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Asymptotic profiles of a nonlocal dispersal SIS epidemic model with saturated incidence

Yan-Xia Feng, Wan-Tong Li and Fei-Ying Yang
School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, P. R. China (fengyx2021@lzu.edu.cn; wtli@lzu.edu.cn; yangfy@lzu.edu.cn)

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Infection mechanism plays a significant role in epidemic models. To investigate the influence of saturation effect, a nonlocal (convolution) dispersal susceptible-infected-susceptible epidemic model with saturated incidence is considered. We first study the impact of dispersal rates and total population size on the basic reproduction number. Yang, Li and Ruan (*J. Differ. Equ.* 267 (2019) 2011–2051) obtained the limit of basic reproduction number as the dispersal rate tends to zero or infinity under the condition that a corresponding weighted eigenvalue problem has a unique positive principal eigenvalue. We remove this additional condition by a different method, which enables us to reduce the problem on the limiting profile of the basic reproduction number into that of the spectral bound of the corresponding operator. Then we establish the existence and uniqueness of endemic steady states by a equivalent equation and finally investigate the asymptotic profiles of the endemic steady states for small and large diffusion rates to provide reference for disease prevention and control, in which the lack of regularity of the endemic steady state and Harnack inequality makes the limit function of the sequence of the endemic steady state hard to get. Finally, we find whether lowing the movements of susceptible individuals can eradicate the disease or not depends on not only the sign of the difference between the transmission rate and the recovery rate but also the total population size, which is different from that of the model with standard or bilinear incidence.

Keywords: Nonlocal dispersal; susceptible-infected-susceptible (SIS) epidemic model; basic reproduction number; endemic steady state; asymptotic profiles

2020 *Mathematics Subject Classification*: 35K57; 35R20; 92D25

1. Introduction

Infection mechanisms such as bilinear incidence, standard incidence and saturated incidence adopted in the epidemic models play a vital role in analysing the spread of infectious diseases. The classical compartmental models such as susceptible-infected-removed (SIR) model and susceptible-infected-susceptible (SIS) model proposed by Kermack and McKendrick [**25**] adopted bilinear incidence of the form βSI assuming that the contact rate is proportional to the total population size. Standard incidence of the form $\frac{\beta SI}{S+I}$ initiated by de Jong *et al.* [**11**] on the condition that the contact rate is a constant. The incidence function $\frac{\beta SI}{m+S+I}$ displays a saturation effect accounting for the fact that the number of contacts an individual can have with other individuals reaches some finite maximal value due to the spatial or social distribution of the population and/or limitation of time (Diekmann and Kretzschmar [**14**]). For other types of incidence functions, one can refer to Anderson and May [**4**], Capasso and Serio [**7**], Heesterbeek and Metz [**21**] and Liu *et al.* [**30**]. For various epidemic models proposed to describe the spatial spread of infectious diseases, we refer to the monograph of Murray [**31**] and surveys by Fitzgibbon and Langlais [**17**], Ruan [**37**], Ruan and Wu [**38**] and the references cited therein.

Earlier models ignore the movements of individuals and the coefficients are constants. Allen *et al.* [**2**] investigated the impact of spatial heterogeneity of environment and movements of individuals on the persistence and extinction of infectious diseases modelled by an SIS reaction-diffusion model with space-dependent coefficients and standard incidence. Since then, the model proposed in [**2**] has attracted much attention. For example, Peng and Liu [**33**] studied global stability of the steady states, Peng [**32**] and Peng and Yi [**35**] considered the asymptotic profiles of endemic steady states. For bilinear incidence, Deng and Wu [**12**, **13**] studied the existence and global attractivity of the steady states. For the same model, Wu and Zou [**43**], Wen *et al.* [**42**] and Castellano and Salako [**8**] further investigated the asymptotic profiles of the endemic steady states for small and large diffusion rates. For saturated incidence of the form $\frac{\beta SI}{m+S+I}$, Suo and Li [**41**], Guo *et al.* [**19**] and Gao *et al.* [**18**] considered the threshold dynamics and asymptotic profiles of endemic steady states. For saturated incidence of the form $\beta SI/(1+mI)$, Cui [**10**], Sun and Cui [**40**] and Huo and Cui [**22**] investigated the effects of diffusion and saturation on asymptotic profiles of the endemic steady states and concluded that the disease will be eliminated if saturated incidence rate is sufficiently large.

Nowadays, convenient transport mechanisms across many scales have changed the way we and organisms travelling with us are distributed across the globe. This causes severe consequences for the spread of epidemics that human infectious diseases rarely remain confined to small spatial regions, but instead spread rapidly across countries and continents by travel of infected individuals (Hallatscheka and Fisher [**20**]). Long-range dispersal of species including humans can be better described by nonlocal convolution operators (Andreu-Vaillo *et al.* [**3**], Brockmann *et al.* [**6**] and Fife [**16**]). Nonlocal epidemic models have been extensively studied since the classical work of Kendall [**23**, **24**], in which he generalized the Kermack–McKendrick model to a space-dependent integro-differential equation and used the integral term $\beta S(x,t)\int_{-\infty}^{\infty} K(x-y)I(y,t)\mathrm{d}y$ to describe how infectious individuals $I(y,t)$ at location y disperse to infect susceptible individuals $S(x,t)$ at location x. For further results on nonlocal epidemic models, we refer to the monograph of Rass and Radcliffe [**36**] and a survey by Ruan [**37**].

Recently, Yang *et al.* [**45**] investigated a nonlocal dispersal SIS epidemic model with standard incidence. They showed the property of the basic reproduction number, the existence, uniqueness and stability of steady states and obtained the asymptotic profiles of endemic steady states for large diffusion rates. Feng *et al.* [**15**] investigated the impact of bilinear incidence, in which they found a concentration phenomenon that the infected individuals concentrate on the sites of

$$\mathcal{S}=\left\{x_*\in\bar{\Omega}:\frac{\gamma(x_*)}{\beta(x_*)}=\min_{x\in\bar{\Omega}}\frac{\gamma(x)}{\beta(x)}\right\}$$

as the movements of infected individuals tend to zero. And limiting the movements of susceptible individuals is not always an effective strategy to eradicate the disease. Only when the total population size is relatively small, this strategy takes effect. For other nonlocal dispersal epidemic models, we refer to Kuniya and Wang [**26**], Yang and Li [**44**] and references cited therein.

Motivated by Feng *et al.* [**15**] and Yang *et al.* [**45**], in this paper we intend to explore the influence of saturated incidence on the asymptotic profiles of endemic steady states of the following SIS epidemic model with nonlocal (convolution) dispersal

$$\begin{cases}\dfrac{\partial S}{\partial t}=d_S\displaystyle\int_\Omega J(x-y)[S(y,t)-S(x,t)]\,\mathrm{d}y-\dfrac{\beta(x)SI}{m(x)+S+I}+\gamma(x)I, & x\in\Omega,\ t>0,\\ \dfrac{\partial I}{\partial t}=d_I\displaystyle\int_\Omega J(x-y)[I(y,t)-I(x,t)]\,\mathrm{d}y+\dfrac{\beta(x)SI}{m(x)+S+I}-\gamma(x)I, & x\in\Omega,\ t>0,\\ S(x,0)=S_0(x),\ I(x,0)=I_0(x), & x\in\Omega,\end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a smooth bounded domain; $S(x,\,t)$ and $I(x,\,t)$ represent the density of susceptible and infectious individuals at location $x\in\Omega$ and time $t>0$, respectively; positive constants d_S and d_I are dispersal coefficients for susceptible and infectious individuals, respectively; $\beta(x)$ and $\gamma(x)$ are positive continuous functions on Ω which denote the transmission rate of susceptible individuals and the recovery rate of infectious individuals at $x\in\Omega$, respectively; positive continuous function $m(x)$ is incorporated to measure the inhibitory effect or model the situation that a vector or intermediate group can absorb the disease burden (Guo *et al.* [**19**]). The convolution integrals describe the nonlocal dispersal of individuals. More specifically, $\int_\Omega J(x-y)S(y,\,t)\mathrm{d}y$ and $\int_\Omega J(x-y)I(y,\,t)\mathrm{d}y$ represent the rates at which susceptible and infectious individuals are arriving at position x from other places, while $\int_\Omega J(x-y)S(x,\,t)\mathrm{d}y$ and $\int_\Omega J(x-y)I(x,\,t)\mathrm{d}y$ are the rates at which susceptible and infectious individuals are leaving location x for other locations, respectively. Throughout the whole paper, we assume that the dispersal kernel function J satisfies

(J) $J(\cdot)\in C(\mathbb{R}^n)$, $J(0)>0$, $J(x)=J(-x)\geqslant 0$, $\int_{\mathbb{R}^n}J(x)\,\mathrm{d}x=1$,

and the initial data satisfy

(H) $S_0(x)$ and $I_0(x)$ are nonnegative continuous functions on $\bar{\Omega}$, and the total number of initial infectious individuals is positive; that is, $\int_\Omega I_0(x)\,\mathrm{d}x>0$.

Nonlocal eigenvalue problems may not admit principal eigenvalues in general (Coville [**9**]). So the basic reproduction number lacks variational characterization. Yang *et al.* [**45**] gave the limits of the basic reproduction number as the dispersal rate tends to zero or infinity under the condition that a corresponding weighted eigenvalue problem admits a principal eigenvalue and only studied the effects of

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large dispersal rates on the disease transmission. In the present paper, we prove the limits of the basic reproduction number removing the additional condition by a different method in Zhang and Zhao [46], which enables us to reduce the problem on the limiting profile of the basic reproduction number into that of the spectral bound of the corresponding operator, and supplement the results about small dispersal rates. It should be pointed out that the basic reproduction number of model in [2, 45] is independent of the total population size and it depends on the total population size for (1.1) with $m \not\equiv 0$. The nonlocal diffusion term brings the lack of regularity of the endemic steady state and Harnack inequality, which makes the limit function of the sequence of the endemic steady state hard to get when we investigate the impacts of small and large dispersal rates. Letting the dispersal rates go to infinity yields spatial homogeneity, which is a trivial conclusion. But small dispersal rates may cause interesting phenomenon. We conclude whether lowering the movements of susceptible individuals can eradicate the disease or not depends on not only the sign of $\beta - \gamma$ but also the total population size, which is different from that of the model with standard incidence [2, 45] or bilinear incidence [15, 43]. Allen *et al.* [2] considered the case $m \equiv 0$ and required that $\beta - \gamma$ changes sign on Ω. They concluded that limiting the movements of susceptible individuals makes the disease die out. Compared with Allen *et al.* [2], in this paper, we consider the following two cases:

(a) the set $\{x \in \Omega : \beta(x) \leqslant \gamma(x)\}$ has interior points in Ω;

(b) $\{x \in \Omega : \beta(x) > \gamma(x)\} = \Omega$.

For case (a), limiting the movements of susceptible individuals makes the disease die out. For case (b), lowering the movements of susceptible individuals can eradicate the disease only when the total population size is relatively small. As for model with standard incidence [2, 45], it does not depend on the total population size and the disease always persists with the total number of infected individuals larger than that of saturated incidence, from which we derive increasing m helps to eliminate the disease or reduce the size of infected individuals. Similar comparisons can be seen when both dispersal rates approach zero and the dispersal rate of susceptible individuals is significantly lower than that of infected individuals.

