THE EFFICIENCY OF $PSL(2,p)^3$ AND OTHER DIRECT PRODUCTS OF GROUPS

by C. M. CAMPBELL, I. MIYAMOTO, E. F. ROBERTSON and P. D. WILLIAMS

(Received 30 October, 1995; revised 1 October, 1996)

1. Introduction. A finite group G is *efficient* if it has a presentation on n generators and $n+m$ relations, where m is the minimal number of generators of the Schur multiplier $M(G)$ of G. The *deficiency of a presentation* of G is $r-n$, where r is the number of relations and n the number of generators. The *deficiency* of G, def G, is the minimum deficiency over all finite presentations of G. Thus a group is efficient if def $G = m$. Both the problem of efficiency and the converse problem of inefficiency have received considerable attention recently; see for example **[1]**, [3], **[14]** and **[15]**.

In particular, in response to a question of Wiegold **[19]** concerning the efficiency of direct powers of groups, several papers have appeared; see **[4]**, **[5]**, **[6]**, **[12]** and **[13]**. An efficient presentation for $PSL(2,5)^2$ was given by Kenne in **[12]** and for $PSL(2,5)^3$ by Campbell, Robertson and Williams in **[5]**. In **[6]**, direct squares of the projective special linear group $PSL(2,p)$ are shown to be efficient for all primes p. In addition to considering direct powers, the efficiency of direct products of some simple groups has been investigated; see **[11]**, **[13]**, where efficient presentations are given for $PSL(2,5)\times A_6$ and $PSL(2,5)\times A_7$.

In this paper we give a new method for obtaining efficient presentations of direct products. This method allows us to give a simpler proof to that given in **[6]** that $PSL(2,p)\times PSL(2,p)$ is efficient. The power of this new method is illustrated, in this case, by leading naturally to an efficient presentation for $SL(2,p)\times PSL(2,p)$. The main result of this paper uses this method to obtain an efficient presentation for $PSL(2,p)\times PSL(2,p)\times PSL(2,p)$, where p is a prime. This leaves us a long way short of answering Wiegold's question in **[19]** which, in essence, asks whether $PSL(2,p)^n$ is efficient for all n. However, it is worth noting that the approach given here gives an efficient presentation for $PSL(2,p)^3$ on six generators that looks more promising than earlier methods which relied heavily on obtaining two-generator presentations. We give two further applications of the method, showing that $A_6\times PSL(2,p)$ and $PSL(2,p)\times J$ are efficient for all primes p, $p\geq 5$, where $J\cong PSL(2,q)\times C_2$ or $J\cong PGL(2,q)$ depending on q. Finally we give an efficient presentation for $PSL(2,5)^4$.

For any group G, we let G' denote the commutator subgroup of G and let $Z(G)$ denote the centre of G. The group H is a *central extension* of G if there is a subgroup $Z\leq Z(H)$ with $H/Z\cong G$. H is *irreducible* if there is no $L\leq H$ with $H=ZL$. In the case that $Z\leq H'$, H is called a *stem extension* of G and a stem extension with $Z\cong M(G)$ is called a *covering group* of G. For perfect groups G, a covering group is uniquely determined. Moreover, if G is perfect and H is a central extension of G, then H' is a perfect irreducible central extension of G with $H'/(Z\cap H')\cong G$; that is, H' is a stem extension of G. For further information on stem extensions see **[17]**.

The Schur multiplier of the direct product $G\times H$ may be calculated by the Schur-Künneth formula $M(G\times H)=M(G)\times M(H)\times(G\otimes H)$; see **[18]**. The Schur multipliers of $SL(2,p)$, $PSL(2,p)$, A_6 and $PGL(2,p)$ are, respectively, trivial, C_2, C_6 and C_2 (where we assume p is prime, $p \geq 5$). Consequently, we have the following results.

Group	Multiplier
$SL(2,p) \times PSL(2,p)$	C_2
$PSL(2,p)^n$	C_2^n
$PSL(2,p) \times A_6$	$C_2 \times C_6$
$PSL(2,p) \times PGL(2,q)$	$C_2 \times C_2$

2. A new approach to efficient presentations. In many of the papers cited above, the technique used to obtain an efficient presentation of $G \times H$ is to write down the standard presentation of $G \times H$ from (not necessarily efficient) presentations of G and H that have small deficiency. This presentation of $G \times H$, however, usually has too many relations (including many commutator relations) to be efficient. Reducing the deficiency is often achieved by showing $G \times H$ is generated by a smaller set of elements of the form gh, $g \in G$, $h \in H$, and rewriting the presentation in terms of the new generators. For example, see **[5]** for details.

