and α and q are constants such that $qe^{-\alpha} = 1$, then equations (3.1) and (3.2) describe a cycloid (Figure 1b) about the line $x = \alpha$, with β as an independent parameter. The cusps of the cycloid are located at $x = \alpha + 1$, $y = \pm(2n + 1)\pi$, $n = 0, 1, 2, \cdots$, and the cycloid has amplitude $qe^{-\alpha} = 1$. For $0 < qe^{-\alpha} < 1$, equations (3.1) and (3.2) represent a curtate cycloid (Figure 1a) where the amplitude is less than 1 and the cusps are replaced by smooth curves. If $qe^{-\alpha} > 1$, then the trochoid is a prolate cycloid and the cusps are replaced by loops (Figure 1c). The cycloid which has a cusp, may be considered as the transition between the curves without loops (curtate) and those containing loops (prolate).

Now consider the case $q = -r < 0$. In this case, we still obtain trochoids



1a; $\Gamma_0(\frac{1}{2})$, curtate cycloid,

1b; $\Gamma_0(1)$, cycloid,

1c; $\Gamma_0(2)$, prolate cycloid

Figure 1. The trochoids for $\alpha = 0$ for $\beta \in [0, 2\pi]$, from equations (3.1) and (3.2).

with the same transition from curtate cycloids (smooth curve) through cycloids (cusps) to prolate cycloids (loops) according as $re^{-\alpha} < 1$, $= 1$, or > 1. However, for $re^{-\alpha} = 1$, the cusps are located at $x = \alpha + 1$, $y = \pm 2n\pi$, $n = 0, 1, 2, \cdots$. Thus there is a phase change of π along the y axis between $q < 0$ and $q > 0$; but the cusps or loops are still located on the right-hand side of the line $x = \alpha$.

For a fixed value of q, the form of the trochoids changes as the value of α varies. Let the curve $\Gamma_\alpha(q)$ be the trochoid defined by (3.1) and (3.2), and we will be particularly interested in the curves $\Gamma_0(q)$. Thus $\Gamma_0(q)$ is the straight line $x = 0$, if $q = 0$, and is a curtate cycloid, cycloid or prolate cycloid about $x = 0$, according as $0 < |q| < 1$, $|q| = 1$, or $|q| > 1$. As α increases, the centre line $x = \alpha$ of the trochoid moves to the right while the nature of the trochoid $\Gamma_\alpha(q)$ depends on whether $|q| e^{-\alpha} < 1$, $= 1$, or > 1.

To obtain the stable region of the complex p-plane for a particular value of q, we need only show that the unstable roots $\alpha \geqq 0$ occur in a particular region of the p-plane. We will prove that all trochoids $\Gamma_\alpha(q)$ for which $\alpha \geqq 0$,

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lie on or to the right of the curve $\Gamma^*(q)$, defined below, and hence that the stable region of the complex p-plane is the region to the left of the curve $\Gamma^*(q)$. Define

$$\Gamma^*(q) = \begin{cases} \Gamma_0(q), & |q| \leqq 1, \\ \Gamma_0(q) \text{ with loop discarded}, & |q| > 1 \end{cases}$$

(see Figures 1 and 2). For $|q| > 1$ and some $\alpha > 0$, the trochoid $\Gamma_\alpha(q)$ intersects the loop of $\Gamma_0(q)$ and hence the region inside this loop is unstable (see Figure 2). Thus $\Gamma^*(q)$ is the envelope of $\Gamma_\alpha(q)$ for $\alpha \geqq 0$, for, from (3.1), we have

$$\begin{aligned} \frac{\partial x}{\partial \alpha} &= 1 + qe^{-\alpha}\cos\beta, \\ &\geqq 1 - |q|e^{-\alpha}, \end{aligned} \tag{3.3}$$