This paper is organized as follows. In § 2.1, we list the main results of this paper including not only the existence, uniqueness of endemic steady state of system (1.1) but also the asymptotic profile of the endemic steady state of system (1.1) for small and large diffusion rates. In § 2.2, we compare the results of system (1.1) with that of model with standard incidence or bilinear incidence. In § 3, we present proofs of the main results stated in § 2.1.

2. Main results and discussion

2.1. Main results

In this section, we state the main results of this paper. Let $N := \int_\Omega (S_0(x) + I_0(x))\,\mathrm{d}x$ be the total number of individuals in Ω at $t = 0$. Define

$$\mathcal{M}[u](x) := d_I \int_\Omega J(x-y)(u(y) - u(x))\,\mathrm{d}y - \gamma(x)u(x).$$

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It is well-known that $\mathcal{M}$ can generate a uniformly continuous semigroup, denoted by $\{T(t)\}_{t\geqslant 0}$. Denote $\theta(x) := \frac{\beta(x)\frac{N}{|\Omega|}}{m(x)+\frac{N}{|\Omega|}}$, $\mathcal{F}[\phi](x) := \theta(x)\phi(x)$ and

$$\mathcal{L}[\phi](x) := \theta(x)\int_0^\infty T(t)\phi\,\mathrm{d}t, \quad \phi \in C(\bar{\Omega}).$$

We define the basic reproduction number of system (1.1) as follows

$$R_0 = r(\mathcal{L}),$$

where $r(\mathcal{L})$ represents the spectral radius of $\mathcal{L}$. Set $\hat{R}_0 = r(\hat{\mathcal{L}})$, where $\hat{\mathcal{L}}$ is defined by replacing $\theta(x)$ by $\beta(x)$ in the definition of $\mathcal{L}$.

First we give the results about the dependence of R_0 on d_I and N.

Theorem 2.1. *The following statements hold.*

(i) *R_0 is a non-increasing function of d_I with $R_0 \to \max\limits_{x\in\bar{\Omega}} \frac{\theta(x)}{\gamma(x)}$ as $d_I \to 0$ and $R_0 \to \frac{\int_\Omega \theta\,\mathrm{d}x}{\int_\Omega \gamma\,\mathrm{d}x}$ as $d_I \to +\infty$.*

(ii) *R_0 is a monotone increasing function of N with $R_0 \to 0$ as $N \to 0$ and $R_0 \to \hat{R}_0$ as $N \to +\infty$.*

Remark 2.2. Allen *et al.* [**2**] obtained the same results as theorem 2.1 (i) for an SIS reaction-diffusion model with the aid of variational characterization for the basic reproduction number because the basic reproduction number equals the reciprocal of the principal eigenvalue of a corresponding eigenvalue problem with random diffusion. But nonlocal eigenvalue problems may not admit principal eigenvalues in general. Under the conditions that the nonlocal weighted eigenvalue problem

$$-d_I\int_\Omega J(x-y)(\phi(y)-\phi(x))dy + \gamma(x)\phi(x) = \mu\theta(x)\phi(x), \quad x\in\Omega$$

admits a unique positive principal eigenvalue μ_p with positive eigenfunction and there exists some positive function $\psi_{d_I}(x) \in L^2(\Omega)$ satisfying $\mathcal{L}[\psi_{d_I}](x) = R_0\psi_{d_I}(x)$, [**45**, Theorem 2.16] gives that $R_0 \to \max\limits_{x\in\bar{\Omega}}\theta(x)/\gamma(x)$ as $d_I \to 0$ and $R_0 \to \int_\Omega \theta\,\mathrm{d}x/\int_\Omega \gamma\,\mathrm{d}x$ as $d_I \to +\infty$. Now we remove these additional conditions and further investigate the impact of N by means of a different method in Zhang and Zhao [**46**], which enables us to reduce the problem on the limiting profile of the basic reproduction number into that of the spectral bound of the corresponding operator.

Next we present the existence and uniqueness of endemic steady state of (1.1).

Theorem 2.3. *Suppose $R_0 > 1$. Then system* (1.1) *has an endemic steady state $(S, I) \in C(\bar{\Omega}) \times C(\bar{\Omega})$. Furthermore, if $d_S \geqslant d_I$, the endemic steady state of* (1.1) *is unique.*

Finally we state results on the asymptotic profile of the endemic steady state for small and large dispersal rates. Denote

$$\Omega^+ = \left\{ x \in \Omega : \frac{N}{|\Omega|}(\beta(x) - \gamma(x)) - \gamma(x)m(x) > 0 \right\},$$
$$H^+ = \{x \in \Omega : \beta(x) > \gamma(x)\}, \quad H^- = \{x \in \Omega : \beta(x) \leqslant \gamma(x)\}.$$

THEOREM 2.4. *Suppose that* $R_0 > 1$ *and the set* H^- *has interior points in* Ω. *For any fixed* $d_I > 0$, *up to a subsequence of* $d_S \to 0$, *the corresponding endemic steady state of* (1.1) *satisfies* $(S, \frac{I}{d_S}) \to (S^*, V^*)$ *in* $C(\bar{\Omega}) \times C(\bar{\Omega})$, *where* $S^*, V^* > 0$ *on* $\bar{\Omega}$ *and* (S^*, V^*) *with* $\int_\Omega S^* \, \mathrm{d}x = N$ *is a positive solution of*

$$\begin{cases} \displaystyle\int_\Omega J(x-y)(S^*(y) - S^*(x)) \, \mathrm{d}y - \frac{\beta(x)S^*V^*}{m(x)+S^*} + \gamma(x)V^* = 0, & x \in \Omega, \\ \displaystyle d_I \int_\Omega J(x-y)(V^*(y) - V^*(x)) \, \mathrm{d}y + \frac{\beta(x)S^*V^*}{m(x)+S^*} - \gamma(x)V^* = 0, & x \in \Omega. \end{cases} \tag{2.1}$$

THEOREM 2.5. *Suppose that* $R_0 > 1$ *and* $H^+ = \Omega$. *For any fixed* $d_I > 0$, *up to a subsequence of* $d_S \to 0$, *the corresponding endemic steady state* (S, I) *of* (1.1) *satisfies one of the following statements:*

(i) $\left(S, \frac{I}{d_S}\right) \to (S^*, V^*)$ *in* $C(\bar{\Omega}) \times C(\bar{\Omega})$, *where* $S^*, V^* > 0$ *on* $\bar{\Omega}$ *and* (S^*, V^*) *with* $\int_\Omega S^* \, \mathrm{d}x = N$ *is a positive solution of* (2.1).

(ii) $(S, I) \to \left(\frac{\gamma m}{\beta - \gamma}, 0\right)$ *in* $C(\bar{\Omega}) \times C(\bar{\Omega})$ *and* $\int_\Omega \frac{\gamma m}{\beta-\gamma} \, \mathrm{d}x = N$.

(iii) $(S, I) \to \left(\frac{\gamma(m+I^*)}{\beta-\gamma}, I^*\right)$ *in* $C(\bar{\Omega}) \times C(\bar{\Omega})$, *where* $I^* = \frac{N - \int_\Omega \frac{\gamma m}{\beta-\gamma} \mathrm{d}x}{\int_\Omega \frac{\beta}{\beta-\gamma} \mathrm{d}x} > 0$.

In addition, when $N \leqslant \int_\Omega \frac{\gamma m}{\beta-\gamma} \, \mathrm{d}x$, $I \to 0$; *when* $\beta - \gamma$ *and* m *are positive constants with* $N > \int_\Omega \frac{\gamma m}{\beta-\gamma} \, \mathrm{d}x$, $I \to \frac{N - \int_\Omega \frac{\gamma m}{\beta-\gamma} \mathrm{d}x}{\int_\Omega \frac{\beta}{\beta-\gamma} \mathrm{d}x}$.

REMARK 2.6. If the set H^- has interior points in Ω, theorem 2.4 implies that $I \to 0$ as $d_S \to 0$. If $H^+ = \Omega$, it follows from theorem 2.5 that $I \to 0$ as $d_S \to 0$ if $N \leqslant \int_\Omega \frac{\gamma m}{\beta-\gamma} \, \mathrm{d}x$. We conjecture that $I \to \frac{N - \int_\Omega \frac{\gamma m}{\beta-\gamma} \mathrm{d}x}{\int_\Omega \frac{\beta}{\beta-\gamma} \mathrm{d}x}$ as $d_S \to 0$ when $N > \int_\Omega \frac{\gamma m}{\beta-\gamma} \, \mathrm{d}x$. We only prove this conjecture when $\beta - \gamma$ and m are positive constants. In figure 1, we perform some numerical simulations to support this conjecture.

THEOREM 2.7. *Suppose that* Ω^+ *is nonempty.*

(i) *Up to subsequences of* $d_I \to 0$ *and* $\frac{d_I}{d_S} \to d \in [0, +\infty)$, *the endemic steady state of* (1.1) *satisfies* $(S, I) \to (S^*, I^*)$ *in* $C(\bar{\Omega}) \times C(\bar{\Omega})$, *where* S^* *is a positive function,* I^* *is nonnegative and not identically zero on* $\bar{\Omega}$, *and* (S^*, I^*)

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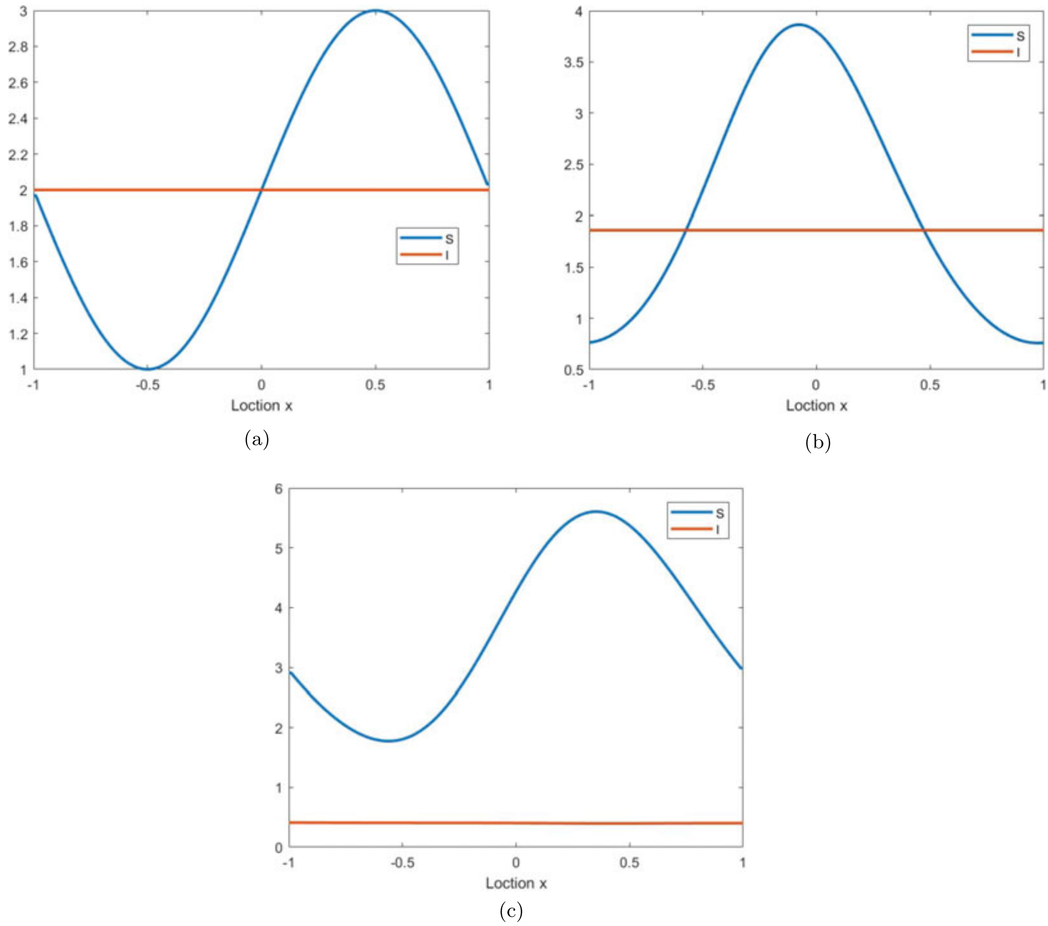

