BULL. AUSTRAL. MATH. SOC. MOS 42A28, 42A40
VOL. 3 (1970), 217-229.

Absolute Riesz summability of Fourier series and their conjugate series

Masako Izumi and Shin-ichi Izumi

This paper contains two theorems. The first theorem treats the $|R, r, 1|$ summability of Fourier series and their associated series of functions of bounded variation. The second concerns the $|R, r, 1|$ summability of Fourier series of functions f such that $\varphi(t)m(1/t)$ is of bounded variation where $m(u)$ increases to infinity as $u \to \infty$. These theorems generalize Mohanty's theorems.

1. Introduction and theorems

1.1. Suppose that $r(x)$ is defined on the interval $x > 0$, $r(x) > 0$, $r(x) \uparrow \infty$ as $x \uparrow \infty$ and $r(x)$ is differentiable continuously. We write $r(n) = r_n$ for integral n. The series $\sum a_n$ is said to be $|R, r, 1|$ summable if

$$(1) \qquad \int_B^\infty \frac{r'(x)}{\left(r(x)\right)^2} dx \left|\sum_{n\le x} r_n a_n\right| < \infty \quad \text{for } aB > 0 .$$

By $|R, r, 1|$ we denote the class of all $|R, r, 1|$ summable series, so that (1) is equivalent to $\sum a_n \in |R, r, 1|$. It is known that $|R, x, 1| = |C, 1|$ and $|R, e^x, 1|$ is the class of all absolutely convergent Fourier series. The following classes, each containing the

Received 9 June 1970.

the next, are usually treated:

$$|R, (\log x)^a, 1| , \quad (a > 0) ,$$

$$|R, x^a, 1| , \quad (a > 0) ,$$

$$|R, e^{(\log x)^a}, 1| , \quad (a > 1) ,$$

$$|R, e^{x^a}, 1| , \quad (0 < a < 1) ,$$

$$\left|R, e^{x/e^{(\log x)^a}}, 1\right| , \quad (0 < a < 1) ,$$

$$\left|R, e^{x/(\log x)^a}, 1\right| , \quad (a > 0) .$$

1.2. Let f be an integrable function periodic with period 2π and its Fourier series and its conjugate series be

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x)$$

and

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x) ,$$

respectively. We write

$$\varphi_x(t) = \varphi(t) = f(x+t) + f(x-t) , \quad \psi_x(t) = \psi(t) = f(x+t) - f(x-t) .$$

If a function g is of bounded variation on the interval $(0, \pi)$, then we write $g \in BV$.

1.3. R. Mohanty [1] proved the following

THEOREM I. $\varphi \in BV \Rightarrow \sum A_n(x)/\log n \in |R, e^{x^a}, 1| \quad (0 < a < 1)$.

We shall prove the following generalization.

THEOREM 1. *(i) The case* $xr'(x)/r(x) \le A$. *If*

(2) $$r(x)/x \uparrow \text{ and } r'(x)/r(x) \downarrow 0 \text{ as } x \uparrow \infty$$

and

https://doi.org/10.1017/S0004972700045895 Published online by Cambridge University Press

(3) $$\int_u^\infty \frac{r'(x)}{xr(x)}\,dx \le A\,\frac{r'(u)}{r(u)} \quad \textit{for all} \quad u > 0\,,$$

then

$$\varphi \in BV \Rightarrow A_n(x) \in |R,\, r,\, 1|\,,$$

$$\psi \in BV \Rightarrow B_n(x) \in |R,\, r,\, 1|\,.$$

(ii) The case

(4) $$\limsup_{x\to\infty}\,\big(xr'(x)/r(x)\big) = \infty\,, \quad \liminf_{x\to\infty}\,\big(xr'(x)/r(x)\big) > 0\,.$$

If

(5) $$m(x) \downarrow 0\,, \quad m(x)r(x)/x \uparrow \quad \textit{and} \quad r'(x)/r(x) \downarrow 0 \quad \textit{as} \quad x \uparrow \infty\,,$$

(6) $$\int_u^\infty \frac{|m'(x)|}{x}\,dt \le \frac{A}{u} \quad \textit{and} \quad \int_u^\infty \frac{m(x)r'(x)}{xr(x)}\,dx \le \frac{Ar'(u)}{r(u)} \quad \textit{for all} \quad u > 0$$

and further if

(7) $$\int_{1/t}^{x_0(t)} \{m(x)/x\}dx \le A \quad \textit{for all} \quad t > 0$$

where $x_0(t)$ *is the root* $> 1/t$ *of the equation* $r'(x)/r(x) = t$ *, then*

$$\varphi \in BV \Rightarrow \sum m_n A_n(x) \in |R,\, r,\, 1|\,,$$

$$\psi \in BV \Rightarrow \sum m_n B_n(x) \in |R,\, r,\, 1|\,,$$

where $m(n) = m_n$ *for integer* n .