Here we introduce a new approach to obtaining an efficient presentation of $G \times H$ that leads naturally to a presentation in terms of the original generators of G and H. We consider G and H each given as a homomorphic image of a triangle group and, from these presentations of G and H, we construct a presentation with a small number of relations defining a group L. The group L has a central subgroup D with $L/D \cong G \times H$. In the cases we consider L is perfect, and we compute D by using the fact that D is a homomorphic image of $M(G \times H)$.

The referee has pointed out that there are other ways of computing D, for instance using computations of generators of second homotopy modules (see **[2]**) in combination with the main theorem of **[9]**.

Our technique is illustrated in the first theorem which gives, as a corollary, an alternative proof of Theorem *B* and Corollary *C* of **[6]**. It is based on the following result.

LEMMA 2.1. *Let G be a group with $a, b \in G$ satisfying $a^\epsilon = (a^m b^\delta)^n$, where ϵ, $\delta = \pm 1$, m and n are integers. Then $\langle a, b\rangle$ is a cyclic subgroup of G and $a^{\epsilon - mn} = b^{\delta n}$.*

Proof. The relation shows that $[a, a^m b^\delta] = 1$, which implies $[a, b] = 1$. Hence $\langle a, b\rangle$ is abelian. Note $a = (a^m b^\delta)^{\epsilon n}$ and $b^\delta = (a^m b^\delta)^{1-\epsilon mn}$, so that $b = (a^m b^\delta)^{\delta(1-\epsilon mn)}$. Thus $\langle a, b\rangle = \langle a^m b^\delta\rangle$. The second part of the assertion follows from rewriting the given relation. □

Throughout the rest of the paper we shall make use of Sunday's presentation of $PSL(2,p)$ in **[16]**; namely

$$PSL(2,p) = \langle a, b \mid a^2 = b^p = (ab)^3 = (ab^4ab^{(p+1)/2})^2 = 1\rangle.$$

We also require the following lemma.

LEMMA 2.2. *If H is the group given by the presentation*

$$H = \langle a, b \mid a^2 = s,\ b^p = (ab)^3 = u,\ (ab^4ab^{(p+1)/2})^2 = t,\ s, u, t \text{ central involutions}\rangle$$

for p prime, $p \geq 5$, *then* $s = t$.

Proof. $H/\langle u\rangle$ is a perfect group and is a stem extension of $PSL(2,p)$ with presentation

$$\langle a, b \mid a^2 = s,\ b^p = (ab)^3 = 1,\ (ab^4ab^{(p+1)/2})^2 = t,\ s, t \text{ central involutions}\rangle.$$

It is $SL(2,p)$ rather than $PSL(2,p)$ since the generators for $SL(2,p)$, given by

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

satisfy these relations. Using these matrix generators, it is immediate that $s = t$ in $H/\langle u\rangle$. Therefore $s \equiv t \pmod u$ in H and so either $t = s$ or $t = su$. In the latter case a presentation for H is

$$\langle a, b \mid a^2 = s,\ b^p = (ab)^3 = u,\ (ab^4ab^{(p+1)/2})^2 = su,\ s, u \text{ central involutions}\rangle$$

and, factoring out by $\langle s\rangle$, we obtain a perfect group with presentation

$$\langle a, b \mid a^2 = 1,\ b^p = (ab)^3 = (ab^4ab^{(p+1)/2})^2 = u,\ u \text{ central involution}\rangle.$$

Since u is non-trivial, this group is not $PSL(2,p)$. Nor is it $SL(2,p)$ since a would be the unique central involution, forcing the group to be abelian. Thus $s = t$ in H as required. □

THEOREM 2.3. *Let G be the group presented by*

$$\langle a, b, x, y \mid y^{\epsilon}(y^{(p-\epsilon)/3}ab)^3 = ab(abx)^2 = b(b^{(p-1)/2}x)^2 = xy(xya)^2 = (ab^4ab^{(p+1)/2})^2(xy^4xy^{(p+1)/2})^2a^2 = 1\rangle,$$

where p is prime, $p \geq 5$ *and* $\epsilon \in \{1, -1\}$ *is such that* $p \equiv \epsilon \pmod 3$. *Then* $G \cong SL(2,p) \times PSL(2,p)$.