Figure 1. Numerical simulations of the profiles of endemic steady states (S, I) for system (1.1) as $d_S \to 0$, where $\Omega = (-1, 1)$, $J(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$, $N = 8$, $d_S = 0.0001$, $d_I = 1$, (a) $m(x) = 6$, $\beta(x) = \sin(\pi x) + 10$, $\gamma(x) = \sin(\pi x) + 2$, (b) $m(x) = \cos(\pi x) + 6$, $\beta(x) = \sin(\pi x) + 10$, $\gamma(x) = \cos(\pi x) + 2$, (c) $m(x) = \cos(\pi x) + 5$, $\beta(x) = \sin(\pi x) + 5$, $\gamma(x) = \sin(\pi x) + 2$.

satisfies

$$\begin{aligned} &I^*(x) = \begin{cases} \dfrac{l(\beta(x) - \gamma(x)) - \gamma(x) m(x)}{d(\beta(x) - \gamma(x)) + \gamma(x)}, & \text{if } \ l(\beta(x) - \gamma(x)) - \gamma(x) m(x) > 0, \ x \in \bar{\Omega}, \\ 0, & \text{if } \ l(\beta(x) - \gamma(x)) - \gamma(x) m(x) \leqslant 0, \ x \in \bar{\Omega}, \end{cases} \\ &S^* = l - dI^*, \\ &\int_\Omega (S^* + I^*) \, \mathrm{d}x = N, \end{aligned} \tag{2.2}$$

in which l is some positive constant. In particular, if $d \in [0, 1]$, then l is uniquely determined.

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(ii) *If the set H^- has interior points in Ω, then up to subsequences of $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$, the endemic steady state of* (1.1) *satisfies $(S, I) \to (S^*, 0)$ in $C(\bar{\Omega}) \times C(\bar{\Omega})$, where*

$$S^*(x) = \begin{cases} l - \dfrac{l(\beta(x)-\gamma(x)) - \gamma(x)m(x)}{\beta(x)-\gamma(x)}, & \text{if } \ l(\beta(x)-\gamma(x)) - \gamma(x)m(x) > 0, \ x \in \bar{\Omega}, \\ l, & \text{if } \ l(\beta(x)-\gamma(x)) - \gamma(x)m(x) \leqslant 0, \ x \in \bar{\Omega}, \end{cases}$$

in which the positive constant l is determined by $\int_\Omega S^* \,\mathrm{d}x = N$.

(iii) *If $H^+ = \Omega$, then up to subsequences of $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$, the endemic steady state of* (1.1) *satisfies $(S, I) \to (S^*, I^*)$ in $C(\bar{\Omega}) \times C(\bar{\Omega})$, where S^* is a positive function and I^* is a nonnegative constant. Moreover, the following conclusions hold.*

(a) *If $N < \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x$, then $(S^*, I^*) = (l - \frac{[l(\beta-\gamma)-\gamma m]^+}{\beta-\gamma}, 0)$, where the positive constant l is determined by $\int_\Omega S^*\,\mathrm{d}x = N$. In addition, there exist positive constants $0 < d_0 \ll 1$, C_1 and C_2 such that*

$$C_1 \frac{d_S}{d_I} \leqslant \|I\|_{L^\infty(\Omega)} \leqslant C_2 \frac{d_S}{d_I} \quad \text{for all } \ 0 < d_I, \frac{d_S}{d_I} < d_0.$$

(b) *If $N = \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x$, then $(S^*, I^*) = (\frac{\gamma m}{\beta-\gamma}, 0)$.*

(c) *If $N > \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x$, then $(S^*, I^*) = (\frac{\gamma(m+I^*)}{\beta-\gamma}, \frac{N-\int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x}{\int_\Omega \frac{\beta}{\beta-\gamma}\,\mathrm{d}x})$.*

THEOREM 2.8. *The following statements hold.*

(i) *Suppose that $R_0 > 1$. If d_I is fixed and $d_S \to +\infty$, then the endemic steady state of* (1.1) *$(S, I) \to (S^*, I^*)$ in $C(\bar{\Omega}) \times C(\bar{\Omega})$, where I^* is the unique positive solution of*

$$d_I \int_\Omega J(x-y)(I(y)-I(x))\,\mathrm{d}y + \frac{\beta(x)\frac{1}{|\Omega|}\left(N - \int_\Omega I\,\mathrm{d}x\right) I}{m(x) + \frac{1}{|\Omega|}\left(N - \int_\Omega I\,\mathrm{d}x\right) + I} - \gamma(x) I = 0 \tag{2.3}$$

and

$$S^* = \frac{N - \int_\Omega I^*\,\mathrm{d}x}{|\Omega|}.$$

(ii) *Suppose that $\int_\Omega \theta\,\mathrm{d}x > \int_\Omega \gamma\,\mathrm{d}x$. For any fixed $d_S > 0$, up to a subsequence of $d_I \to +\infty$, the corresponding endemic steady state of* (1.1) *satisfies $(S, I) \to (S^*, I^*)$ in $C(\bar{\Omega}) \times C(\bar{\Omega})$, where I^* is a positive constant and S^* is the positive solution of*

$$\begin{cases} d_S \displaystyle\int_\Omega J(x-y)(\tilde{S}(y) - \tilde{S}(x))\,\mathrm{d}y - \frac{\beta(x)\tilde{S}I^*}{m(x)+\tilde{S}+I^*} + \gamma(x)I^* = 0, & x \in \Omega, \\ \int_\Omega \tilde{S}\,\mathrm{d}x = N - I^*|\Omega|. \end{cases} \tag{2.4}$$

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(iii) *Suppose that* $\int_\Omega \theta \, \mathrm{d}x > \int_\Omega \gamma \, \mathrm{d}x$. *If* $d_S \to +\infty$ *and* $d_I \to +\infty$, *then the endemic steady state of* (1.1) *satisfies*

$$(S, I) \to \left(\frac{N}{|\Omega|} \frac{\int_\Omega \gamma \, \mathrm{d}x}{\int_\Omega \theta \, \mathrm{d}x}, \frac{N}{|\Omega|} \frac{\int_\Omega (\theta - \gamma) \, \mathrm{d}x}{\int_\Omega \theta \, \mathrm{d}x} \right) \quad \text{in} \quad C(\bar{\Omega}) \times C(\bar{\Omega}).$$

REMARK 2.9. If $m \equiv 0$, system (1.1) becomes the following model with standard incidence

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \displaystyle\int_\Omega J(x-y)[S(y,t) - S(x,t)] \, \mathrm{d}y - \dfrac{\beta(x) SI}{S+I} + \gamma(x) I, & x \in \Omega, \ t > 0, \\ \dfrac{\partial I}{\partial t} = d_I \displaystyle\int_\Omega J(x-y)[I(y,t) - I(x,t)] \, \mathrm{d}y + \dfrac{\beta(x) SI}{S+I} - \gamma(x) I, & x \in \Omega, \ t > 0, \\ S(x,0) = S_0(x), \ I(x,0) = I_0(x), & x \in \Omega. \end{cases} \tag{2.5}$$

Yang *et al.* [**45**, Section 4] only proved asymptotic profiles of endemic steady states of (2.5) for large dispersal rates. But the proofs in [**45**, Section 4] contain some errors. The proof of theorem 2.8 is still correct for $m \equiv 0$ and the results of theorem 2.8 are consistent with that in [**45**, Section 4]. Since the endemic steady states of nonlocal dispersal system (2.5) lack regularity and Harnack inequality, the methods for reaction-diffusion model (Allen *et al.* [**2**] and Peng [**32**]) are not applicable to (2.5). Applying similar arguments to the proofs of theorems 2.4, 2.5 and 2.7, we supplement the following results on asymptotic profiles of endemic steady states of (2.5) for small dispersal rates.

PROPOSITION 2.10. *Suppose that* $\hat{R}_0 > 1$.

(i) *Assume the set* H^- *has interior points in* Ω. *For any fixed* $d_I > 0$, *up to a subsequence of* $d_S \to 0$, *the corresponding endemic steady state of* (2.5) *satisfies* $(S, \frac{I}{d_S}) \to (S^*, V^*)$ *in* $C(\bar{\Omega}) \times C(\bar{\Omega})$, *where* $S^* \geqslant 0$, $V^* > 0$ *on* $\bar{\Omega}$ *and* $\int_\Omega S^* \, \mathrm{d}x = N$.

(ii) *Assume* $H^+ = \Omega$. *For any fixed* $d_I > 0$, *if* $d_S \to 0$, *then the corresponding endemic steady state of* (2.5) *satisfies* $(S, I) \to \left(\frac{N\gamma}{(\beta - \gamma) \int_\Omega \frac{\beta}{\beta-\gamma} \, \mathrm{d}x}, \frac{N}{\int_\Omega \frac{\beta}{\beta-\gamma} \, \mathrm{d}x} \right)$ *in* $C(\bar{\Omega}) \times C(\bar{\Omega})$.