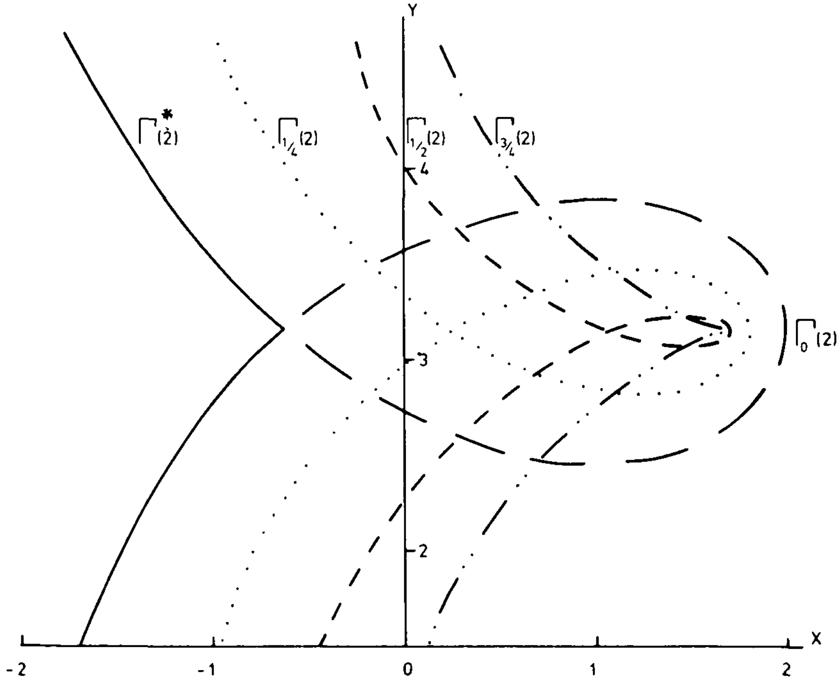

Figure 2. Showing the behaviour of the trochoids for $q = 2$, $\alpha = 0, 0.25, 0.5$, and 0.75. The envelope $\Gamma^*(2)$ is shown and note that for $\alpha > 0$, the trochoids $\Gamma_\alpha(2)$ pass through the loop of $\Gamma_0(2)$.

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for all β, and this derivative is non-negative provided that

$$1 \geqq e^{-\alpha} |q|,$$

$$\alpha \geqq \alpha^* = \ln |q|.$$

Thus for $|q| \leqq 1$, it follows directly from (3.3), that $\partial x/\partial \alpha \geqq 0$ for all β and all $\alpha \geqq 0$, so that all the trochoids $\Gamma_\alpha(q)$ lie on or to the right of $\Gamma^*(q)$. Further, α^* is defined for all $q \neq 0$, and the trochoids $\Gamma_\alpha(q)$ lie on or to the right of $\Gamma_{\alpha^*}(q)$ provided that $\alpha \geqq \alpha^*$.

In considering the case $|q| > 1$, we note that the trochoids $\Gamma_\alpha(q)$ for $q > 1$ and $q < -1$ are identical except for a phase shift of π along the y-axis. It is thus sufficient to consider only $q > 1$. For fixed α and $q > 1$ such that $qe^{-\alpha} > 1$, the trochoid contains loops and the gradient $dy/dx = (dy/d\beta)/(dx/d\beta)$, will be zero at two points symmetrically placed on either side of each loop. Substituting from (3.1) and (3.2),

$$\frac{dy}{dx} = (1 + qe^{-\alpha}\cos\beta)(qe^{-\alpha}\sin\beta)^{-1},$$

$$= 0,$$

which implies that $\cos\beta = -e^{\alpha}/q$. In the first period $0 \leqq \beta < 2\pi$, of the trochoid, the roots of this equation are given by

$$\beta_{1,2} = \pi \pm \theta,$$

where $\theta = \arccos e^{\alpha}/q$, $0 < \theta < \pi/2$. The pattern of roots is repeated in each period 2π of the trochoid. Obviously the section of the trochoid $\Gamma_\alpha(q)$ for which $\pi - \theta \leqq \beta \leqq \pi + \theta$, corresponds to the right-hand extremity of the loop and hence is to the right of the sections of the trochoid corresponding to $0 \leqq \beta < \beta_1$, and $\beta_2 < \beta \leqq 2\pi$.

We now return to the problem of finding the envelope of the trochoids $\Gamma_\alpha(q)$ for $q > 1$ and $qe^{-\alpha} > 1$. From (3.1), $\partial x/\partial \alpha \geqq 0$, provided that $\cos\beta \geqq -e^{\alpha}/q$, i.e., provided that $0 \leqq \beta < \beta_1$, or $\beta_2 \leqq \beta \leqq 2\pi$ with obvious periodic extensions. Thus except for $\pi - \theta < \beta < \pi + \theta$, the trochoid $\Gamma_\alpha(q)$ moves to the right as α increases. Obviously the loop can be ignored in determining the envelope $\Gamma^*(q)$.

In summary, the solutions of (2.1) with $A = cI$, are stable provided that all of the complex $p = T\xi_B$ lie to the left of the envelope $\Gamma^*(q)$.