Proof. Let $H = \langle a, b\rangle$, $K = \langle x, y\rangle$. From Lemma 2.1, the relations of G imply

$$[ab, y] = [ab, x] = [b, x] = [a, xy] = 1, \qquad (ab)^{-3} = y^p = x^2 = b^{-p}, \qquad (xy)^3 = a^{-2}.$$

From the first four relations we obtain $[a,x] = [a,y] = [b,y] = 1$, and so $[H,K] = 1$, H, $K \lhd G$, and G is a central product of H and K. Let $D = \langle a^2, x^2, (xy^4xy^{(p+1)/2})^2\rangle$. Clearly $D \leq H \cap K \leq Z(G)$ but, in fact, $D = Z(G)$ as $G/D \cong PSL(2,p) \times PSL(2,p)$ has trivial centre. Moreover, as G/D is finite then G' is finite. It follows that G is finite since G is perfect. By [**19**], D is an epimorphic image of $M(PSL(2,p) \times PSL(2,p))$. Note that D is not trivial since $PSL(2,p) \times PSL(2,p)$ cannot have a deficiency 1 presentation. Hence D is isomorphic to C_2 or $C_2 \times C_2$. Now $G/H \cong K/D \cong PSL(2,p) \cong H/D \cong G/K$ and so H has a presentation of the following form:

$$H = \langle a, b \mid a^2 = s,\ b^p = (ab)^3 = u,\ (ab^4ab^{(p+1)/2})^2 = t,\ s, u, t \text{ central involutions}\rangle.$$

From Lemma 2.2 we have $s = t$ in H. Thus, K has a presentation of the form

$$K = \langle x, y \mid x^2 = y^p = u,\ (xy)^3 = s,\ (xy^4xy^{(p+1)/2})^2 = 1,\ s,\ u \text{ central involutions}\rangle.$$

Now H and K are central extensions of $PSL(2,p)$ and so H' and K' are perfect extensions of $PSL(2,p)$ giving

$$PSL(2,p) \cong H'/H' \cap D \cong K'/K' \cap D,$$

where $D = \langle s, u\rangle$. Since a is trivial in H/H' and x^2 is trivial in K/K' we now have $s \in H'$ and $u \in K'$. We claim that $H' \cong SL(2,p)$. Suppose, by way of contradiction, $H' \cong PSL(2,p)$. Since $PSL(2,p) \cong H'/H' \cap D$ and $s \in H' \cap D$ we must have $s = 1$. Now K' is either $PSL(2,p)$ or $SL(2,p)$. If K' is $PSL(2,p)$, then $u = 1$ and hence $D = 1$, which is a contradiction, or K' is $SL(2,p)$ in which case $K'/\langle xy^4xy^{(p+1)/2}\rangle \cong PSL(2,p)$, which is again a contradiction. Therefore $H' \cong SL(2,p)$.

Further, as above, K' cannot be $SL(2,p)$ so that $K' \cong PSL(2,p)$ and $u = 1$, $s \neq 1$. Now

$$H = \langle a, b \mid a^2 = (ab^4ab^{(p+1)/2})^2 = s,\ b^p = (ab)^3 = 1,\ s \text{ a central involution}\rangle,$$

which is perfect and so $H \cong SL(2,p)$. Also

$$K = \langle x, y \mid x^2 = y^p = (xy^4xy^{(p+1)/2})^2 = 1,\ (xy)^3 = s,\ s \text{ a central involution}\rangle,$$

which is not perfect and $PSL(2,p) \cong K/\langle s\rangle \cong K'$. Again, from K/K', we see that $(xy)^3$ has order 2 in K/K' and so $(xy)^3$ is not trivial in K/K'. Therefore $(xy)^3 \notin K'$, giving $s \notin K'$.

Now $D = \langle s\rangle = H \cap K$ and so $H \cap K' = 1$. Thus

$$SL(2,p) \times PSL(2,p) \cong H \times K' \leq G$$

and

$$G/\langle s\rangle \cong PSL(2,p) \times PSL(2,p).$$

Hence $G \cong SL(2,p) \times PSL(2,p)$. □

COROLLARY 2.4. *For p a prime, $p \geq 5$, $PSL(2,p) \times PSL(2,p)$ is efficient.*

Proof. Add the relation $a^2 = 1$ to the relations defining G in Theorem 2.3. □

The method we have used yields an efficient presentation for $SL(2,p) \times PSL(2,p)$ on four generators. We may now rewrite this presentation on two generators as shown in the next result.

COROLLARY 2.5. *The group $SL(2,p) \times PSL(2,p)$ may be presented as*

$$\langle u, w \mid u^{-3}w^p(u^{p-3}w^{p+2})^2 = (w^{p-6}u^{p-12}wu^{-(p+3)/2})^2(u^pw^2)^2 = u^p(u^pw^p)^2 = 1\rangle$$

and $PSL(2,p) \times PSL(2,p)$ as

$$\langle u, w \mid u^p(u^pw^p)^2 = u^{-3}w^p(u^{p-3}w^{p+2})^2 = (w^{p-6}u^{p-12}wu^{-(p+3)/2})^2 = (u^pw^2)^2 = 1\rangle,$$

where p is a prime, $p \geq 5$.