We shall consider special cases of Theorem 1. We take first $r(x) = x^a$ $(a > 1)$; then $r'(x)/r(x) = a/x$. Therefore, Theorem 1 *(i)* gives

COROLLARY 1. $\varphi \in BV \Rightarrow \sum A_n(x) \in |R,\, x^a,\, 1|$ $(a > 1)$. *The corresponding result holds for conjugate series.*

We shall next take $r(x) = e^{(\log x)^a}$ $(a > 1)$, then

https://doi.org/10.1017/S0004972700045895 Published online by Cambridge University Press

$r'(x)/r(x) = a(\log x)^{a-1}/x$ and $x_0(t) \cong \frac{1}{t}\left(\log\frac{1}{t}\right)^{a-1}$, and then (7) gives $m(x) = 1/\log\log x$. Therefore, Theorem 1 *(ii)* gives

COROLLARY 2. $\varphi \in BV \Rightarrow \sum A_n(x)/\log\log n \in \left|R, e^{(\log x)^a}, 1\right|$ $(a > 1)$. *The corresponding result holds for conjugate series.*

In the case $r(x) = e^{x^a}$ $(0 < a < 1)$, we have $r'(x)/r(x) = a/x^{1-a}$, $x_0(t) \cong 1/t^{1/(1-a)}$ and then $m(x) \cong 1/\log x$. Therefore, Theorem 1 *(ii)* gives

COROLLARY 3. $\varphi \in BV \Rightarrow \sum A_n(x)/\log n \in \left|R, e^{x^a}, 1\right|$ $(0 < a < 1)$. *The corresponding result holds for conjugate series.*

The first part of this corollary is Theorem I.

Similarly,

COROLLARY 4. $\varphi \in BV \Rightarrow \sum A_n(x)/\log n \log\log n \in \left|R, e^{x/e^{(\log x)^a}}, 1\right|$ $(0 < a < 1)$. *The corresponding result holds for conjugate series.*

Finally, consider the case $r(x) = e^{x/(\log x)^a}$ $(a > 0)$. Then $r'(x)/r(x) \cong 1/(\log x)^a$, $x_0(t) \cong e^{1/t^{1/a}}$ and $m(x) \cong 1/\log x(\log\log x)^b$ $(b > 1)$. Therefore

COROLLARY 5. $\varphi \in BV \Rightarrow \sum A_n(x)/\log n(\log\log n)^b \in \left|R, e^{x/(\log x)^a}, 1\right|$ $(a > 0, b > 1)$. *The corresponding result holds for conjugate series.*

1.4. We know the following theorem due to R. Mohanty [*1*].

THEOREM II.

(i) $\varphi(t)\log\log\frac{4\pi}{t} \in BV \Rightarrow \sum A_n(x) \in \left|R, e^{(\log x)^a}, 1\right|$ $(a > 1)$.

(ii) $\varphi(t)\log\frac{2\pi}{t} \in BV \Rightarrow \sum A_n(x) \in \left|R, e^{x^a}, 1\right|$ $(0 < a < 1)$.

(iii) $\psi(t)\log\frac{2\pi}{t} \in BV$, $\psi(t)/t \in L(0, \pi) \Rightarrow \sum B_n(x) \in \left|R, e^{x^a}, 1\right|$ $(0 < a < 1)$.

(iv) $\varphi(t)t^{-a} \in BV \Rightarrow \sum A_n(x) \in \left|R, e^{x/(\log x)^{1+1/a}}, 1\right|$ $(a > 0)$.

We prove the following generalization.