4. *p* real and *q* complex

We now consider the equation (2.4) under the assumptions that p is real, and in polar form, $q = Re^{i\phi}$, is complex. Rearranging (2.4) leads to

$$\mu e^{\mu} = qe^{-p}, \tag{4.1}$$

where $\mu = z - p$. Set

$$\begin{aligned} \mu &= \alpha + i\beta, \\ &= \rho e^{i\theta}, \end{aligned} \tag{4.2}$$

substitute into (4.1) and extract real and imaginary parts to obtain

$$\begin{aligned} R &= \rho e^{\alpha + p}, \\ \phi &= \theta + \beta + 2m\pi, \end{aligned} \tag{4.3}$$

where m is an integer. The polar representations of both q and μ imply that both phase angles ϕ and θ are restricted to the range $[0, 2\pi)$. However, $\beta = \mathrm{Im}(\mu)$ is not restricted in any way and the term $2m\pi$ is required to satisfy the restriction on ϕ. From (4.2), we have $\beta = \rho \sin \theta$, so that (4.3) simplifies to

$$\phi = \theta + Re^{-(\alpha + p)} \sin \theta, \tag{4.4}$$

and we have taken $m = 0$. We now seek that region of the complex ξ_A, or (R, ϕ), plane for which

$$\mathrm{Re}(z) = \alpha + p < 0.$$

In the complex q-plane using the natural polar variables (R, ϕ), equation (4.4) represents an Archimedean spiral, where α and θ are constant parameters. However, it will be convenient to express (R, ϕ) as Cartesian coordinates with R as the independent variable, and to treat α and θ as parameters. In this more convenient Cartesian representation, equation (4.4) is a straight line. The transformation from the convenient Cartesian representation back to the q-plane in polar form, is made later in this discussion.

First consider the problem of finding the unstable or neutrally stable region, $\alpha + p \geqq 0$, of the q-plane for the special case $p \equiv 0$. From (4.2)

$$\alpha = \rho \cos \theta,$$

and there is no restriction on the values of either ρ or β. Hence, for $\alpha = 0$, $\theta = \pi/2$ or $3\pi/2$; and for $\alpha > 0$, θ is restricted to the ranges $\theta \in [0, \pi/2)$, or $\theta \in (3\pi/2, 2\pi)$. In its Cartesian representation, equation (4.4) describes a straight line with a ϕ-intercept $\phi_I = \theta$, and gradient $e^{-\alpha} \sin \theta$. For $\theta \in [0, \pi/2)$, then $\phi_I(\theta) < \phi_I(\pi/2)$, and the value $\phi_I = \pi/2$ is attained only at $\alpha = 0$. Further, the gradient is bounded by

$$0 \leqq e^{-\alpha} \sin \theta \leqq 1,$$

where the upper bound is obtained only for $\alpha = 0$, at which $\theta = \pi/2$. Thus all the straight lines for this range of θ lie under the bounding line given by equation (4.4) with $\alpha = 0$, $\theta = \pi/2$, i.e.,

$$\phi = \pi/2 + R \tag{4.5}$$

(see Figure 3). It is readily apparent that a similar situation arises for $\alpha \geqq 0$ and $\theta \in (3\pi/2, 2\pi)$, except that all the straight lines for $\alpha > 0$ and $\theta \in (3\pi/2, 2\pi)$, lie above the bounding curve for which $\alpha = 0$, $\theta = 3\pi/2$, i.e. the curve described by

$$\phi = \frac{3\pi}{2} - R. \tag{4.6}$$

Obviously the neutral roots of equation (2.4) for $p \equiv 0$ lie on these two bounding lines. Further, none of the straight lines defined by equation (4.4) for $\alpha \geqq 0$, enter the interior of the triangle E with vertices $(0, \pi/2)$, $(0, 3\pi/2)$ and $(\pi/2, \pi)$.

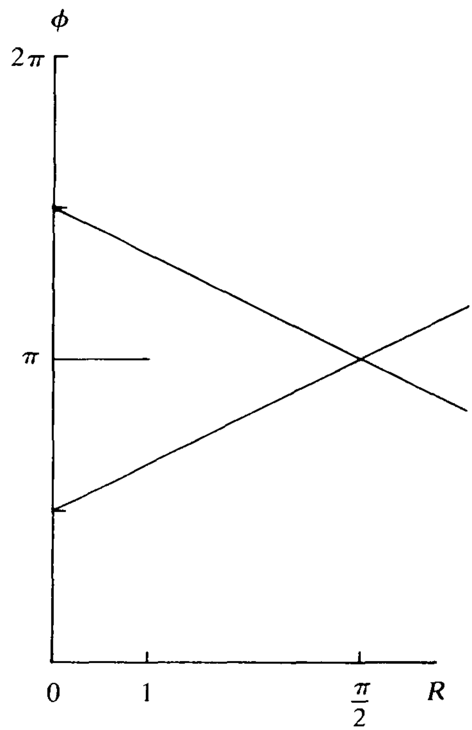

Figure 3. Showing the triangle E and the line segment $-1 \leqq R \leqq 0$,

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If $\alpha < 0$, the phase angle θ is restricted to the range $(\pi/2, 3\pi/2)$, and hence