Proof. In the presentation given in Theorem 2.3, put $u = y^{(p-\epsilon)/3}ab$ and $w = b^{(p-1)/2}x$, so that $y = u^{-\epsilon 3}$, $ab = u^{\epsilon p}$, $b = w^{-2}$, $x = w^p$ and $a = u^{\epsilon p}w^2$. Rewriting the presentation in Theorem 2.3 in terms of these generators gives the required result.

Finally, adding the relation $(u^p w^2)^2 = 1$ (corresponding to $a^2 = 1$) gives the presentation of $PSL(2,p) \times PSL(2,p)$. □

THEOREM 2.6. *$SL(2,p) \times PSL(2,p)$ is efficient for all primes p.*

Proof. Since $SL(2,2) \cong PSL(2,2) \cong D_6$, an efficient presentation of $SL(2,2) \times SL(2,2)$ is given in [5] or [6]. For $p = 3$, it may be verified using GAP or MAGMA that the following is a presentation of $SL(2,3) \times PSL(2,3)$:

$$\langle x, y \mid x^2y^4x^4y^8 = (x^3y)^3y^9 = yx^{-2}y^{-1}x^{-2}y^3x^{-2} = 1 \rangle.$$

For $p \geq 5$ the result follows from Theorem 2.3. □

3. An efficient presentation for $PSL(2,p)^3$. We use an extension of the method illustrated in the previous section. We consider G, H and K, each given as a homomorphic image of a triangle group, and construct a presentation with a small number of relations defining a group L with a central subgroup D such that $L/D \cong G \times H \times K$. We compute D in the case where L is perfect by using the fact that it is a homomorphic image of $M(G \times H \times K)$. In this section we apply the method to show that the direct cube of $PSL(2,p)$ is efficient for $p \geq 5$. To do this we first generalize Lemma 2.1.

LEMMA 3.1. *Let G be a group and $a, b, c \in G$ satisfy the relations*

$$a(ab^{-1})^2 = 1, \qquad c^\gamma = (c^k ab^{-1})^6,$$

where $\gamma = \pm 1$ and k is an integer. Then $\langle a, b, c \rangle$ is cyclic and the relations $b^2 = a^3 = c^{(6k-\gamma)}$ hold in G.

Proof. From the proof of Lemma 2.1 the relation $a(ab^{-1})^2 = 1$ gives $a = (ab^{-1})^{-2}$, $b = (ab^{-1})^{-3}$. Furthermore, $\langle a, b \rangle = \langle ab^{-1} \rangle$ and $\langle c, ab^{-1} \rangle = \langle c^k ab^{-1} \rangle$. This shows that $\langle a, b, c \rangle$ is cyclic. Since $\langle a, b, c \rangle$ is abelian, $c^\gamma = (c^k ab^{-1})^6$ gives $c^{\gamma - 6k} = (ab^{-1})^6$ and the result follows. □

THEOREM 3.2. *The group $PSL(2,p)^3$ is efficient, p prime, $p \geq 5$.*

Proof. Consider the group G generated by $\{a, b, x, y, u, v\}$ subject to the relations

$$xy(xya^{-1})^2 = 1, \tag{1}$$

$$v^\epsilon(v^{(p-\epsilon)/6}xya^{-1})^6 = 1, \tag{2}$$

$$uv(uvx^{-1})^2 = 1, \tag{3}$$

$$b^\epsilon(b^{(p-\epsilon)/6}uvx^{-1})^6 = 1, \tag{4}$$

$$ab(abu^{-1})^2 = 1, \tag{5}$$

$$y^\epsilon(y^{(p-\epsilon)/6}abu^{-1})^6 = 1, \tag{6}$$

$$xy(xyu^{-1})^2 = 1, \tag{7}$$

$$b^\epsilon(b^{(p-\epsilon)/6}xyu^{-1})^6 = 1, \tag{8}$$

$$(ab^4ab^{(p+1)/2})^{-2}(xy^4xy^{(p+1)/2})^2(uv^4uv^{(p+1)/2})^2 = 1, \tag{9}$$

where $\epsilon \in \{1, -1\}$, $p \equiv \epsilon \pmod 3$.

Applying Lemma 3.1 to each of the pairs of relations (i), ($i+1$), where $i = 1, 3, 5, 7$, we see that the following relations hold in G:

$$[a, xy] = [a, v] = [xy, v] = 1, \qquad [b, x] = [b, uv] = [x, uv] = 1,$$
$$[ab, y] = [ab, u] = [y, u] = 1, \qquad [b, xy] = [b, u] = [xy, u] = 1,$$
$$v^p = a^2 = (xy)^3 = b^p = x^2 = (uv)^3 = y^p = u^2 = (ab)^3.$$

The first 12 relations above show that any element from one of the pairs $\{a, b\}$, $\{x, y\}$, $\{u, v\}$ commutes with each of the four generators in the other two pairs.