PROPOSITION 2.11. *Assume* H^+ *is nonempty.*

(i) *If* $d_I \to 0$ *and* $\frac{d_I}{d_S} \to d \in [0, +\infty)$, *then the endemic steady state of* (2.5) *satisfies* $(S, I) \to (S^*, I^*)$ *in* $C(\bar{\Omega}) \times C(\bar{\Omega})$, *where*

$$I^*(x) = \begin{cases} \dfrac{N}{|\Omega| + (1-d) \int_{H^+} \frac{\beta - \gamma}{d(\beta-\gamma)+\gamma} \, \mathrm{d}x} \dfrac{\beta - \gamma}{d(\beta - \gamma) + \gamma}, & \text{if } x \in H^+, \\ 0, & \text{if } x \in H^-, \end{cases}$$

$$S^* = \frac{N}{|\Omega| + (1-d) \int_{H^+} \frac{\beta-\gamma}{d(\beta-\gamma)+\gamma} \, \mathrm{d}x} - d I^*.$$

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(ii) *If* $d_I \to 0$ *and* $\frac{d_I}{d_S} \to +\infty$, *then the endemic steady state of* (2.5) *satisfies the following statements.*

(a) *If the set* H^- *has interior points in* Ω, *then* $I \to 0$ *in* $C(\bar{\Omega})$ *and* $S \to S^*$ *uniformly on any compact subset of* H^- *and* H^+, *respectively, where*

$$S^*(x) = \begin{cases} 0, & \text{if } \ x \in H^+, \\ \dfrac{N}{|H^-|}, & \text{if } \ x \in H^-. \end{cases}$$

(b) *If* $H^+ = \Omega$, *then* $(S, I) \to \left(\frac{N\gamma}{(\beta-\gamma)\int_\Omega \frac{\beta}{\beta-\gamma}\,\mathrm{d}x}, \frac{N}{\int_\Omega \frac{\beta}{\beta-\gamma}\,\mathrm{d}x} \right)$ *in* $C(\bar{\Omega}) \times C(\bar{\Omega})$.

2.2. Discussion

This paper is mainly concerned with the asymptotic profiles of the endemic steady states of a nonlocal (convolution) dispersal SIS epidemic model (1.1) with saturated incidence. Yang *et al.* [**45**] considered (2.5) with standard incidence and Feng *et al.* [**15**] studied the following model with bilinear incidence

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \displaystyle\int_\Omega J(x-y)[S(y,t) - S(x,t)]\,\mathrm{d}y - \beta(x)SI + \gamma(x)I, & x \in \Omega, \ t > 0, \\ \dfrac{\partial I}{\partial t} = d_I \displaystyle\int_\Omega J(x-y)[I(y,t) - I(x,t)]\,\mathrm{d}y + \beta(x)SI - \gamma(x)I, & x \in \Omega, \ t > 0, \\ S(x,0) = S_0(x), \ I(x,0) = I_0(x), & x \in \Omega. \end{cases} \tag{2.6}$$

When an infectious disease breaks out in an area, people look forward to eradicating it as soon as possible for their health and normal lives. As a consequence, developing effective prevention and control strategies is especially important. For the sake of revealing the impacts of infection mechanism and dispersal rates on the elimination of infectious disease and providing possible options for disease control, we make a comparison between our main results about (1.1) and those about models (2.5) and (2.6). One can see table 1 for a quick understanding about the extinction or persistence of disease with small dispersal rates of susceptible individuals or/and infected individuals. Recall

$$\mathcal{S} = \left\{ x_* \in \bar{\Omega} : \frac{\gamma(x_*)}{\beta(x_*)} = \min_{x \in \bar{\Omega}} \frac{\gamma(x)}{\beta(x)} \right\}.$$

Comparison of basic reproduction number

The basic reproduction number of (2.5) is independent of the total population size N, but that of (2.6) and (1.1) depends on N and monotonically increases with respect to N by theorem 2.1. This implies that an infectious disease is more likely to be endemic in a large population. And the basic reproduction number of (1.1) is smaller than that of (2.5); that is, the transmission ability of (1.1) is weaker than that of (2.5). When the total population size is relatively large, the basic reproduction number of (2.6) is the biggest one corresponding to the strongest infection ability.

Table 1. *Asymptotic behaviour of endemic steady states* (S, I) *for small dispersal rates*

Model	$I \to I^*$ as $d_S \to 0$	$I \to I^*$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to d$
(1.1)	If $\mathrm{Int}(H^-) \neq \emptyset$, $I^* \equiv 0$; if $H^+ = \Omega$, $I^* \equiv 0$ for small N and $I^* > 0$ for large N	If $d \in [0, +\infty)$, $I^* = 0$ in H^- and $I^* \geqslant 0$ in H^+; if $d = +\infty$, behave like $d_S \to 0$
(2.5)	If $\mathrm{Int}(H^-) \neq \emptyset$, $I^* \equiv 0$; if $H^+ = \Omega$, $I^* > 0$	If $d \in [0, +\infty)$, $I^* = 0$ in H^- and $I^* > 0$ in H^+; if $d = +\infty$, behave like $d_S \to 0$
(2.6)	$I^* \equiv 0$ for small N and $I^* > 0$ for large N	If $d \in (0, +\infty)$, $I^* \geqslant 0$ but $I^* \not\equiv 0$; if $d = 0$, concentrate on $\mathcal{S}$; if $d = +\infty$, behave like $d_S \to 0$

Comparison of asymptotic profiles of endemic steady states

Case I. $d_S \to 0$. The asymptotic profiles of endemic steady states of these three models are different. For (2.5), in view of theorem 2.10, if there are low-risk sites which belong to H^- (see Allen *et al.* [**2**]), the total size of infected individuals tends to zero. But if the habitats only consist of high-risk sites which belong to H^+, the total size of infected individuals tends to $\frac{N}{\int_\Omega \frac{\beta}{\beta-\gamma}\,\mathrm{d}x}$; that is, the disease will always persist. For (2.6), by [**15**, Theorems 2.3-2.5], if $N < \int_\Omega \frac{\gamma}{\beta}\,\mathrm{d}x$, the disease can be eradicated; if $N > \int_\Omega \frac{\gamma}{\beta}\,\mathrm{d}x$, the total size of infected individuals tends to $\frac{N}{|\Omega|} - \frac{1}{|\Omega|}\int_\Omega \frac{\gamma}{\beta}\,\mathrm{d}x$. As to (1.1), it follows from theorem 2.4 that if H^- is nonempty, then limiting the movements of susceptible individuals yields extinction of the disease; if $H^+ = \Omega$, then the disease is either extinct or persistent by theorem 2.5. The disease can be eliminated when $N \leqslant \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x$ and we conjecture that the total size of infected individuals tends to $\frac{N - \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x}{\int_\Omega \frac{\beta}{\beta-\gamma}\,\mathrm{d}x}$ when $N > \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x$, from which we derive that increasing m helps to eliminate the disease or reduce the size of infected individuals. In figure 1, we perform some numerical simulations to support this conjecture. So systems (1.1) and (2.5) behave similarly when H^- is nonempty; model (1.1) is analogous to (2.6) when $H^+ = \Omega$.

Case II. $d_I \to 0$ and $\frac{d_I}{d_S} \to d \in [0, +\infty]$. If $d \in (0, +\infty)$, the infected individuals do not reside in H^- for (1.1) and (2.5). The infected individuals reside in all sites of H^+ for (2.5) but do not distribute in some sites of H^+ for (1.1). If $d = +\infty$ and H^- is nonempty, then the disease is eradicated for (1.1) and (2.5). If $d = +\infty$ and $H^+ = \Omega$, then the disease must be persistent for (2.5) but it depends on the total population size N for (1.1) and (2.6). The disease is extinct when N is relatively small and persistent when N is large. If $d = 0$, system (2.6) represents a particular concentration phenomenon that the infected individuals concentrate on the sites of $\mathcal{S}$. Systems (2.5) and (1.1) behave similarly. The infected individuals do not distribute in H^-. For (2.5), the infected individuals live exactly in H^+. But in some sites of H^+, there are not infected individuals for (1.1) implying that (1.1) tends to reduce the scale of disease. As a result, the individuals with significantly lowered dispersal rates play a dominant role.

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To sum up, once a low-risk region is created or N is relatively small, limiting the movements of susceptible individuals is an effective method to make the disease die out. When N is large or all sites of Ω are high-risk, the disease cannot be totally eradicated by limiting the movements of susceptible individuals and/or that of infected individuals. But limiting the movements of susceptible individuals and that of infected individuals can prevent the disease from being extensive over the whole area.

REMARK 2.12. Models (1.1), (2.5) and (2.6) do not incorporate the birth–death effect and crowding effect such as linear source $\Lambda - S$ and logistic source $S(a - bS)$. For random diffusion models with birth–death effect or crowding effect, appealing to the uniform persistence theory or topological degree argument yields the existence of endemic steady states. One can see Guo *et al.* [**19**], Li *et al.* [**28**], Li *et al.* [**29**], Li *et al.* [**27**], Peng *et al.* [**34**] and references cited therein for related studies. For nonlocal dispersal model, the lack of compactness of the solution semiflow makes the uniform persistence theory and topological degree argument unapplicable and a varying total population results in the difficulty that the stationary system can not be transformed into a single equation. One needs to find new methods to solve the existence, uniqueness, stability and asymptotic profiles of endemic steady states. We leave these for further study.

3. Preliminaries and proof of main results

In this section, we present the proofs of main theorems stated in § 2.1. Before that, we give some useful results.

3.1. Preliminaries

Adding up the two equations of (1.1) and integrating it on Ω yield that the total population size is constant, that is,

$$\int_\Omega (S(x,t) + I(x,t))\,\mathrm{d}x = N \quad \text{for all } t \geqslant 0.$$

We investigate the existence and uniqueness of the steady state of system (1.1). That is, we consider the following stationary problem

$$\begin{cases} d_S \displaystyle\int_\Omega J(x-y)(S(y)-S(x))\,\mathrm{d}y - \frac{\beta(x)SI}{m(x)+S+I} + \gamma(x)I = 0, & x \in \Omega, \\ d_I \displaystyle\int_\Omega J(x-y)(I(y)-I(x))\,\mathrm{d}y + \frac{\beta(x)SI}{m(x)+S+I} - \gamma(x)I = 0, & x \in \Omega. \end{cases} \tag{3.1}$$

The solutions of (3.1) also satisfy

$$\int_\Omega (S(x) + I(x))\,\mathrm{d}x = N. \tag{3.2}$$

By the same proof as [**45**, Lemma 3.3], we have the following lemma.

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LEMMA 3.1. *System* (1.1) *admits a unique disease-free steady state* $(\frac{N}{|\Omega|}, 0)$.

Define

$$\lambda_v(d_I) := \inf_{\substack{\varphi\in L^2(\Omega)\\ \varphi\neq 0}} \frac{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega(\gamma(x)-\theta(x))\varphi^2(x)\,\mathrm{d}x}{\int_\Omega \varphi^2(x)\,\mathrm{d}x}.$$

The nonlocal dispersal eigenvalue problem may not admit a principal eigenvalue. See a counterexample in [**9**]. In general, λ_v defined above may not be the principal eigenvalue.

LEMMA 3.2 Yang *et al.* [**45**, Lemma 2.5]. *Set* $v(x) = -d_I\int_\Omega J(x-y)dy + \theta(x) - \gamma(x)$. *Suppose there is some* $x_0 \in \mathrm{Int}(\Omega)$ *satisfying that* $v(x_0) = \max_{x\in\bar\Omega} v(x)$, *and the partial derivatives of* $v(x)$ *up to order* $n-1$ *at* x_0 *are zero. Then* $\lambda_v(d_I)$ *is the unique principal eigenvalue of*

$$d_I\int_\Omega J(x-y)(u(y)-u(x))dy + \theta(x)u(x) - \gamma(x)u(x) = -\lambda u(x).$$

See other sufficient conditions for the existence of the principal eigenvalue in Coville [**9**] and Shen and Xie [**39**].