$$\frac{\pi}{2} < \phi_l(\theta) < \frac{3\pi}{2}.$$

Thus the straight lines of (4.4) for $\alpha < 0$ all intersect the interior $\mathscr{E}$ of the triangle $E(\mathscr{E} = \operatorname{Int} E)$. Finally we investigate the purely real roots corresponding to $\beta = 0$, $\theta = 0$ or $\theta = \pi$, for which $\alpha > 0$ or $\alpha < 0$. The unstable real roots correspond to $\theta = 0$, and yield the straight line $\phi = 0$, which does not intersect E. Thus $\mathscr{E}$ is the region of stability.

In the more general cases of $p \neq 0$; consider first $p < 0$ and let $s = -p > 0$. The stability condition $\alpha + p < 0$ implies that $\alpha < -p = s$, so that the unstable region corresponds to $\alpha \geqq s > 0$. From (4.2), the value of $\beta = \operatorname{Im}(\mu)$ is unrestricted. Provided that $s < \infty$, the phase angle θ is restricted to the ranges $\theta \in [0, \pi/2)$ and $\theta \in (3\pi/2, 2\pi)$ for $\alpha \geqq s$. Thus the ϕ-intercepts of the straight line (4.4) are the same as for the previous case $p \equiv 0$. Further, the gradient of the straight line is restricted by $0 \leqq e^{s-\alpha} \sin\theta \leqq 1$, with the upper limit being attained only at $\alpha = s$, $\beta \to \infty$ and $\theta \to \pi/2$. These observations are identical to the case $p \equiv 0$, and the stability region is still $\mathscr{E} = \operatorname{Int}(E)$.

Now consider the case $p > 0$, for which the stability condition requires that $\alpha < -p < 0$. The description of the region of stability is facilitated by first discussing the possible location of all the unstable lines (4.4) for which $\alpha \geqq -p$. The case $\beta = 0$, which yields purely real roots μ corresponding to exponential type solutions of equation (2.1), is considered now. With $\alpha > 0$, $\theta = 0$ and $\phi = 0$ from equation (4.3). These unstable real roots lie in the unstable region outside of the triangle E and are relatively unimportant. For $\alpha < 0$, $\theta = \pi$ and $\phi = \pi$ but here unstable real roots are obtained for $-p \leqq \alpha < 0$, while stable real roots are obtained for $\alpha < -p$. With $\alpha < 0$, $\beta = 0$, equation (4.3) yields

$$R = -e^{p}\alpha e^{\alpha} = -e^{p} f(\alpha),$$

say, where it is readily shown that $f(\alpha)$ is monotonic increasing for $-1 \leqq \alpha < 0$. Thus if $0 < p \leqq 1$ and $-p \leqq \alpha < 0$, then

$$R \leqq -e^{p}(-pe^{-p}) = p, \tag{4.7}$$

and these roots are unstable. Further, the function $f(\alpha)$ is monotonic decreasing for $-\infty < \alpha < -1$, and $f(\alpha)$ also has a minimum value of $-e^{-1}$ at $\alpha = -1$. Thus if $p > 1$, and $-p \leqq \alpha < 0$, then $f(\alpha) \geqq f(-1)$ and

$$R \leqq e^{p-1}. \tag{4.8}$$

Obviously the stable regions correspond to the remaining sections of the line $\phi = \pi$. We will return to these results after discussing the complex roots.

Assuming that $\beta \neq 0$, then the phase angle θ is restricted to the ranges $\theta \in (0, \pi)$ or $\theta \in (\pi, 2\pi)$, and the intercepts $\phi_l(\theta)$ are similarly limited. Since $-p \leqq \alpha < 0$ is an unstable root, there are no restrictions on the range of θ for these roots (cf., the restrictions on θ for $p \geqq 0$). The straight lines (4.4) cut the line $\phi = \pi$ at the point (R_1, π) where

$$R_1 = \left(\frac{\pi - \theta}{\sin \theta}\right) e^{\alpha + p} = \frac{\pi - \theta}{\sin(\pi - \theta)} e^{\alpha + p} \geqq 1,$$

where $R \geqq 0$, and equality occurs in (4.9) only at $\alpha = -p$, and $\theta = \pi$. Thus the line segment $R \geqq 1$, $\phi = \pi$ contains unstable eigenvalues.

