Residual finiteness of cyclic extensions

of certain free products

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We prove that a cyclic extension of a free product of finite
p-groups is residually finite, p a fixed prime. The question
of cyclic extensions of free products of arbitrary cyclic groups
remains open.

1. Introduction

In [2] Gilbert Baumslag proved that any finitely generated cyclic
extension of a free group is residually finite. This raises the question
of cyclic extensions of arbitrary free products of cyclic groups. In this
note we prove that if H is a free product of finite p-groups for the
same prime p, then any cyclic extension of H is residually finite.
This follows from a more general result: if H is a free product of
finitely generated nilpotent groups, none of which are infinite cyclic, and
all of whose torsion subgroups, if any, are p-groups for the same prime
p, then any cyclic extension of H is residually finite (Theorem A). The
proof is similar to that used by Gilbert Baumslag for cyclic extensions of
free groups. An important difference is that we do not require that the
extension be finitely generated.

Our proof requires that H be residually nilpotent, which is not, for
example, the case when the primes involved are different. In this
direction however, we have a partial result (a special case of Theorem B):
Let H be a free product of finitely generated p-groups where, with the

possible exception of one of the primes involved, there are at most finitely many factors to each prime. Then cyclic extensions of H are residually finite. The problem of arbitrary free products remains open.

2. Proofs

For each positive integer c let $\prod (c, H_i, I)$ **denote** the cth nilpotent product of the groups H_i, $i \in I$ (see, for instance, [3]). In addition we will allow c to take on the value ∞ by defining $\prod (\infty, H_i, I)$ to be the free product of the groups H_i, $i \in I$.

DEFINITION. Let G be a cyclic extension of H, $G = \langle t \rangle H$, where $H = \prod (c, H_i, I)$, as above, such that for each $i \in I$ there exist elements $j \in I$ and $h \in H$ satisfying $H_i^t = H_j^h$. Then G is called a *cof extension of H relative to the decomposition* $\prod (c, H_i, I)$. In the special case where $H_i^t = H_j$ for each $i \in I$ we call G a *regular cof extension* relative to the given decomposition.

Given a cof extension we define a t-action on I by defining $it = j$ if H_i^t is conjugate in H to H_j. We denote the orbit containing i by $[i]$; that is,

$$[i] = \{ it^r \mid r \in \mathbb{Z} \} .$$

More generally, if $S \subseteq I$, we define $[S] = \bigcup_{i \in S} [i]$.

We recall that the orbits form a partition of I.

The following result is almost immediate.

LEMMA 1. *Let G be a cof extension of H relative to* $\prod (c, H_i, I)$ *and let S be any subset of I. Define K to be the normal closure in H of the subgroup* $\prod (c, H_i, I \backslash [S])$. *Then K is normal in G and G/K is a cof extension of* $H/K \cong \prod (c, H_i, [S])$.

COROLLARY 1. *To show that G is residually finite it may be assumed, with no loss in generality, that the number of orbits is finite.*

Proof. Let $g \neq 1$ be an element of G. Then $g = t^r h$ for some integer r and $h \in H$, and h is expressible as the product of elements from say H_{i_1}, H_{i_2}, ..., H_{i_k}. Let $S = \{i_1, i_2, \ldots, i_k\}$ and construct K as in the lemma. Then $g \notin K$, so if G/K is residually finite there is a finite homomorphic image of G/K, and hence of G, in which g is non-trivial. Since the number of orbits of H/K in G/K is finite, the result follows.

COROLLARY 2. *Let G be a cof extension of H as above, and suppose that the components H_i of H are finitely generated and residually finite. Furthermore, suppose that all orbits are finite. Then G is residually finite.*

Proof. By Corollary 1, we may assume that the number of orbits is finite, and hence, H is finitely generated. Since a free product of residually finite groups is residually finite (see [1] for a short proof), G is a cyclic extension of a finitely generated residually finite group and, hence, is residually finite.

An interesting application of Corollary 2 is to the case where H is the free product of finite groups H_i such that for each integer n at most a finite number of the groups H_i are of order n.

LEMMA 2. *Let G be a regular cof extension of H. To show that G is residually finite it may be assumed, with no loss in generality, that every orbit is finite.*

Proof. We consider the case where I contains only one infinite orbit, since the more general case requires only further notation but no change in proof. Thus, let

$$(1) \qquad\qquad H = \prod \left(c, H_i, \mathbb{Z} \right)$$

with $\left(H_0 \right)^{t^r} = H_r$, for all integers r.

For any integer m greater than 1 let

$$H(m) = \prod \left(c, H_i, \{1, 2, \ldots, m\} \right)$$

and define the action of an element $\overline{t}$ on $H(m)$ by

$$\overline{H_i^t} = H_{\widehat{(i+1)}}$$

where $\widehat{i+1}$ is the least positive residue congruent to $i + 1$ modulo m. In this way we obtain a cof extension G_m, say, of $H(m)$.

Now $t^{i-\hat{i}}$ induces an isomorphism ψ_i of H_i onto $H_{\hat{i}}$. The maps $t \to \overline{t}$ and ψ_i induce a homomorphism θ_m, say, of G onto G_m. For each g in G, $g \neq 1$, we can choose m sufficiently large so that $g\theta_m \neq 1$, and this establishes the result.