REMARK 3.3. Although the principal eigenvalue does not always exist, we can use the principal eigenfunction of the approximation problem to construct lower or upper solution. Since $v(x)$ defined in lemma 3.2 is continuous on $\bar\Omega$, there exists some $x_0 \in \bar\Omega$ such that $v(x_0) = \max_{x\in\bar\Omega} v(x)$. Define a function sequence as follows:

$$v_n(x) = \begin{cases} v(x_0), & x \in B_{x_0}\left(\frac{1}{n}\right), \\ v_{n,1}(x), & x \in \left(B_{x_0}\left(\frac{2}{n}\right)\backslash B_{x_0}\left(\frac{1}{n}\right)\right), \\ v(x), & x \in \Omega\backslash B_{x_0}\left(\frac{2}{n}\right), \end{cases}$$

where $B_{x_0}(\frac{1}{n}) = \left\{x\in\Omega : |x-x_0| < \frac{1}{n}\right\}$, $v_{n,1}(x)$ satisfies $v_{n,1} \leqslant v(x_0)$ and $v_{n,1}(x)$ is continuous on Ω. Indeed, $v_{n,1}(x)$ exists if only we take n is large enough, denoted by $n \geqslant n_0 > 0$. Thus, lemma 3.2 implies that the eigenvalue problem

$$d_I\int_\Omega J(x-y)\phi(y)dy + v_n(x)\phi(x) = -\lambda\phi(x)$$

admits a principal eigenpair, denoted by $(\lambda_v^n(d_I), \phi_n)$. And $|\lambda_v^n - \lambda_v| \to 0$ as $n \to +\infty$ due to [**45**, Remark 2.7]. In the following arguments, we sometimes use ϕ_n to construct lower or upper solution.

In view of [**45**, Corollary 2.11], we have the following relation between $\lambda_v(d_I)$ and R_0.

LEMMA 3.4. $\lambda_v(d_I)$ *has the same sign as* $1 - R_0$.

Following the same proof as [**15**, Lemma 3.12], an equivalent problem to (3.1) is obtained.

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LEMMA 3.5. *Suppose that $d_I > 0$ and $d_S > 0$ are fixed. The following conclusions hold.*

(i) *Let $(S,\, I)$ be a nonnegative solution of (3.1). Then the function*

$$\kappa = d_S S + d_I I \tag{3.3}$$

is a constant function. Furthermore, by letting

$$\tilde{S} = \frac{S}{\kappa} \quad \text{and} \quad \tilde{I} = \frac{d_I I}{\kappa}, \tag{3.4}$$

$(\kappa,\, \tilde{S},\, \tilde{I})$ satisfies

$$\tilde{S} = \frac{1}{d_S}\left(1 - \tilde{I}\right), \tag{3.5}$$

$$\frac{\kappa}{d_S}\int_\Omega \left[\left(1-\tilde{I}\right) + \frac{d_S}{d_I}\tilde{I}\right] \mathrm{d}x = N \tag{3.6}$$

and

$$\begin{cases} d_I \displaystyle\int_\Omega J(x-y)(\tilde{I}(y) - \tilde{I}(x))\,\mathrm{d}y + \frac{\beta(x)\frac{\kappa}{d_S}(1-\tilde{I})\tilde{I}}{m(x) + \frac{\kappa}{d_S}(1-\tilde{I}) + \frac{\kappa}{d_I}\tilde{I}} - \gamma(x)\tilde{I} = 0, & x \in \Omega, \\ 0 \leqslant \tilde{I} < 1, & x \in \Omega. \end{cases} \tag{3.7}$$

(ii) *If $(\kappa,\, \tilde{S},\, \tilde{I})$ solves (3.5), (3.6) and (3.7), then $(S,\, I) = (\kappa\tilde{S},\, \frac{\kappa}{d_I}\tilde{I})$ is a nonnegative solution of (3.1).*

Define

$$\lambda_v^*(\kappa) := \inf_{\substack{\varphi \in L^2(\Omega) \\ \varphi \neq 0}} \frac{\begin{array}{c}\displaystyle\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2\,\mathrm{d}y\,\mathrm{d}x \\ \displaystyle + \int_\Omega \left(\gamma(x) - \frac{\frac{\kappa}{d_S}\beta(x)}{m(x) + \frac{\kappa}{d_S}}\right)\varphi^2(x)\,\mathrm{d}x\end{array}}{\displaystyle\int_\Omega \varphi^2(x)\,\mathrm{d}x}$$

and

$$\lambda_v^*(\infty) := \inf_{\substack{\varphi \in L^2(\Omega) \\ \varphi \neq 0}} \frac{\begin{array}{c}\displaystyle\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2\,\mathrm{d}y\,\mathrm{d}x \\ \displaystyle + \int_\Omega (\gamma(x) - \beta(x))\,\varphi^2(x)\,\mathrm{d}x\end{array}}{\displaystyle\int_\Omega \varphi^2(x)\,\mathrm{d}x}.$$

Recall that the spectral bound of a closed operator A is defined by

$$s(A) = \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\},$$

in which $\sigma(A)$ and $\rho(A)$ denote the spectrum and resolvent set of A, respectively.

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LEMMA 3.6. *The following statements hold.*

(i) *If $\lambda_v^*(\infty) \geqslant 0$, then 0 is the only nonnegative solution of* (3.7).

(ii) *If $\lambda_v^*(\infty) < 0$, then there exists some $\kappa^* > 0$ such that* (3.7) *admits a unique positive solution $\tilde{I}_\kappa$ when $\kappa > \kappa^*$ and 0 is the only nonnegative solution of* (3.7) *when $0 \leqslant \kappa \leqslant \kappa^*$. Moreover, the positive solution satisfies $0 < \tilde{I}_\kappa < 1$ and $\tilde{I}_\kappa$ is strictly increasing and continuously differentiable on $\kappa \in (\kappa^*, +\infty)$.*

Proof. It is easy to see that $\lambda_v^*(0) \geqslant \min_{x\in\bar{\Omega}} \gamma(x) > 0$. In view of the continuity and monotonicity of $\lambda_v^*(\kappa)$ on κ, if $\lambda_v^*(\infty) \geqslant 0$, then $\lambda_v^*(\kappa) \geqslant 0$ for all $\kappa \geqslant 0$; if $\lambda_v^*(\infty) < 0$, then there exists some $\kappa^* > 0$ such that $\lambda_v^*(\kappa) < 0$ when $\kappa > \kappa^*$ and $\lambda_v^*(\kappa) \geqslant 0$ when $0 \leqslant \kappa \leqslant \kappa^*$. By [**9**, Theorem 1.6], one can get that (3.7) admits a unique positive solution if and only if $\lambda_v^*(\kappa) < 0$. Then we derive the conclusions of lemma 3.6 about the existence of nonnegative solutions of (3.7). By similar arguments as the proof of [**45**, Theorem 3.8], we derive that the positive solution satisfies $0 < \tilde{I}_\kappa < 1$. Define a mapping $G : \mathbb{R}^+ \times C(\bar{\Omega}) \to C(\bar{\Omega})$ by

$$G(\kappa, \tilde{I}) = d_I \int_\Omega J(x-y)(\tilde{I}(y) - \tilde{I}(x))\,\mathrm{d}y + \frac{\beta(x)\frac{\kappa}{d_S}(1-\tilde{I})\tilde{I}}{m(x) + \frac{\kappa}{d_S}(1-\tilde{I}) + \frac{\kappa}{d_I}\tilde{I}} - \gamma(x)\tilde{I}.$$

Assume $\kappa_0,\, \kappa_1 \in (\kappa^*, +\infty)$ and $\kappa_0 < \kappa_1$. It is verified that $G(\kappa_1, \tilde{I}_{\kappa_0}) > 0$, which implies that $\tilde{I}_{\kappa_0}$ is a lower solution of $G(\kappa_1, \tilde{I}) = 0$. Clearly, 1 is an upper solution. By the method of upper and lower solutions and the uniqueness of a positive solution of $G(\kappa_1, \tilde{I}) = 0$, we have $\tilde{I}_{\kappa_0} < \tilde{I}_{\kappa_1}$. By the positivity of $\tilde{I}_{\kappa_0}$, 0 is the principal eigenvalue of

$$d_I \int_\Omega J(x-y)(\varphi(y) - \varphi(x))\,\mathrm{d}y + \frac{\frac{\kappa_0}{d_S}\beta(x)(1-\tilde{I}_{\kappa_0}(x))}{m(x) + \frac{\kappa_0}{d_S}(1-\tilde{I}_{\kappa_0}(x)) + \frac{\kappa_0}{d_I}\tilde{I}_{\kappa_0}(x)}\varphi(x) \\ - \gamma(x)\varphi(x) = \lambda\varphi(x).$$

The Fréchet derivative of G with respect to the second variable at $(\kappa_0, \tilde{I}_{\kappa_0})$ is

$$\begin{aligned} G_{\tilde{I}}(\kappa_0, \tilde{I}_{\kappa_0})u = d_I \int_\Omega & J(x-y)(u(y) - u(x))\,\mathrm{d}y \\ & + \frac{\frac{\kappa_0}{d_S}\beta(x)(1-\tilde{I}_{\kappa_0}(x))}{m(x) + \frac{\kappa_0}{d_S}(1-\tilde{I}_{\kappa_0}(x)) + \frac{\kappa_0}{d_I}\tilde{I}_{\kappa_0}(x)}u(x) \\ & - \gamma(x)u(x) - \frac{\frac{\kappa_0}{d_S}\beta(x)\tilde{I}_{\kappa_0}(x)\left(\frac{\kappa_0}{d_I} + m(x)\right)}{\left[m(x) + \frac{\kappa_0}{d_S}(1-\tilde{I}_{\kappa_0}(x)) + \frac{\kappa_0}{d_I}\tilde{I}_{\kappa_0}(x)\right]^2}u(x). \end{aligned}$$

Then we have $s(G_{\tilde{I}}(\kappa_0, \tilde{I}_{\kappa_0})) < 0$ and $0 \in \rho(G_{\tilde{I}}(\kappa_0, \tilde{I}_{\kappa_0}))$. It follows from the Implicit Function Theorem that there exists a unique $\tilde{I}_\kappa \in C(\bar{\Omega})$ such that $G(\kappa, \tilde{I}_\kappa) = 0$ for $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$ with $\epsilon > 0$ and this $\tilde{I}_\kappa$ is continuously differentiable with respect to κ. The proof is completed. □