THEOREM 2. *If* (2), (3) *and* (4) *hold and*

$$\int_1^\infty \frac{1}{x}\left\{\frac{r''(x)r(x)}{\left(r'(x)\right)^2} - 1\right\}dx < \infty , \tag{8}$$

$$m(x) \uparrow \text{ as } x \uparrow \infty \text{ and } \left(1/m(1/x)\right)' \downarrow \text{ as } x \uparrow , \tag{9}$$

and further if

$$t^{-1} < x_0(t) \le t^{-1}e^{Am(1/t)} \text{ for all } t > 0 \tag{10}$$

where $x_0(t)$ *is the root of the equation* $r'(x)/r(x) = t$ *, then*

$$\varphi(t)m(1/t) \in BV \Rightarrow \sum A_n(x) \in |R, r, 1| ,$$

$$\psi(t)m(1/t) \in BV \Rightarrow \sum B_n(x) \in |R, r, 1| .$$

Theorem II, *(i)* and *(ii)* are deduced from Theorem 2 and *(iii)* is also, without the second assumption. Further we have

COROLLARY 6.

(i) $\varphi(t)(\log 2\pi/t)^{1/a} \in BV \Rightarrow \sum A_n(x) \in \left|R, e^{x/e^{(\log x)^a}}, 1\right|$ $(0 < a < 1)$.

(ii) $\varphi(t)t^{-1/a} \in BV \Rightarrow \sum A_n(x) \in \left|R, e^{x/(\log x)^a}, 1\right|$ $(a \ge 1)$.

The corresponding result holds for conjugate series.

Corollary 6, *(ii)* is an improvement of Theorem II, *(iv)*.

1.5. For the proof of the above theorems, we use the following known lemma. (See [2].)

LEMMA. *If* $p(x) \uparrow$ *as* $x \uparrow$, *then*

$$\left| \sum_{n \le x} p(n)\sin nt - \int_1^x p(u)\sin ut\,du \right| \le Ap(x)$$

for any $x > 1$ *and any* $t \in (0, \pi)$.

2. Proof of the theorems

2.1. Proof of Theorem 1. We shall prove the theorem for Fourier series only, since our method of proof is also applicable to conjugate series.

We shall take $B = 1$ in (1). By assumption and integration by parts,

$$A_n(x) = \frac{1}{\pi}\int_0^\pi \varphi(t)\cos nt\,dt = \frac{-1}{\pi n}\int_0^\pi \sin nt\,d\varphi(t) ,$$

and then we have to prove, by (1), that

$$\int_1^\infty \frac{r'(x)}{\left(r(x)\right)^2}\,dx \left| \sum_{n \le x} \frac{m_n r_n}{n} \int_0^\pi \sin nt\,d\varphi(t) \right| < \infty .$$

We suppose φ is of bounded variation, so that it is sufficient to prove that

$$\int_1^\infty \frac{r'(x)}{\left(r(x)\right)^2}\,dx \left| \sum_{n \le x} \frac{m_n r_n}{n} \sin nt \right| \le A \quad \text{for all} \quad t > 0 .$$

By (2), (3), (5), (6) and the lemma, this is equivalent to

$$\int_1^\infty \frac{r'(x)}{\left(r(x)\right)^2}\,dx \left| \int_1^x \frac{m(u)r(u)}{u} \sin ut\,du \right| \le A \quad \text{for all} \quad t > 0 . \tag{11}$$

The left side integral is

$$\int_1^\infty dx \left|\int_1^x du\right| \le \int_1^{1/t} dx \left|\int_1^x du\right| + \int_{1/t}^\infty dx \left|\int_1^{1/t} du\right| + \int_{1/t}^\infty dx \left|\int_{1/t}^x du\right|$$
$$= P + Q + R$$

where

https://doi.org/10.1017/S0004972700045895 Published online by Cambridge University Press

$$P \le t \int_1^{1/t} \frac{r'(x)}{(r(x))^2}\,dx \int_1^x m(u)r(u)du$$
$$= t\int_1^{1/t} m(u)r(u)du \int_u^{1/t} \frac{r'(x)}{(r(x))^2}\,dx \le At\int_1^{1/t} m(u)du \le A$$

and

$$Q \le t\int_{1/t}^{\infty} \frac{r'(x)}{(r(x))^2}\,dx \int_1^{1/t} m(u)r(u)du = \frac{t}{r(1/t)}\int_1^{1/t} m(u)r(u)du \le A\ .$$

We shall now estimate R . The inner integral of R is

$$\int_{1/t}^{x} \frac{m(u)r(u)}{u}\sin ut\,du = I\int_{1/t}^{x} \frac{m(u)}{u}\,r(u)e^{iut}du\ .$$