Let $D = \langle a^2, (ab^4ab^{(p+1)/2})^2, (xy^4xy^{(p+1)/2})^2 \rangle$. Clearly $G/D \cong PSL(2,p)^3$ and so $D = Z(G)$. A straightforward matrix reduction shows that G is perfect. Then $D \le G'$ and so D, being a homomorphic image of $M(PSL(2,p)^3)$, is elementary abelian of order dividing 8. Letting $H = \langle a, b \rangle$, $K = \langle x, y \rangle$ and $L = \langle u, v \rangle$ we see that $[H, K] = [H, L] = [K, L] = 1$ and $H, K, L \lhd G$. We have $H = H'D$, $K = K'D$, $L = L'D$ and, arguing as in the proof of Theorem 2.3, each of H', K' and L' is isomorphic to $SL(2,p)$ or $PSL(2,p)$. Also $a^2 = (ab^4ab^{(p+1)/2})^2 \in Z(H')$, $x^2 = (xy^4xy^{(p+1)/2})^2 \in Z(K')$ and $u^2 = (uv^4uv^{(p+1)/2})^2 \in Z(L')$. Now as $a^2 = x^2 = u^2$ we see from (9) that $a^2 = 1$. Hence $Z(G) = 1$. □

We may also give an efficient presentation of $PSL(2,p)^3$, where $p \ge 5$, on two generators as follows. In the presentation given in Theorem 3.2, let $c = v^{(p-\epsilon)/6}xya^{-1}$. Then $v = c^{-\epsilon 6}$ and $xya^{-1} = c^{\epsilon p}$. As $xy(xya^{-1})^2 = 1$ we obtain $xy = c^{-\epsilon 2p}$ and $a = c^{-\epsilon p}xy = c^{-\epsilon 3p}$. Put $w = b^{(p-\epsilon)/6}uvx^{-1}$, so that $b = w^{-\epsilon 6}$ and $uvx^{-1} = w^{\epsilon p}$. Use $uv(uvx^{-1})^2 = 1$ to obtain $uv = w^{-\epsilon 2p}$ and $x = w^{-\epsilon p}uv = w^{-\epsilon 3p}$. Hence $y = w^{\epsilon 3p}c^{-\epsilon 2p}$ and $u = w^{-\epsilon 2p}c^{\epsilon 6}$. Now rewrite the presentation in Theorem 3.2 in terms of the generators c and w and apply Tietze transformations to obtain the following defining relations.

For $p \equiv 1 \pmod 6$ the defining relations become

$$c^{2p}(c^{2p+6}w^{-2p})^2 = 1,$$
$$w^{-6}(w^{p+1}c^{-2p-6})^6 = 1,$$
$$c^{-3p}w^{-6}(c^{-3p}w^{-6}c^{-6}w^{2p})^2 = 1,$$
$$w^{3p}c^{-2p}((w^{3p}c^{-2p})^{(p-1)/6}c^{-3p}w^{-6}c^{-6}w^{2p})^6 = 1,$$
$$(w^{-2p+24}c^{3p-18}w^{p+3}c)^2(c^{-2p}(w^{3p}c^{-2p})^3c^{-2p}(w^{3p}c^{-2p})^{(p+1)/2})^2 = 1.$$

For $p \equiv -1 \pmod 6$ similar defining relations may be given.

Since $PSL(2,2)$ is isomorphic to the dihedral group of order six, an efficient presentation of $PSL(2,2)^3$ is given in [**5**]. An efficient presentation of $PSL(2,3)^3 \cong A_4^3$ is given in [**7**]. Combining these results with Theorem 3.2 we have the following result.

THEOREM 3.3. *For all primes p, $PSL(2,p)^3$ is efficient.*

4. Some other applications. We further illustrate the power of the method by proving that $PSL(2,p) \times A_6$ is efficient for $p \ge 5$. We also show that, for all primes $p \ge 5$ and certain primes $q \ge 5$, $PSL(2,p) \times PGL(2,q)$ is efficient. Other applications are given in [**3**]. For $PSL(2,p) \times A_6$ we make use of the following presentation of A_6; see [**8**].

$$A_6 = \langle a, b \mid a^2 = b^4 = (ab)^5 = (ab^2)^5 = 1 \rangle.$$

THEOREM 4.1. *For any prime* $p \geq 5$, $PSL(2,p) \times A_6$ *is efficiently presented as*

$$\langle a,b,x,y \mid ab((ab)^2x)^2 = b(bxy)^3 = y(y^{(p-1)/2}a)^2 = x^2 = xy(xya)^2a^{-2} = (ab^2)^5(xy^4xy^{(p+1)/2})^2a^2 = 1\rangle.$$