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3.2. Proof of main results

Proof of theorem 2.1. Set $\mathcal{A}_\mu = \mathcal{M} + \frac{1}{\mu}\mathcal{F}$. By virtue of [**45**, Proposition 2.9], we have $\mathcal{M}$ and $\mathcal{A}_\mu$ are resolvent positive operators and $s(\mathcal{M}) < 0$. (i) For any $\mu > 0$, it follows from [**1**, Theorem 6] that $s(\mathcal{A}_\mu)$ is non-increasing in d_I. We derive from the proof of [**45**, Theorem 2.10] that $R_0 = r(-\mathcal{F}\mathcal{M}^{-1})$ and [**46**, Lemma 2.5 (ii)] gives that $r(-\mathcal{F}\mathcal{M}^{-1})$ is the unique solution of $s(\mathcal{A}_\mu) = 0$. As a result, R_0 is non-increasing in d_I. In view of [**39**, Theorem 2.2], $\lim_{d_I \to 0} s(\mathcal{A}_\mu) = \max_{x\in\bar{\Omega}}[-\gamma(x) + \frac{1}{\mu}\theta(x)]$ and $\lim_{d_I\to+\infty} s(\mathcal{A}_\mu) = \frac{1}{|\Omega|}\int_\Omega(-\gamma + \frac{1}{\mu}\theta)\,\mathrm{d}x$. Combined with [**46**, Lemma 2.5 (ii)], we conclude that $R_0 \to \max_{x\in\bar{\Omega}} \frac{\theta(x)}{\gamma(x)}$ as $d_I \to 0$ and $R_0 \to \frac{\int_\Omega \theta\,\mathrm{d}x}{\int_\Omega \gamma\,\mathrm{d}x}$ as $d_I \to +\infty$. (ii) Define

$$\lambda_p(\mathcal{A}_\mu) := \sup\left\{\lambda \in \mathbb{R} : \exists\, \varphi \in C(\bar{\Omega}), \varphi > 0 \ \text{ s.t. } \ (-\mathcal{A}_\mu + \lambda)[\varphi] \leqslant 0 \ \text{ in } \ \bar{\Omega}\right\}$$

and

$$\lambda_p'(\mathcal{A}_\mu) := \inf\left\{\lambda \in \mathbb{R} : \exists\, \varphi \in C(\bar{\Omega}), \varphi > 0 \ \text{ s.t. } \ (-\mathcal{A}_\mu + \lambda)[\varphi] \geqslant 0 \ \text{ in } \ \bar{\Omega}\right\}.$$

By virtue of [**39**, Proposition 3.9] and [**5**, Theorem 1.1], $s(\mathcal{A}_\mu) = \lambda_p(\mathcal{A}_\mu) = \lambda_p'(\mathcal{A}_\mu)$. Combined with [**9**, Proposition 1.1 (ii)], we conclude that $s(\mathcal{A}_\mu)$ is increasing in N. Next we prove $s(\mathcal{A}_\mu) \to s(\mathcal{M})$ as $N \to 0$ for any $\mu > 0$. If $s(\mathcal{M})$ is the principal eigenvalue, that is, there exists some $\varphi \in C(\bar{\Omega})$, $\varphi > 0$ such that

$$d_I \int_\Omega J(x-y)(\varphi(y) - \varphi(x))\,\mathrm{d}y - \gamma(x)\varphi(x) = s(\mathcal{M})\varphi(x), \quad x \in \bar{\Omega}.$$

For any $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that $|\frac{\theta(x)}{\mu}| < \frac{\varepsilon}{2}$ for all $N \leqslant N_\varepsilon$. Then, for $N \leqslant N_\varepsilon$, we have

$$\begin{aligned} d_I \int_\Omega J(x-y)(\varphi(y) - \varphi(x))\,\mathrm{d}y + \left(\frac{\theta(x)}{\mu} - \gamma(x)\right)\varphi(x) \\ - (s(\mathcal{M}) + \varepsilon)\varphi(x) \leqslant 0, \quad x \in \bar{\Omega}. \end{aligned}$$

By the definition of $\lambda_p'(\mathcal{A}_\mu)$, we have $s(\mathcal{A}_\mu) = \lambda_p'(\mathcal{A}_\mu) \leqslant s(\mathcal{M}) + \varepsilon$ for $N \leqslant N_\varepsilon$. Meanwhile, it follows from [**9**, Proposition 1.1 (ii)] that $s(\mathcal{A}_\mu) = \lambda_p(\mathcal{A}_\mu) \geqslant s(\mathcal{M})$ for $N \leqslant N_\varepsilon$. Hence $s(\mathcal{A}_\mu) \to s(\mathcal{M})$ as $N \to 0$ for any $\mu > 0$.

If $s(\mathcal{M})$ is not the principal eigenvalue, then we can use an approximation argument. Set $h(x) = -d_I \int_\Omega J(x-y)\,\mathrm{d}y - \gamma(x)$. For each $\epsilon > 0$, we derive from [**39**, Lemma 3.1 and Theorem 2.1 (2)] that there exists $h_\epsilon \in C(\bar{\Omega})$ such that $\|h_\epsilon - h\|_{C(\bar{\Omega})} < \epsilon$ and $s(\mathcal{M}_\epsilon)$ is the principal eigenvalue of $\mathcal{M}_\epsilon$, where $\mathcal{M}_\epsilon$ is defined by replacing h by h_ϵ in the definition of $\mathcal{M}$. By the above arguments, we have $s(\mathcal{M}_\epsilon + \frac{1}{\mu}\mathcal{F}) \to s(\mathcal{M}_\epsilon)$ as $N \to 0$. Combined with [**39**, Theorem 2.1 (5)], we get $s(\mathcal{A}_\mu) \to s(\mathcal{M})$ as $N \to 0$. It follows from [**46**, Theorem 2.7] that $R_0 \to 0$ as $N \to 0$. By similar arguments as above, we can prove $s(\mathcal{A}_\mu) \to s(\mathcal{M} + \frac{1}{\mu}\hat{\mathcal{F}})$ as $N \to +\infty$, where $\hat{\mathcal{F}}$ is defined by $\hat{\mathcal{F}}[\phi](x) := \beta(x)\phi(x)$ for $\phi \in C(\bar{\Omega})$. Then we derive from [**46**, Theorem 2.6] that $R_0 \to r(\hat{\mathcal{L}})$ as $N \to +\infty$. The proof is completed. □

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Proof of theorem 2.3. Define

$$\mathcal{N}(\kappa) := \frac{\kappa}{d_S}\int_\Omega \left[\left(1-\tilde{I}_\kappa\right)+\frac{d_S}{d_I}\tilde{I}_\kappa\right]\mathrm{d}x,$$

where $\tilde{I}_\kappa$ is the unique nonnegative solution of (3.7). Since $R_0 > 1$, we have $\lambda_v^*(d_S\frac{N}{|\Omega|}) < 0$ due to lemma 3.4. By virtue of the continuity and monotonicity of $\lambda_v^*(\kappa)$ with respect to κ, we have $d_S\frac{N}{|\Omega|} > \kappa^*$, where κ^* is mentioned in lemma 3.6. Then, $N > \frac{|\Omega|}{d_S}\kappa^* = \mathcal{N}(\kappa^*)$. In view of lemma 3.6, $\mathcal{N}(\kappa)$ is continuous on κ. Obviously, $\mathcal{N}(0) = 0$ and $\mathcal{N}(\kappa) \to +\infty$ as $\kappa \to +\infty$. Then there exists $\kappa_N > \kappa^*$ such that $\mathcal{N}(\kappa_N) = N$. As a result, there exists a positive solution of (3.7) satisfying (3.6) for $\kappa = \kappa_N$, which implies that system (3.1) admits a positive solution by lemma 3.5.

Next we prove the uniqueness of the positive solution of (3.1). By lemma 3.5, it suffices to prove the uniqueness of the positive solution of (3.7) satisfying (3.6). If $d_S \geqslant d_I$, in view of Lemma 3.6, $\mathcal{N}(\kappa)$ is strictly increasing with respect to κ. Thus, κ_N is unique implying the uniqueness of the positive solution of (3.7) satisfying (3.6). The proof is completed. □

Proof of theorem 2.4. Since $R_0 > 1$, we have $\lambda_v < 0$ and (1.1) admits an endemic steady state (S, I). It follows from lemma 3.5 that the function $\kappa = d_S S + d_I I$ is a constant function and $\tilde{I} = \frac{d_I I}{\kappa}$ satisfies (3.7). Noting $0 < \tilde{I} < 1$, up to a subsequence, there exists $0 \leqslant \hat{I} \leqslant 1$ such that $\tilde{I}(x) \to \hat{I}(x)$ as $d_S \to 0$ for every $x \in \bar{\Omega}$. Suppose that there is some $x_0 \in H^-$ such that $\hat{I}(x_0) = 1$. By (3.7), we have $d_I\int_\Omega J(x-y)(\tilde{I}(y)-\tilde{I}(x))\,\mathrm{d}y > 0$ for all $x \in H^-$. Letting $d_S \to 0$ gives $d_I\int_\Omega J(x_0-y)(\hat{I}(y)-\hat{I}(x_0))\,\mathrm{d}y \geqslant 0$. Since $0 \leqslant \hat{I} \leqslant 1$ and $\hat{I}(x_0) = 1$, we have $\int_\Omega J(x_0-y)(\hat{I}(y)-1)\,\mathrm{d}y = 0$, which implies $\hat{I}(y) = 1$ almost everywhere in $B_\sigma(x_0) \subset H^-$, where $B_\sigma(x_0)$ is a ball centred at x_0 with radius σ. If $\beta \equiv \gamma$ on $B_\sigma(x_0)$, then we derive from (3.7) that

$$d_I\int_\Omega J(x-y)(\tilde{I}(y)-\tilde{I}(x))\,\mathrm{d}y - \frac{\gamma(x)\left(m(x)+\frac{\kappa}{d_I}\tilde{I}\right)\tilde{I}}{m(x)+\frac{\kappa}{d_S}(1-\tilde{I})+\frac{\kappa}{d_I}\tilde{I}} = 0, \quad x \in B_\sigma(x_0),$$

from which we derive that $\frac{\gamma(x)(m(x)+\frac{\kappa}{d_I}\tilde{I}(x))\tilde{I}(x)}{m(x)+\frac{\kappa}{d_S}(1-\tilde{I}(x))+\frac{\kappa}{d_I}\tilde{I}(x)} \to 0$ for all $x \in B_\sigma(x_0)$ as $d_S \to 0$. In view of (3.3) and (3.2), we get

$$\kappa = \frac{1}{|\Omega|}\left[d_S N - (d_S - d_I)\int_\Omega I\,\mathrm{d}x\right]. \tag{3.8}$$

This implies that there is some constant $d_* > 0$ such that κ is uniformly bounded for all $0 < d_S < d_*$. As a result, $S(x) = \frac{\kappa}{d_S}(1-\tilde{I}(x)) \to +\infty$ for all $x \in B_\sigma(x_0)$ as $d_S \to 0$. This contradicts (3.2).