Since

$$\text{(12)}\qquad \left(r(u)e^{iut}\right)' = r(u)e^{iut}\left(r'(u)/r(u)+it\right)\ ,$$

we have, using integration by parts,

$$\int_{1/t}^{x} \frac{m(u)}{u}\,r(u)e^{iut}du = \int_{1/t}^{x} \frac{m(u)\left(r(u)e^{iut}\right)'}{u\left(r'(u)/r(u)+it\right)}\,du$$
$$= \left[\frac{m(u)r(u)e^{iut}}{u\left(r'(u)/r(u)+it\right)}\right]_{u=1/t}^{x} - \int_{1/t}^{x} \frac{m'(u)r(u)e^{iut}}{u\left(r'(u)/r(u)+it\right)}\,du$$
$$+ \int_{1/t}^{x} \frac{m(u)r(u)e^{iut}}{u^2\left(r'(u)/r(u)+it\right)^2}\left(\left(\frac{r'(u)}{r(u)} + it\right)du+ud\left(\frac{r'(u)}{r(u)}\right)\right)$$
$$= S(x) - T - U(x) + V(x)\ ,$$

where

$$IS(x) = \frac{m(x)r(x)}{x\left(\left(r'(x)/r(x)\right)^2+t^2\right)}\left(\frac{r'(x)}{r(x)}\sin xt - t\cos xt\right)\ ,$$

$$T = \frac{tm(1/t)r(1/t)e^{i}}{r'(1/t)/r(1/t)+it}\ .$$

In case (i), $m(x) = 1$ and $r'(x)/r(x) \le A/x$ and then

$$W = \int_{1/t}^{\infty} \frac{r'(x)}{(r(x))^2}\,|IS(x)|dx \le \frac{A}{t}\int_{1/t}^{\infty} \frac{r'(x)}{xr(x)}\,dx \le A$$

https://doi.org/10.1017/S0004972700045895 Published online by Cambridge University Press

by (3). In case *(ii)*,

$$W \le \int_{1/t}^{x_0(t)} dx + \int_{x_0(t)}^{\infty} dx$$

$$\le A \int_{1/t}^{x_0(t)} \frac{r'(x)}{(r(x))^2} \frac{m(x)r(x)}{x\{r'(x)/r(x)\}} dx + \frac{A}{t} \int_{x_0(t)}^{\infty} \frac{r'(x)}{(r(x))^2} \frac{m(x)r(x)}{x} dx$$

$$\le A \int_{1/t}^{x_0(t)} \frac{m(x)}{x} dx + A \le A$$

by (6) and (7).

In each case *(i)* and *(ii)*,

$$X = \int_{1/t}^{\infty} \frac{r'(x)}{(r(x))^2} |T| dx \le m(1/t)r(1/t) \int_{1/t}^{\infty} \frac{r'(x)}{(r(x))^2} dx \le A ,$$

$$Y = \int_{1/t}^{\infty} \frac{r'(x)}{(r(x))^2} |U(x)| dx \le \frac{A}{t} \int_{1/t}^{\infty} \frac{|m'(u)|}{u} du \le A$$

by (6). Finally,

$$Z = \int_{1/t}^{\infty} \frac{r'(x)}{(r(x))^2} |V(x)| dx$$

$$\le \frac{A}{t} \int_{1/t}^{\infty} \frac{m(u)}{u^2} du + A \int_{1/t}^{\infty} \frac{r'(x)}{(r(x))^2} dx \int_{1/t}^{x} \frac{m(u)r(u)}{u\left((r'(u)/r(u))^2+t^2\right)} \left| d\left\{\frac{r'(u)}{r(u)}\right\}\right|$$

$$= A + AZ' .$$

In case *(i)*,

$$Z' \le \frac{A}{t} \int_{1/t}^{\infty} \left| d\left\{\frac{r'(u)}{r(u)}\right\}\right| \le \frac{A}{t} \frac{r'(1/t)}{r(1/t)} \le A .$$

In case *(ii)*,

$$Z' \le A \int_{1/t}^{x_0(t)} \frac{m(u)}{u(r'(u)/r(u))^2} \left| d\left\{\frac{r'(u)}{r(u)}\right\}\right| + \frac{A}{t^2} \int_{x_0(t)}^{\infty} \frac{m(u)}{u} \left| d\left\{\frac{r'(u)}{r(u)}\right\}\right|$$

$$\le At \frac{r(x_0(t))}{r'(x_0(t))} + \frac{A}{t^2 x_0(t)} \frac{r'(x_0(t))}{r(x_0(t))} \le A ,$$

https://doi.org/10.1017/S0004972700045895 Published online by Cambridge University Press

by (5), since $(1/t)r'(1/t)/r(1/t) \geq A > 0$ as $t \to 0$ by (4), $r'(x_0(t))/r(x_0(t)) = t$ and $tx_0(t) = x_0(t)r'(x_0(t))/r(x_0(t)) \geq A > 0$ as $t \to 0$.