Proof. Let G be the group presented in the statement of the theorem. From Lemma 2.1 we obtain immediately that the relations

$$[x,ab] = [b,xy] = [a,y] = (ab)^5 = b^4(xy)^3 = y^pa^2 = 1$$

hold in G. We rewrite $xy(xya)^2a^{-2} = 1$ as $axya^{-1} = (xy)^{-2}$ and raise this to the third power to obtain $a(xy)^3a^{-1} = (xy)^{-6}$. Hence

$$ab^4a^{-1} = b^{-8}. \tag{10}$$

Also $yxyxyaxa^{-1} = 1$ and so, using $b^4(xy)^3 = 1$, we obtain $axa^{-1} = b^4x$. Squaring this relation gives

$$(b^4x)^2 = 1. \tag{11}$$

Hence $b^4xb^4xy = y$ and, since $[b,xy] = 1$ and $x^2 = 1$, we obtain

$$yb^4y^{-1} = b^{-4}.$$

Therefore $[y^2, b^4] = 1$. Then $b^4y^{p-1}b^{-4} = y^{p-1}$ and, as $y^pa^2 = 1$, we have $b^4y^{-1}a^{-2}b^{-4} = y^{-1}a^{-2}$, which implies $a^{-2}b^{-4}a^2 = yb^{-4}y^{-1} = b^4$. From (10) we have $a^2b^4a^{-2} = b^{16}$ and therefore $b^{20} = 1$.

Using (11) and $[y^2, b^4] = 1$ we see that $[b^4, (xy^4xy^{(p+1)/2})^2] = 1$. Also $a^2b^4 = b^{-4}a^2$. From (10) we have $ab^8 = b^4a$ and so $b^4(ab^2) = (ab^2)b^8$, which shows that $b^4(ab^2)^5 = (ab^2)^5b^8$. From these results we see that the relation

$$b^4(ab^2)^5(xy^4xy^{(p+1)/2})^2a^2 = b^4$$

implies $(ab^2)^5(xy^4xy^{(p+1)/2})^2a^2b^{-8} = b^4$, so that $b^{12} = 1$. As $b^{20} = 1$ we deduce that $b^4 = 1$. Hence $[a,x] = [x,b] = [y,b] = (xy)^3 = 1$.

Let $D = \langle a^2, (xy^4xy^{(p+1)/2})^2\rangle$, $K = \langle a,b\rangle$ and $H = \langle x,y\rangle$. Then $G/D \cong PSL(2,p) \times A_6$ and, since G is perfect, D is a homomorphic image of $C_2 \times C_6$. Note that $a^2 \in D \leq Z(G)$, so that $a^{12} = 1$. Now $(ab^2)^5 = a^{-2}(xy^4xy^{(p+1)/2})^{-2}$ and we obtain

$$(ab^2)^2a(ab^2)^{-2} = a^{-6}(xy^4xy^{(p+1)/2})^{-2}b^{-2}. \tag{12}$$

We square relation (12) to obtain $a^2 = (xy^4xy^{(p+1)/2})^{-4}$. Therefore

$$(ab^2)^5 = (xy^4xy^{(p+1)/2})^2.$$

Thus D is a cyclic group generated by $(xy^4xy^{(p+1)/2})^2$. However, $(xy^4xy^{(p+1)/2})^2 \in H'$ and $D \leq Z(H)$. Furthermore, $G/K \cong H/D \cong PSL(2,p)$. Thus H is $PSL(2,p)$ or $SL(2,p)$ but, as $x^2 = 1$, we conclude that H is $PSL(2,p)$. Therefore $(xy^4xy^{(p+1)/2})^2 = 1$ and $G \cong PSL(2,p) \times A_6$. □

We now show that an efficient presentation of $PSL(2,p) \times A_6$ exists for any prime p. It remains to show that $PSL(2,2) \times A_6$ and $PSL(2,3) \times A_6$ are efficient. It may be proved using GAP or MAGMA that

$$PSL(2,2) \times A_6 = \langle r,s \mid s^4 = rs^8r^{-1}, (r^5s^3)^2 = s^6, (s^{-3}r^4)^2(r^4s^3)^5 = 1\rangle$$

https://doi.org/10.1017/S0017089500032195 Published online by Cambridge University Press

and

$$PSL(2,3)\times A_6 = \langle a,b,x,y \mid ab((ab)^2x)^2 = b(bxy)^3 = y(ya)^2 = x^2 = xy(xya)^2a^{-2} = (ab^2)^5(xyxy^2)^2a^2 = 1\rangle.$$

Both of these are efficient presentations which were found using an approach similar to that described in [**5**].

Combining these computational results with Theorem 4.1 gives the following theorem.