If $\beta \not\equiv \gamma$ on $B_\sigma(x_0)$, integrating (3.7) over $B_\sigma(x_0)$ yields

$$d_I\int_{B_\sigma(x_0)}\int_\Omega J(x-y)(\tilde{I}(y)-\tilde{I}(x))\,\mathrm{d}y\,\mathrm{d}x + \int_{B_\sigma(x_0)}(\beta(x)-\gamma(x))\tilde{I}(x)\,\mathrm{d}x \geqslant 0.$$

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Sending $d_S \to 0$ gives

$$d_I \int_{B_\sigma(x_0)} \int_\Omega J(x-y)(\hat{I}(y) - \hat{I}(x))\, \mathrm{d}y\, \mathrm{d}x + \int_{B_\sigma(x_0)} (\beta(x) - \gamma(x))\hat{I}(x)\, \mathrm{d}x \geqslant 0,$$

which implies $\int_{B_\sigma(x_0)} (\beta(x) - \gamma(x))\hat{I}(x)\, \mathrm{d}x \geqslant 0$. A contradiction occurs. Thus,

$$H^- \subset \{x \in \bar{\Omega} : 0 \leqslant \hat{I}(x) < 1\}. \tag{3.9}$$

By (3.6), we have

$$N = \frac{\kappa}{d_S} \int_\Omega \left[\left(1 - \tilde{I}\right) + \frac{d_S}{d_I}\tilde{I} \right] \mathrm{d}x > \frac{\kappa}{d_S} \int_\Omega \left(1 - \tilde{I}\right) \mathrm{d}x.$$

If d_S is sufficiently small, then $\int_\Omega (1-\tilde{I})\, \mathrm{d}x \geqslant \frac{1}{2}\int_\Omega (1-\hat{I})\, \mathrm{d}x \geqslant \frac{1}{2}\int_{H^-} (1-\hat{I})\, \mathrm{d}x > 0$ due to (3.9). As a result,

$$0 < \frac{\kappa}{d_S} < \frac{N}{\int_\Omega (1-\tilde{I})\, \mathrm{d}x} \leqslant \frac{2N}{\int_\Omega (1-\hat{I})\, \mathrm{d}x} \quad \text{for all} \quad 0 < d_S < d_*.$$

As the right-hand side is constant, we have $\kappa \to 0$ as $d_S \to 0$. Then, $I = \frac{\kappa}{d_I}\tilde{I} \to 0$ uniformly on $\bar{\Omega}$ as $d_S \to 0$. By (3.6), we have

$$\frac{\kappa}{d_S} = \frac{N}{\int_\Omega (1-\tilde{I}) + \frac{d_S}{d_I}\tilde{I}\, \mathrm{d}x} \to \frac{N}{\int_\Omega (1-\hat{I})\, \mathrm{d}x} \quad \text{as} \quad d_S \to 0.$$

Set $v(x) = -d_I \int_\Omega J(x-y)\, \mathrm{d}y + \theta(x) - \gamma(x)$. Find a sequence $\{v_n\}$ such that

$$\|v_n - v\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad n \to +\infty$$

and the eigenvalue problem

$$d_I \int_\Omega J(x-y)\varphi_n(y)\, \mathrm{d}y + v_n(x)\varphi_n(x) = -\lambda\varphi_n(x), \quad x \in \Omega$$

admits a principal eigenpair $(\lambda_n, \varphi_n(x))$. There exists $n_1 > 0$ large enough such that

$$\lambda_n \leqslant \frac{1}{2}\lambda_v - \|v_n - v\|_{L^\infty(\Omega)} \quad \text{for all} \quad n \geqslant n_1.$$

By (3.6), we have

$$\frac{\kappa}{d_S} = \frac{N}{\int_\Omega (1-\tilde{I}) + \frac{d_S}{d_I}\tilde{I}\, \mathrm{d}x} \geqslant \frac{N}{|\Omega|} \quad \text{for all} \quad 0 < d_S < d_*. \tag{3.10}$$

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Set $\underline{\tilde{I}} = \delta\varphi_n$ and a simple computation gives that

$$\begin{aligned}
& d_I \int_\Omega J(x-y)(\underline{\tilde{I}}(y) - \underline{\tilde{I}}(x))\,\mathrm{d}y + \frac{\beta(x)\frac{\kappa}{d_S}(1-\underline{\tilde{I}})\underline{\tilde{I}}}{m(x)+\frac{\kappa}{d_S}(1-\underline{\tilde{I}})+\frac{\kappa}{d_I}\underline{\tilde{I}}} - \gamma(x)\underline{\tilde{I}} \\
&\quad = d_I \int_\Omega J(x-y)(\underline{\tilde{I}}(y) - \underline{\tilde{I}}(x))\,\mathrm{d}y + \left(\frac{\beta(x)\frac{\kappa}{d_S}}{m(x)+\frac{\kappa}{d_S}} - \gamma(x)\right)\underline{\tilde{I}} \\
&\qquad - \beta(x)\frac{\kappa}{d_S}\underline{\tilde{I}}^2\left[\frac{m(x)+\frac{\kappa}{d_I}}{\left(m(x)+\frac{\kappa}{d_S}(1-\underline{\tilde{I}})+\frac{\kappa}{d_I}\underline{\tilde{I}}\right)\left(m(x)+\frac{\kappa}{d_S}\right)}\right] \\
&\quad \geqslant d_I \int_\Omega J(x-y)(\underline{\tilde{I}}(y) - \underline{\tilde{I}}(x))\,\mathrm{d}y + (\theta(x)-\gamma(x))\underline{\tilde{I}} \\
&\qquad - \beta(x)\frac{\kappa}{d_S}\underline{\tilde{I}}^2\left[\frac{m(x)+\frac{\kappa}{d_I}}{\left(m(x)+\frac{\kappa}{d_S}(1-\underline{\tilde{I}})+\frac{\kappa}{d_I}\underline{\tilde{I}}\right)\left(m(x)+\frac{\kappa}{d_S}\right)}\right] \\
&\quad = (-\lambda_n + v(x) - v_n(x))\underline{\tilde{I}} - \beta(x)\frac{\kappa}{d_S}\underline{\tilde{I}}^2\left[\frac{m(x)+\frac{\kappa}{d_I}}{\left(m(x)+\frac{\kappa}{d_S}(1-\underline{\tilde{I}})+\frac{\kappa}{d_I}\underline{\tilde{I}}\right)\left(m(x)+\frac{\kappa}{d_S}\right)}\right] \\
&\quad \geqslant -\frac{1}{2}\lambda_v\underline{\tilde{I}} - \beta(x)\frac{\kappa}{d_S}\underline{\tilde{I}}^2\left[\frac{m(x)+\frac{\kappa}{d_I}}{\left(m(x)+\frac{\kappa}{d_S}(1-\underline{\tilde{I}})+\frac{\kappa}{d_I}\underline{\tilde{I}}\right)\left(m(x)+\frac{\kappa}{d_S}\right)}\right] \\
&\quad \geqslant 0,
\end{aligned}$$

provided δ small enough for all $0 < d_S < d_*$. Hence $0 < \delta\varphi_n \leqslant \tilde{I} \leqslant 1$ for all $0 < d_S < d_*$. Denote $\Phi(x,\,\tilde{I}) = \frac{\beta(x)\frac{\kappa}{d_S}(1-\tilde{I})}{m(x)+\frac{\kappa}{d_S}(1-\tilde{I})+\frac{\kappa}{d_I}\tilde{I}} - \gamma(x)$ and $\Theta(x,\,\tilde{I}) = \Phi(x,\,\tilde{I})\tilde{I}$. It is easy to check that $\partial_{\tilde{I}}\Phi(x,\,s) < 0$ for all $s > 0$. For any $x_1,\,x_2 \in \bar{\Omega}$, we find that

$$\begin{aligned}
& d_I \int_\Omega (J(x_1-y) - J(x_2-y))\tilde{I}(y)\,\mathrm{d}y - d_I\tilde{I}(x_1)\int_\Omega (J(x_1-y)-J(x_2-y))\,\mathrm{d}y \\
&\quad + [\Theta(x_1,\tilde{I}(x_1)) - \Theta(x_2,\tilde{I}(x_1))] \\
&\qquad = -\left[-d_I\int_\Omega J(x_2-y)\,\mathrm{d}y + \partial_{\tilde{I}}\Theta(x_2,\tau\tilde{I}(x_1)+(1-\tau)\tilde{I}(x_2))\right](\tilde{I}(x_1)-\tilde{I}(x_2)),
\end{aligned} \tag{3.11}$$

in which $0 \leqslant \tau \leqslant 1$. Without loss of generality, assume $\tilde{I}(x_1) \geqslant \tilde{I}(x_2)$. Since $\partial_{\tilde{I}}\Theta(x,\,s) = \Phi(x,\,s) + \partial_{\tilde{I}}\Phi(x,\,s)s < \Phi(x,\,s)$ for all $s > 0$, we have

$$\partial_{\tilde{I}}\Theta(x_2,\tau\tilde{I}(x_1)+(1-\tau)\tilde{I}(x_2)) \leqslant \Phi(x_2,\tau\tilde{I}(x_1)+(1-\tau)\tilde{I}(x_2)) < \Phi(x_2,\tilde{I}(x_2)). \tag{3.12}$$

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Note that $\tilde{I}(x_2)$ satisfies

$$d_I \int_\Omega J(x_2-y)\tilde{I}(y)\,\mathrm{d}y + \left[-d_I \int_\Omega J(x_2-y)\,\mathrm{d}y + \Phi(x_2,\tilde{I}(x_2))\right]\tilde{I}(x_2) = 0.$$

Since $0<\delta\varphi_n \leqslant \tilde{I} \leqslant 1$ for all $0<d_S<d_*$ and $\delta\varphi_n$ is independent of d_S, there exists $\eta>0$ such that

$$-d_I \int_\Omega J(x_2-y)\,\mathrm{d}y + \Phi(x_2,\tilde{I}(x_2)) < -\eta. \tag{3.13}$$

We derive from (3.12) and (3.13) that

$$-d_I \int_\Omega J(x_2-y)\,\mathrm{d}y + \partial_{\tilde{I}}\Theta(x_2,\tau\tilde{I}(x_1)+(1-\tau)\tilde{I}(x_2)) < -\eta. \tag{3.14}$$

It follows from (3.11) and (3.14) that $\tilde{I}$ is equicontinuous with respect to $0<d_S<d_*$. Applying the Arzelà–Ascoli Theorem yields $\tilde{I}\to\hat{I}$ uniformly on $\bar{\Omega}$ as $d_S\to 0$ and $0<\hat{I}\leqslant 1$. Then, $\frac{I}{d_S}=\frac{\tilde{I}}{d_I}\frac{\kappa}{d_S}\to\frac{\hat{I}}{d_I}\frac{N}{\int_\Omega(1-\hat{I})\,\mathrm{d}x}:=V^*$ and $S=\frac{\kappa}{d_S}(1-\tilde{I})\to\frac{N}{\int_\Omega(1-\hat{I})\,\mathrm{d}x}(1-\hat{I}):=S^*$ uniformly on $\bar{\Omega}$ as $d_S\to 0$. Dividing two equations of (3.1) by d_S and letting $d_S\to 0$ yield that $(S^*,\,V^*)$ satisfies (2.1). The proof is completed. □

Proof of theorem 2.5. Since $R_0>1$, we derive from theorem 2.3 that there exists an endemic steady state $(S,\,I)$ of (1.1) for any $d_S>0$ with R_0 independent of d_S. Note that $\int_\Omega I\,\mathrm{d}x\leqslant N$. Then there exists a sequence $\{d_{S_n}\}$ with $d_{S_n}\to 0$ as $n\to+\infty$ such that the corresponding endemic steady state $(S_n,\,I_n)$ satisfies $\int_\Omega I_n\,\mathrm{d}x\to l$ for some $0\leqslant l\leqslant N$. By (3.8), we get that

$$\kappa_n = \frac{1}{|\Omega|}\left[d_{S_n}N-(d_{S_n}-d_I)\int_\Omega I_n\,\mathrm{d}x\right]\to\frac{d_I}{|\Omega|}l \quad \text{as} \quad n\to+\infty$$

and $\{\kappa_n\}$ is bounded. We derive from (3.3) that $0<I_n\leqslant\frac{\kappa_n}{d_I}$, which implies that $\{I_n\}$ is uniformly bounded. Since S_n is continuous on $\bar{\Omega}$, there exist $x_n\in\bar{\Omega}$ such that $S_n(x_n)=\max\limits_{x\in\bar{\Omega}}S_n(x)$. By the first equation of (3.1), we have $-\frac{\beta(x_n)S_n(x_n)}{m(x_n)+S_n(x_n)+I_n(x_n)}+\gamma(x_n)\geqslant 0$. Then we derive that

$$S_n(x_n)\leqslant\frac{\gamma(x_n)}{\beta(x_n)-\gamma(x_n)}(m(x_n)+I_n(x_n)).$$

Thus, $\{S_n\}$ is uniformly bounded. By (3.3), $I_n=\frac{\kappa_n-d_{S_n}S_n}{d_I}\to\frac{l}{|\Omega|}$ as $n\to+\infty$.