Thus we get $R \leq W + X + Y + Z$, where R is bounded uniformly in t and then the required inequality (11) is proved. This completes the proof of the theorem.

2.2. Proof of Theorem 2. We put $m(1/t)\varphi(t) = g(t)$ and we can suppose $\varphi(\pi) = g(\pi) = 0$ without loss of generality. Then

$$A_n(x) = \frac{1}{\pi}\int_0^\pi \varphi(t)\cos nt\,dt = -\frac{1}{\pi}\int_0^\pi dg(t)\int_0^t \frac{\cos nu}{m(1/u)}\,du$$

where

$$\int_0^t \frac{\cos nu}{m(1/u)}\,du = \frac{\sin nt}{nm(1/t)} - \frac{1}{n}\int_0^t \left(\frac{1}{m(1/u)}\right)'\sin nu\,du\ .$$

It is sufficient to prove that

$$\int_1^\infty \frac{r'(x)}{(r(x))^2}\,dx\left|\sum_{n\leq x} r_n\int_0^t \frac{\cos nu}{m(1/u)}\,du\right| \leq A \quad \text{for all} \quad t > 0\ ,$$

since $g(t)$ is of bounded variation. The left side integral is not greater than

$$\frac{1}{m(1/t)}\int_1^\infty \frac{r'(x)}{(r(x))^2}\,dx\left|\sum_{n\leq x}\frac{r_n\sin nt}{n}\right| + \int_1^\infty \frac{r'(x)}{(r(x))^2}\,dx\left|\int_0^t \left(\frac{1}{m(1/u)}\right)'\left(\sum_{n\leq x}\frac{r_n\sin nu}{n}\right)du\right| = M + N\ .$$

M can be estimated similarly to the proof of Theorem 1 and we get, by (2) and (3),

$$M \leq \frac{A}{m(1/t)}\int_{1/t}^{x_0(t)} \frac{dx}{x} + A = \frac{A}{m(1/t)}\log\frac{x_0(t)}{1/t} + A \leq A\ ,$$

by (10). Using the lemma and (2), (3),

https://doi.org/10.1017/S0004972700045895 Published online by Cambridge University Press

$$N \leq \int_1^\infty \frac{r'(x)}{(r(x))^2}\,dx \left| \int_0^t \left(\frac{1}{m(1/u)}\right)' du \int_1^x \frac{r(v)\sin uv}{v}\,dv \right| + A$$

$$\leq \int_0^{1/t} dx \left| \int_0^t du \int_1^x dv \right| + \int_{1/t}^\infty dx \left| \int_0^t du \int_1^{1/t} dv \right|$$

$$+ \int_{1/t}^\infty dx \left| \int_0^t du \int_{1/t}^x dv \right| + A$$

$$= P + Q + R + A ,$$

where

$$P \leq \int_1^{1/t} \frac{r'(x)}{(r(x))^2}\,dx \int_1^x r(v)dv \int_0^t u\left(\frac{1}{m(1/u)}\right)' du \leq A$$

by (9), and

$$Q \leq \int_{1/t}^\infty \frac{r'(x)}{(r(x))^2}\,dx \int_0^t u\left(\frac{1}{m(1/u)}\right)' du \int_1^{1/t} r(v)dv \leq \frac{At}{m(1/t)r(1/t)} \int_1^{1/t} r(v)dv$$

$$\leq A .$$

We shall now estimate R . The inner integral of R is

$$\int_{1/t}^x \frac{r(v)\sin uv}{v}\,dv = I \int_{1/t}^x \frac{r(v)e^{iuv}}{v}\,dv$$