Let $\gamma_c(H)$ denote the cth term of the lower central series of H. We then have the following result.

LEMMA 3. *Let G be a cof extension of $H = \prod (\infty, H_i, I)$. Then for any integer c, $G/\gamma_c(H)$ is a regular cof extension of $\overline{H} = H/\gamma_c(H)$ with respect to the decomposition into a cth nilpotent product with factors conjugate to the $H_i\gamma_c(H)/\gamma_c(H)$.*

Proof. We denote the images under the natural map $G \to G/\gamma_c(H)$ by bars. Once again we assume that I is a single infinite orbit, the other cases being similar. Thus, suppose (1) holds and that

$$H_0^{t^r} = H_r^{h_r},$$

where h_r is an element of H. Then

$$\overline{H} = \prod (c, \overline{H}_i, \mathbb{Z}).$$

Using generators and defining relators it is easy to see that if a group A is the cth nilpotent product of groups A_i of nilpotent class at most c, then for arbitrary elements g_i of A, A is also the cth nilpotent product of the groups $A_i^{g_i}$. Thus if we put $\overline{(H^i)}^{\overline{h}_i} = B_i$, then

(2) $\overline{H} = \prod (c, B_i, \mathbb{Z})$

and $\overline{G}$ is a regular cof extension of $\overline{H}$ relative to (2).

THEOREM A. *Let $G = H\langle t \rangle$ be a cyclic extension of*
$H = \prod\ (\infty,\ H_i,\ I)$ *where each H_i is a finitely generated nilpotent*
group, not infinite cyclic, whose torsion subgroup, if any, is a p-group
for the same prime p . Then G is residually finite.

Proof. It is easy to see that the non-infinite-cyclic maximal
nilpotent subgroups of H are the conjugates of the H_i in H . Thus G
is a cof extension of H . By Corollary 1 it suffices to deal with the
case of a finite number of orbits only.

Now if g is a non-unit element of G we must find a normal subgroup
of finite index which excludes g . This is easily done if g is not in
H , so we may assume that $g \in H$. By a result of Mal'cev [4], H is
residually nilpotent. Thus there exists an integer c such that
$g \notin \gamma_c(H)$. By Lemma 3, $\overline{G} = G/\gamma_c(H)$ is a regular cof extension, and by
Lemma 2 we may assume that $\overline{G}$ has finite orbits only. Thus, $\overline{H} = H/\gamma_c(H)$
is a finitely generated nilpotent group and, hence is residually finite.
Since a cyclic extension of a finitely generated residually finite group is
residually finite, the result follows.

To prove Theorem B we shall need the following obvious result.

LEMMA 4. *Let G be a cof extension of $H = \prod\ (c,\ H_i,\ I)$ and let*
$i_1,\ i_2,\ \dots,\ i_k$ *be elements of I from distinct finite orbits of*
cardinality $m_1,\ m_2,\ \dots,\ m_k$, respectively. Let $m = m_1 \dots m_k$. Then if
$s = t^m$, $H\langle s \rangle$ *is a cof extension of H in which the orbits containing*
$i_1,\ \dots,\ i_k$ *each consist of precisely one element. Furthermore, since*
$H\langle s \rangle$ *is of finite index in G , G is residually finite if and only if*
$H\langle s \rangle$ *is.*

THEOREM B. *Let p be a given prime and H a free product of*
finitely generated groups H_i , none of which is infinite cyclic. Suppose
that if H_i is infinite, it is nilpotent with torsion subgroup (if any) a
p-group. Further, suppose that for each integer n not a power of p the
number of factors H_i of order n is finite. Then any cyclic extension
G of H is residually finite.

Proof. Clearly G is a cof extension. Let Q be the free product of all the infinite H_i and of those which are p-groups and P be the free product of the remaining factors. Then $H = P*Q$. To show that G is residually finite it suffices to show that a non-trivial element g in H is non-trivial in some finite image of G . We may assume, by Corollary 1, that G has finitely many orbits and by Lemma 4, that if H_i is a factor of P , then the orbit containing i consists precisely of i .

If Q^H denotes the normal closure of Q in H , then G/Q^H is residually finite by Lemma 1 and Corollary 2, while G/P^H is residually finite by Lemma 1 and Theorem A. Thus, we may assume that g is in $P^H \cap Q^H$. We define $l(g)$ to be the number of occurrences of elements from the factors of P when g is expressed in normal form, that is, $l(g)$ is the syllable length of g with respect to the factors of P . The remainder of the proof is by induction on $l(g)$ in any group of the type described in the theorem.

If $l(g) = 0$, then $g \notin P^H \cap Q^H$. Thus, assume the result true for $l(g)$ less than n and let $l(g) = n$. Suppose that g involves in its normal form elements from a factor, H_1 say, of P . Let K be the normal closure in H of all the H_i with the exception of H_1 . We note, by Lemma 1, that K is normal in G and H/K is isomorphic to the finite group H_1 . It is easy to see that K is a free product satisfying the same hypotheses that H does in the statement of the theorem. Clearly g is in K , and when g is expressed in normal form in K , $l(g) < n$. Now $\overline{H} = K\langle t \rangle$ is an extension of the type described in the theorem, so by the induction hypothesis there is a finite factor of $\overline{H}$ excluding g . Since $\overline{H}$ is of finite index in G , the result follows.

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