The stability analysis for the case $p > 0$ is readily completed by combining the results (4.7), (4.8) and (4.9). Obviously (4.8) and (4.9) imply that there are no stable regions for $p \geqq 1$. Further, (4.7) and (4.9) imply that the line segment

$$\phi = \pi, \qquad p < R < 1, \tag{4.9}$$

is stable for $0 < p < 1$, and this is the only stable region.

It is necessary to transfer these results from the Cartesian representation to the complex $q = Re^{i\phi}$ plane (in polar form). The stable region defined by (4.9) is just the section $(-1, -p)$ of the negative real axis $(0 < p < 1)$. The sides of the triangle E are defined by equations (4.5) and (4.6), together with the third side $R = 0$. In the complex q-plane, the side $R = 0$ reduces to a point at the origin while the other two curves represent Archimedean spirals, one turning clockwise and the other anticlockwise. These spirals intersect at $R = \pi/2$, $\phi = \pi$, and the included region is the region of stability for $p \leqq 0$. This teardrop shaped region was derived by Goel et al. [3], for the case $p = 0$.

Finally, let us briefly consider the generalization of (2.4) in which $q = Re^{i\phi}$ and $p = x + iy$ are both complex. Setting $\mu = z - p$, (2.4) reduces to

$$\mu e^{\mu} = Re^{i(\phi - y)} e^{-x},$$

and if $\mu = \alpha + i\beta = \rho e^{i\theta}$, then on separating real and imaginary parts

$$R = \rho e^{\alpha + x}, \tag{4.10}$$

$$\begin{aligned} \phi - y &= \theta + \beta + 2m\pi, \\ &= \theta + Re^{-(\alpha + x)} \sin \theta + 2m\pi. \end{aligned} \tag{4.11}$$

Apart from the obvious rotation about the origin, these equations are identical with (4.3) and (4.4). Thus we are able to use the complex p plane parametrically and for each point (x, y), determine the stable region of the complex q plane. Obviously varying y merely rotates the teardrop, or straight line segment, in the q plane.

5. Discussion

The results regarding the stability regions of the parameter spaces p and q, for equation (2.4), are remarkably simple and elegant. They are summarised in Tables 1a and 1b.

Table 1a (q real, p complex)

Envelope curve		Value of q	Stable region
Straight line	$x = 0$	$q = 0$	Left hand half of complex p-plane
Curtate cycloid Fig. 1a	$\Gamma^*(q)$	$0 < \lvert q \rvert < 1$	That section of the complex p-plane to the left of the envelope $\Gamma^*(q)$.
Cycloid Fig. 1b.	$\Gamma^*(q)$	$\lvert q \rvert = 1$	
prolate cycloid Fig. 1c.	$\Gamma^*(q)$	$\lvert q \rvert > 1$	

Table 1b (q complex, p real)

Envelope curve	Value of p	Stable region
Archimedean spirals eqns (4.5) and (4.6) Fig. 3	$p \leqq 0$	Area of complex q-plane enclosed by the spirals.
Straight line segment $p < R < 1$, $\phi = \pi$	$1 > p > 0$	The line segment $p < R < 1$, $\phi = \pi$
Nil	$p \geqq 1$	Nil

In addition to these tables, if $p = x + iy$ is complex, then the three cases in Table 1b, also apply to $x \leqq 0$, $1 > x > 0$ and $x \geqq 1$, except that the envelopes and stability regions are rotated clockwise through the angle y. These results indicate a greater region of stability if q is real and that a stability region does not exist for $\text{Re}(p) = x \geqq 1$. Thus a complex value of q is destabilising, but this is not surprising since q arises from the time delay term in equation (2.1). Thus the stability regions of (2.4) have been completely described although we have not attempted to calculate the actual eigenvalues.

However, this does not complete the stability analysis of equation (2.1) for various simplifying assumptions are made in obtaining (2.4). Specifically these assumptions relate to the structure of the coefficient matrices in (2.1), and, in particular, we assume that at least one of these matrices can be represented as a scalar multiple of the identity matrix I. By means of this step, the unknown variable λ can be collected into the term $(\lambda - ce^{-\lambda T})I$, where c is a scalar, and then related to the eigenvalues of the remaining coefficient matrix. This separation of the unknown is crucial to our obtaining the relatively simple equation (2.4). If this separation cannot be carried out, the approach fails.

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