and, by (12),

$$\int_{1/t}^x \frac{r(v)e^{iuv}}{v}\,dv = \left[\frac{r(v)e^{iuv}}{v\left(r'(v)/r(v)+iu\right)}\right]_{v=1/t}^x + \int_{1/t}^x \frac{r(v)e^{iuv}}{v^2\left(r'(v)/r(v)+iu\right)}\,dv$$

$$+ \int_{1/t}^x \frac{r(v)e^{iuv}}{v\left(r'(v)/r(v)+iu\right)^2}\,d\left(\frac{r'(v)}{r(v)}\right)$$

$$= S(x) - T + U(x) + V(x) .$$

Now

https://doi.org/10.1017/S0004972700045895 Published online by Cambridge University Press

$$W = \int_{1/t}^{\infty} \frac{r'(x)}{(r(x))^2}\,dx \left|\int_0^t \left(\frac{1}{m(1/u)}\right)' \, IS(x)du\right|$$

$$= \int_{1/t}^{\infty} \frac{r'(x)}{xr(x)}\,dx \left|\int_0^t \left(\frac{1}{m(1/u)}\right)' \frac{(r'(x)/r(x))\sin ux - u\cos ux}{(r'(x)/r(x))^2+u^2}\,du\right|$$

$$\le \int_{1/t}^{\infty} \left(\frac{r'(x)}{r(x)}\right)^2 \frac{dx}{x} \left|\int_0^t \left(\frac{1}{m(1/u)}\right)' \frac{\sin ux}{(r'(x)/r(x))^2+u^2}\,du\right|$$

$$+ \int_{1/t}^{\infty} \frac{r'(x)}{xr(x)}\,dx \left|\int_0^t \left(\frac{1}{m(1/u)}\right)' \frac{u\cos ux}{(r'(x)/r(x))^2+u^2}\,du\right|$$

$$= W_1 + W_2$$

where

$$W_1 \le \int_{1/t}^{\infty} dx \left|\int_0^{1/x} du\right| + \int_{1/t}^{\infty} dx \left|\int_{1/x}^{t} du\right|$$

$$\le \int_{1/t}^{\infty} dx \int_0^{1/x} \left(\frac{1}{m(1/u)}\right)' u du + \int_{1/t}^{\infty} \frac{m'(x)}{(m(x))^2}\, xdx \left|\int_{1/x}^{t'} \sin uxdu\right|$$

$$\le \int_{1/t}^{\infty} dx \int_x^{\infty} \frac{m'(v)}{(m(v))^2}\frac{dv}{v} + \int_{1/t}^{\infty} \frac{m'(x)}{(m(x))^2}\,dx \le A$$

and

$$W_2 \le \int_{1/t}^{\infty} \frac{r(x)}{xr'(x)}\,dx \int_0^{1/x} \left(\frac{1}{m(1/u)}\right)' u du + \int_{1/t}^{\infty} \frac{r(x)}{xr'(x)} \frac{m'(x)}{(m(x))^2}\,dx$$

$$\le A \int_{1/t}^{\infty} \frac{m'(u)}{(m(u))^2}\,du \le A$$

by (4). Further we have

$$X = \int_{1/t}^{\infty} \frac{r'(x)}{(r(x))^2}\,dx \left|\int_0^t \left(\frac{1}{m(1/t)}\right)' IT du\right| \le A\,\frac{r(1/t)}{m(1/t)} \int_{1/t}^{\infty} \frac{r'(x)}{(r(x))^2}\,dx \le A \,,$$

https://doi.org/10.1017/S0004972700045895 Published online by Cambridge University Press

$$Y = \int_{1/t}^{\infty} \frac{r'(x)}{(r(x))^2}\,dx \left|\int_0^t \left(\frac{1}{m(1/u)}\right)' IU(x)dx\right|$$

$$\le A \int_{1/t}^{\infty} \frac{r'(x)}{(r(x))^2}\,dx \left|\int_{1/t}^{x} \frac{r(v)}{v^2}\,dv \int_0^t \left(\frac{1}{m(1/u)}\right)' \frac{(r'(v)/r(v))\sin uv - u\cos uv}{(r'(v)/r(v))^2+u^2}\,du\right|$$

$$\le A \int_{1/t}^{\infty} \frac{r'(x)}{(r(x))^2}\,dx \int_{1/t}^{x} \frac{r(v)}{v^2}\,dv \left\{\int_0^{1/v} \left(\frac{1}{m(1/u)}\right)' \frac{uv}{r'(v)/r(v)}\,du\right.$$