THEOREM 4.2. *For any prime p, $PSL(2,p)\times A_6$ is efficient.*

We note that the case $p=5$ of Theorem 4.2 was proved by Jamali in [**11**].

We conclude our examples using the new method by proving that $PSL(2,p)\times PGL(2,q)$ is efficient for all primes p, with $p\geq 5$, and certain primes q. We make use of the presentation

$$\langle a,b \mid a^2 = b^q = (ab^2)^4 = (abab^2)^3 = 1\rangle,$$

which presents $PGL(2,q)$ whenever the congruence $x^2\equiv 2 \pmod q$ has no solution in the integers and presents $C_2\times PSL(2,q)$ otherwise (details of this result are provided in [**10**]). For now, we denote this group by J and note that, in both cases, the multiplier is C_2.

THEOREM 4.3. *The group $PSL(2,p)\times J$ may be efficiently presented as*

$$\langle a,b,x,y \mid ab^2(ab^2xy)^3 = xy(xya)^2 = y(y^{(p-1)/2}a)^2 = b(b^{(q-1)/2}x)^2 = x^2 = (xy^4xy^{(p+1)/2})^2a^2(abab^2)^3 = 1\rangle,$$

where p, q are primes ≥ 5.

Proof. From Lemma 2.1 we have that the relations

$$[ab^2,xy] = [a,xy] = [a,y] = [b,x] = (ab^2)^4(xy)^3 = (xy)^3a^2 = y^pa^2 = b^q = 1$$

hold in G. Then $[a,x]=[b^2,y]=1$ but, as $b=(b^2)^{(q-1)/2}$, we have $[y,b]=1$. Letting $H=\langle x,y\rangle$, $K=\langle a,b\rangle$ we see that $[H,K]=1$, H, $K\lhd G$ and $G=HK$. Let $D=\langle a^2,(xy^4xy^{(p+1)/2})^2\rangle\leq H\cap K\leq Z(G)$. Then $G/D\cong PSL(2,p)\times J$, where J is $PGL(2,q)$ or $C_2\times PSL(2,q)$. Hence G' is finite and, as G/G' is finite, then G is finite. Now $G'\cap D$ is elementary abelian of order 1, 2 or 4, since the Schur-Künneth formula shows $M(PSL(2,p)\times J)$ is $C_2\times C_2$. Let $z=(xy^4xy^{(p+1)/2})^2$ and note that $z^2=a^4=1$. From $(abab^2)^3=a^{-2}z^{-1}$ we see $bab^2abab^2abab^2=az^{-1}$. Multiply on the left by b and use $(ab^2)^4=a^2$ to obtain $ab^{-2}a^{-1}b^{-1}ab^2abab^2=baz^{-1}$. Hence $b^{-2}a^{-1}b^{-1}a^{-1}a^2b^2abab^2=a^{-1}baz^{-1}$ and, as a^2 is central, we have

$$(abab^2)^{-1}b^2(abab^2) = a^{-1}baz^{-1}a^2.$$

Raise this relation to the qth power to obtain $1=z^{-q}a^{2q}$, which simplifies to $z=a^2$. Hence, $(abab^2)^3=1$ and $(xy^4xy^{(p+1)/2})^2=a^2$. Thus D is cyclic of order 1 or 2. Now $H/D\cong G/K\cong PSL(2,p)$ and $H=DH'$. However, $(xy^4xy^{(p+1)/2})^2\in H'$, and therefore $D\leq H'$ and $H=H'$. Hence H is $PSL(2,p)$ or $SL(2,p)$ but, as $x^2=1$, we conclude that $H\cong PSL(2,p)$ and $G=H\times K$. □

With the above stated restrictions on the prime q, we have the following result.

COROLLARY 4.4. *Let p and q be primes with $p, q \geq 5$. The group $PSL(2,p) \times PGL(2,q)$ is efficient whenever the congruence $x^2 \equiv 2 \pmod q$ has no integer solution.*

Finally we show that $PSL(2,5)^4$ is efficient.

THEOREM 4.5. *The group $PSL(2,5)^4$ is efficient.*

Proof. From the presentation for $PSL(2,5)^2$ given in [6],

$$\langle a, b, c, d \mid a^{15} = (ba^{-1})^2 = (ab^{-3}a^{-1}b^{-4})^2 b^5 = (ba^{-3}b^{-1}a^{-4})^2 a^5 = b^4 a^{-6} b a^{-6} = 1 \rangle,$$

the methods described in [5] were used to find the presentation for $PSL(2,5)^4$:

$$\langle x, y \mid x^{30} = [x^{-2}, y^{-2}], y^{30} = [x^{15}, y^{15}], (x^{-15}y^{12}x^{15}y^{-4})^2 y^{-10} = 1, (x^{-15}y^{16})^4 y^{-6} x^{-15} y^{10} = 1,$$
$$(y^{-15}x^{12}y^{15}x^{-4})^2 x^{-10} = 1, (y^{-15}x^{16})^4 x^{-6} y^{-15} x^{10} = 1 \rangle.$$

That this is a presentation for $PSL(2,5)^4$ may be verified using GAP or MAGMA. □

ACKNOWLEDGEMENT. The authors would like to thank the referee for his helpful comments. The referee has pointed out that he believes that the results of this paper form part of a more general setting.

