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Some aspects of Carmichael's Conjecture

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INTRODUCTION

Let ϕ denote Euler's function; that is $\phi(x)$ is the number of natural numbers not exceeding x and relatively prime to x. Also recall that $\phi(1) = 1$ by definition. It is more than ninety years since Carmichael conjectured that, for any natural number A, the equation $\phi(x) = A$ never has a unique solution [**1**]. Since then this conjecture has received a considerable amount of attention but few theoretical results have been obtained (see Pomerance [**6**] and Hagis [**3**]). Recent computer searches showed that Carmichael's Conjecture is valid below $10^{10000000}$ (see Schlafly and Wagon [**7**]).

While mathematicians are still looking for more prime divisors of a counterexample to this conjecture, a new approach is offered in this thesis to deal with this problem and its theoretical aspects. The purpose of this thesis is to look at the conditions on Carmichael's Conjecture with an approach which relies on the concept of the set of solutions of the equation $\phi(x) = A$.

MAIN RESULTS

The following definitions are of importance.

(i) $\mathbb{F}_A = \{x : \phi(x) = A\}$, the set of solutions of $\phi(x) = A$.

(ii) $\{\mathbb{F}_{A_{(i)}}\}$, the sequences of sets of solutions $\mathbb{F}_{A_{(i)}}$, where $A_{(i)} = 2^i N$ and N is an odd composite natural number.

(iii) $\mathbb{F}_{A_{(k)}}$, the minimal set $\mathbb{F}_{A_{(i)}}$, where k is the minimal index for which $\phi(x) = A_{(i)}$ has solutions.

In this thesis it is shown that for a given N the cardinality of the minimal set $\mathbb{F}_{A_{(k)}}$ is even and the number of its odd solutions is equal to the number of its even solutions. It is also shown that Carmichael's Conjecture is proved unconditionally for infinitely many sets $\mathbb{F}_{A_{(i)}}$ if the cardinality of $\mathbb{F}_{A_{(k)}}$ is either strictly greater than 2 or

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equal to 2 and $3 \nmid x$, for $x \in \mathbb{F}_{A_{(k)}}$. While if the cardinality is 2 and $3|x$, for $x \in \mathbb{F}_{A_{(k)}}$, then Carmichael's Conjecture holds with some restrictions on k and N.

In this research we have been able to show that for a given N, a counterexample to Carmichael's Conjecture occurs first in $\mathbb{F}_{A_{(k+1)}}$ or never, and if it occurs, then it is equal to 2^2x where $x \in \mathbb{F}_{A_{(k)}}$ is odd and the cardinality of $\mathbb{F}_{A_{(k)}}$ is 2. If such a counterexample occurs, then it is proved that $x = 3^a y$, where $a \geqslant 2$ and $\gcd(3, y) = 1$, the cardinality of $\mathbb{F}_{\phi(y)}$ is 2 and $\mathbb{F}_{\phi(y)}$ is minimal. Then other related minimal sets of cardinality 2, which are necessary consequences of the minimality of $\mathbb{F}_{\phi(y)}$, are found and the existence of this counterexample is discussed.

A necessary and sufficient condition for the existence of a counterexample to Carmichael's Conjecture is also given and a great number of necessary conditions for this existence are established. In addition, some introductory lemmas and theorems are proved.

Furthermore, progress is made by defining new prime sequences and related sequences of sets of solutions, which are connected to a possible counterexample to Carmichael's Conjecture. A general investigation of some prime divisors of a possible unique solution obtained by Donnelly [**2**] is used together with [**4**, Corollary 1.2] to find a specific sequence of prime divisors of a possible unique solution.

The general structure of a solution x of $\phi(x) = A$ for $a \neq 2^i$, $\forall i \geqslant 1$, is also given in this thesis. This structure of a solution of $\phi(x) = A$ is employed to extend the bound of a possible unique solution. For example, if x is a unique solution of $\phi(x)$ and satisfies the equation $x - \phi(x) = a$, then by [**4**, Corollary 1.2] all prime divisors of x divide $\phi(x)$. Consequently they divide a, and therefore the bound of x is increased by a.

In addition, the thesis contains an original result which disproves Mendelsohn's Theorem [**5**] by means of counterexamples, and it supplies a corrected version of this theorem.

Conclusion

In the present work, several theorems and lemmas based on the concept of minimal sets and the existence of a possible counterexample to Carmichael's Conjecture are put forward. Then the prime divisors exponents of such a counterexample are used to extract more necessary conditions. Finally, recommendations for further research are outlined and further conjectures are set out.

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