$$+ \left|\int_{1/v}^{t} \left(\frac{1}{m(1/u)}\right)' \frac{(r'(v)/r(v))\sin uv}{(r'(v)/r(v))^2+u^2}\,dv\right| + \int_0^{1/v} \left(\frac{1}{m(1/u)}\right)' \frac{udu}{(r'(v)/r(v))^2}$$

$$\left. + \left|\left(\int_{1/v}^{r(v)/r(v)} + \int_{r'(v)/r(v)}^{t}\right)\left(\frac{1}{m(1/u)}\right)' \frac{u\cos uv}{(r'(v)/r(v))^2+u^2}\,du\right|\right\}$$

$$= Y_1 + Y_2 + Y_3 + |Y_4+Y_5|$$

where

$$Y_1 + Y_3 \le A \int_{1/t}^{\infty} \frac{dv}{vr'(v)/r(v)} \int_v^{\infty} \frac{m'(w)}{(m(w))^2}\frac{dw}{w} \le A \int_{1/t}^{\infty} \frac{m'(w)}{(m(w))^2}\,dw \le A$$

by (4). By (9) and the fact that $u/((r'(u)/r(u))^2+u^2) \uparrow$ as $u \uparrow$, $(1/v < u < r'(v)/r(v))$, and using the second mean value theorem for the integral concerning u, we get

$$Y_2 + |Y_4| \le A \int_{1/t}^{\infty} \frac{1}{v^2}\frac{v^2 m'(v)}{(m(v))^2}\frac{dv}{v(r'(v)/r(v))} \le A \int_{1/t}^{\infty} \frac{m'(v)}{(m(v))^2}\,dv \le A\,.$$

On the other hand, by (9) and the fact that $u/((r'(v)/r(v))^2+u^2) \downarrow$ as $u \uparrow$, $(r'(u)/r(u) < u < t)$, and again using the second mean value theorem for the integral concerning u, then

$$|Y_5| \le A \int_{1/t}^{\infty} \frac{r'(x)}{(r(x))^2}\,dx \int_{1/t}^{x} \frac{r(v)}{v^2} \left[\left(\frac{1}{m(1/v)}\right)'\right]_{u=r'(v)/r(v)} \frac{dv}{vr'(v)/r(v)}\,.$$

Since $r'(v)/r(v) > A/v$,

$$\left[\left(\frac{1}{m(1/u)}\right)'\right]_{u=r'(v)/r(v)} \le \left[\left(\frac{1}{m(1/u)}\right)'\right]_{u=A/v} = A\frac{m'(Av)}{(m(Av))^2}v^2$$

and then

$$|Y_5| \le A \int_{1/t}^{\infty} \frac{m'(Av)}{\{m(Av)\}^2}\, dv \le A .$$

Thus we have proved that $Y \le A$. Finally

$$Z = \int_{1/t}^{\infty} \frac{r'(x)}{\{r(x)\}^2}\, dx \left|\int_0^t \left(\frac{1}{m(1/u)}\right)' IV(x)du\right|$$

$$\le \frac{A}{m(1/t)} \int_{1/t}^{\infty} \frac{r'(x)}{\{r(x)\}^2}\, dx \int_{1/t}^{x} \frac{r(v)}{v\{r'(v)/r(v)\}^2} \left|d\left(\frac{r'(v)}{r(v)}\right)\right|$$

$$\le \frac{A}{m(1/t)} \int_{1/t}^{\infty} \frac{1}{v\{r'(v)/r(v)\}^2} \left|d\left(\frac{r'(v)}{r(v)}\right)\right|$$

$$\le \frac{A}{m(1/t)} \int_{1/t}^{\infty} \frac{1}{v} \left(\frac{r''(v)r(v)}{\{r'(v)\}^2} - 1\right) dv \le A ,$$

by (8). Since $R \le W + X + Y + Z$, R is bounded and then the theorem is proved.

References

[1] R. Mohanty, "On the absolute Riesz summability of Fourier series and allied series", *Proc. London Math. Soc.* (2) 52 (1951), 295-320.

[2] Shin-ichi Izumi and Masako Satô (= Masako Izumi), "Integrability of trigonometric series. I", *Tôhoku Math. J.* (2) 6 (1954), 258-263.

Department of Mathematics,
Institute of Advanced Studies,
The Australian National University,
Canberra, ACT.