REFERENCES

1. Y. G. Baik and S. J. Pride, On the efficiency of Coxeter groups, *Bull. London Math. Soc.* **29** (1997), 32–36.

2. W. A. Bogley and S. J. Pride, Computing generators of π_2, in *Two-dimensional Homotopy and Combinatorial Group theory* (C. Hog-Angeloni, W. Metzler and A. J. Sieradski eds.), London Math. Soc. Lecture Notes **197** (Cambridge University Press, Cambridge, 1995), 157–188.

3. M. J. Brookes, C. M. Campbell and E. F. Robertson, Efficiency and direct products of groups, in *Proc. Groups – Korea 1994* (A. C. Kim and D. L. Johnson eds.), (Walter de Gruyter, 1995) 25–33.

4. C. M. Campbell, T. Kawamata, I. Miyamoto, E. F. Robertson and P. D. Williams, Deficiency zero presentations for certain perfect groups, *Proc. Roy. Soc. Edinburgh Sect. A* **103** (1986), 63–71.

5. C. M. Campbell, E. F. Robertson and P. D. Williams, On the efficiency of some direct powers of groups, in *Groups-Canberra 1989* (L. G. Kovács ed.), Lecture Notes in Mathematics No. 1456 (Springer-Verlag, Berlin, 1990), 106–113.

6. C. M. Campbell, E. F. Robertson and P. D. Williams, Efficient presentations of the groups $PSL(2,p) \times PSL(2,p)$, p prime, *J. London Math. Soc.* (2) **41** (1990), 69–77.

7. C. M. Campbell, E. F. Robertson and P. D. Williams, Efficient presentations of direct powers of imperfect groups, *Algebra Colloq.* **4** (1997), 21–27.

8. J. J. Cannon, J. McKay and K. C. Young, The non-abelian simple groups G, $|G| < 10^5$—presentations, *Comm. Algebra* **7** (1979), 1397–1406.

9. J. H. Conway, H. S. M. Coxeter and G. C. Shephard, The centre of a finitely generated group, *Tensor* **25** (1972), 405–418.

10. T. Hert and P. D. Williams, A note on a presentation of $PGL(2,p)$ to appear.

11. A. Jamali, *Computing with simple groups: maximal subgroups and presentations*, Ph.D. thesis, University of St Andrews (1988).

12. P. E. Kenne, Presentations for some direct products of groups, *Bull. Austral. Math. Soc.* **28** (1983), 131–133.

13. P. E. Kenne, *Minimal group presentations: a computational approach*, Ph.D. Thesis, Australian National University, 1991.

14. L. G. Kovács, Finite groups with trivial multiplicator and large deficiency, in *Proc. Groups – Korea 1994* (A. C. Kim and D. L. Johnson eds.), (Walter de Gruyter, 1995), 211–225.

15. E. F. Robertson, R. M. Thomas and C. I. Wotherspoon, A class of inefficient groups with symmetric presentations, in *Proc. Groups – Korea 1994* (A. C. Kim and D. L. Johnson eds.), (Walter de Gruyter, 1995), 277–284.

16. J. G. Sunday, Presentations of the groups $SL(2, m)$ and $PSL(2, m)$, *Canad. J. Math* **24** (1972), 1129–1131.

17. M. Suzuki, *Group theory I*, Grundlehren der Mathematischen Wissenschaften 247 (Springer-Verlag, 1982).

18. J. Wiegold, The multiplicator of a direct product, *Quart. J. Oxford* (2) **22** (1971), 103–105.

19. J. Wiegold, The Schur multiplier, in *Groups – St Andrews 1981*, (C. M. Campbell and E. F. Robertson eds.), London Mathematical Society Lecture Notes 71 (Cambridge University Press, Cambridge, 1982), 137–154.

C. M. Campbell
University of St. Andrews
North Haugh
St. Andrews
Fife KY16 9SS
Scotland

I. Miyamoto
Faculty of Engineering
Yamanashi University
Takeda-4
Kofu
Japan

E. F. Robertson
University of St. Andrews
North Haugh
St. Andrews
Fife KY16 9SS
Scotland

P. D. Williams
Department of Mathematics
California State University
San Bernardino
California 92407
U.S.A.