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If $l > 0$, the first equation of (3.1) gives

$$S_n = \frac{d_{S_n} \int_\Omega J(x-y)(S_n(y) - S_n(x))\,\mathrm{d}y(m(x) + S_n + I_n) + \gamma(x)(m(x) + I_n)I_n}{(\beta(x) - \gamma(x))I_n},$$

which implies that $S_n \to \frac{\gamma(m+\frac{l}{|\Omega|})}{\beta-\gamma}$ in $C(\bar{\Omega})$ as $n \to +\infty$. By (3.2), we have

$$I_n \to \frac{l}{|\Omega|} = \frac{N - \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x}{\int_\Omega \frac{\beta}{\beta-\gamma}\,\mathrm{d}x} \quad \text{uniformly on} \quad \bar{\Omega} \quad \text{as} \quad n \to +\infty.$$

Now, we are in a position to consider the case $l = 0$ implying $\kappa_n \to 0$ as $n \to +\infty$. Up to a subsequence if needed, one of the following three statements must hold:

(a) $\frac{\kappa_n}{d_{S_n}} \to 0$ as $n \to +\infty$;

(b) $\frac{\kappa_n}{d_{S_n}} \to C$ with C being a positive constant as $n \to +\infty$;

(c) $\frac{\kappa_n}{d_{S_n}} \to +\infty$ as $n \to +\infty$.

In view of (3.10), (a) is impossible to occur. By similar arguments as the proof of theorem 2.4, there exists some function $0 < \hat{I} \leqslant 1$ such that $\tilde{I}_n = \frac{d_I I_n}{\kappa_n} \to \hat{I}$ uniformly on $\bar{\Omega}$ as $n \to +\infty$. If (b) holds, then $\frac{I_n}{d_{S_n}} = \frac{\tilde{I}_n}{d_I}\frac{\kappa_n}{d_{S_n}} \to \frac{\hat{I}}{d_I}C := V^*$ and $S_n = \frac{\kappa_n}{d_{S_n}}(1 - \tilde{I}_n) \to C(1 - \hat{I}) := S^*$ in $C(\bar{\Omega})$ as $n \to +\infty$. Dividing the two equations of (3.1) by d_S and letting $n \to +\infty$ yield (S^*, V^*) satisfies (2.1). If (c) holds, then $S_n = \frac{\kappa_n}{d_{S_n}}(1 - \tilde{I}_n)$ satisfies

$$\frac{d_{S_n}}{\kappa_n} d_I \int_\Omega J(x-y)(S_n(y) - S_n(x))\,\mathrm{d}y - \frac{\beta(x)S_n(x)\tilde{I}_n(x)}{m(x) + S_n(x) + \frac{\kappa_n}{d_I}\tilde{I}_n(x)} + \gamma(x)\tilde{I}_n(x) = 0. \tag{3.15}$$

We derive from (3.15) that

$$\begin{aligned} S_n(x) = {} & \frac{\frac{d_{S_n}}{\kappa_n} d_I \int_\Omega J(x-y)(S_n(y) - S_n(x))\,\mathrm{d}y(m(x) + S_n(x) + \frac{\kappa_n}{d_I}\tilde{I}_n(x))}{(\beta(x) - \gamma(x))\tilde{I}_n(x)} \\ & + \frac{\gamma(x)\tilde{I}_n(x)(m(x) + \frac{\kappa_n}{d_I}\tilde{I}_n(x))}{(\beta(x) - \gamma(x))\tilde{I}_n(x)}, \end{aligned}$$

which implies that $S_n \to \frac{\gamma m}{\beta-\gamma}$ in $C(\bar{\Omega})$ as $n \to +\infty$. If $\int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x \neq N$, a contradiction occurs due to $I_n \to 0$ as $n \to +\infty$.

If $N \leqslant \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x$, by above arguments, it is easy to see that $I_n \to 0$ as $n \to +\infty$. If $\beta - \gamma$ and m are positive constants with $N > \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x$, assume on the contrary that $I_n \to 0$ as $n \to +\infty$. We derive from the above arguments that $\frac{\kappa_n}{d_{S_n}} \to C$ as

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$n \to +\infty$ and $\hat{I}$ satisfies

$$d_I \int_\Omega J(x-y)(\hat{I}(y)-\hat{I}(x))\,\mathrm{d}y + \frac{\beta(x)C(1-\hat{I})\hat{I}}{m(x)+C(1-\hat{I})} - \gamma(x)\hat{I} = 0. \tag{3.16}$$

Note that $C\int_\Omega (1-\hat{I})\,\mathrm{d}x = N$. In view of (3.16), we have

$$N - \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x = -\frac{d_I}{2}\frac{m+C}{\beta-\gamma}\int_\Omega J(x-y)\left(\sqrt{\frac{\hat{I}(y)}{\hat{I}(x)}} - \sqrt{\frac{\hat{I}(x)}{\hat{I}(y)}}\right)^2 \mathrm{d}y \leqslant 0,$$

which is a contradiction. The proof is completed. □

Proof of theorem 2.7. Since Ω^+ is nonempty, [**45**, Corollaries 2.12 and 2.13] implies that $R_0 > 1$ for all small d_I. Then by theorem 2.3, the endemic steady state (S, I) exists for small d_I.

(i) **Case I.** $d \in [0, 1]$. By (3.6), there exists some positive constant d_1 such that

$$\frac{\kappa}{d_S} = \frac{N}{\int_\Omega \left[(1-\tilde{I}) + \frac{d_S}{d_I}\tilde{I}\right]\mathrm{d}x} \leqslant \frac{N}{|\Omega|} \quad \text{for all } \; 0 < d_I, \left|\frac{d_I}{d_S} - d\right| \leqslant d_1. \tag{3.17}$$

Up to a subsequence, we assume $\frac{\kappa}{d_S} \to l \leqslant \frac{N}{|\Omega|}$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to d$. Note that $I = \frac{\kappa}{d_I}\tilde{I}$ and $\tilde{I}$ satisfies (3.7). One can get that I satisfies

$$d_I \int_\Omega J(x-y)(I(y)-I(x))\,\mathrm{d}y + \frac{\beta(x)\frac{\kappa}{d_S}(1-\frac{d_I}{\kappa}I)I}{m(x)+\frac{\kappa}{d_S}(1-\frac{d_I}{\kappa}I)+I} - \gamma(x)I = 0,$$

which implies that

$$I(x) = \frac{-B(x) - \sqrt{B^2(x) - 4A(x)C(x)}}{2A(x)}, \tag{3.18}$$

where

$$\begin{aligned}
A(x) &= -\frac{d_I}{d_S}\beta(x) + \frac{d_I}{d_S}\gamma(x) - \gamma(x) - \left(1 - \frac{d_I}{d_S}\right) d_I \int_\Omega J(x-y)\,\mathrm{d}y, \\
B(x) &= -\left(m(x) + \frac{\kappa}{d_S}\right) d_I \int_\Omega J(x-y)\,\mathrm{d}y + \frac{\kappa}{d_S}\beta(x) - \gamma(x)m(x) - \frac{\kappa}{d_S}\gamma(x) \\
&\quad + \left(1 - \frac{d_I}{d_S}\right) d_I \int_\Omega J(x-y)I(y)\,\mathrm{d}y, \\
C(x) &= \left(m(x) + \frac{\kappa}{d_S}\right) d_I \int_\Omega J(x-y)I(y)\,\mathrm{d}y.
\end{aligned}$$

By (3.3) and (3.17), S is uniformly bounded with respect to $0 < d_I, \left|\frac{d_I}{d_S} - d\right| \leqslant d_1$. Let $I(x_0) = \max_{x\in\bar{\Omega}} I(x)$. It follows from the second equation

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of (3.1) that

$$I(x_0) \leqslant \max_{x\in\bar{\Omega}} \frac{\beta(x)S(x)}{\gamma(x)}, \tag{3.19}$$

which implies that I is uniformly bounded with respect to $0 < d_I$, $\left|\frac{d_I}{d_S} - d\right| \leqslant d_1$. Let $S(x_1) = \min_{x\in\bar{\Omega}} S(x)$. By the first equation of (3.1) that $S(x_1) \geqslant \min_{x\in\bar{\Omega}} \frac{\gamma(x)m(x)}{\beta(x)}$. If $l = 0$, then $I = \frac{\kappa}{d_S}\frac{d_S}{d_I}\tilde{I} \to 0$ uniformly on $\bar{\Omega}$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to d \in (0,\, 1]$. Then, $S = \frac{\kappa - d_I I}{d_S} \to 0$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to d \in [0,\, 1]$, which is a contradiction. Thus, $0 < l \leqslant \frac{N}{|\Omega|}$. By (3.18), up to a subsequence, we have

$$I \to \frac{(l(\beta-\gamma)-\gamma m)^+}{d\beta + (1-d)\gamma} := I^* \ \text{ in } \ C(\bar{\Omega}) \ \text{ as } \ d_I \to 0 \ \text{ and } \ \frac{d_I}{d_S} \to d.$$

Then, up to a subsequence, $S = \frac{\kappa - d_I I}{d_S} \to l - dI^*$ in $C(\bar{\Omega})$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to d$. Moreover,

$$\int_\Omega [(1-d)I^* + l]\,\mathrm{d}x = N. \tag{3.20}$$

Set $g(l) = \int_\Omega [(1-d)I^* + l]\,\mathrm{d}x$. Obviously, $g(0) = 0$, $g(\frac{N}{|\Omega|}) > N$ and $g(l)$ is strictly increasing with respect to $l \in [0,\, \frac{N}{|\Omega|}]$. Then l is uniquely determined by (3.20). Hence the limits of S and I are independent of any chosen subsequence.

Case II. $d \in (1, +\infty)$. By (3.6), there exists some positive constant d_2 such that

$$\frac{\kappa}{d_S} = \frac{N}{\int_\Omega \left[(1-\tilde{I}) + \frac{d_S}{d_I}\tilde{I}\right]\,\mathrm{d}x} \geqslant \frac{N}{|\Omega|} \ \text{ for all } \ 0 < d_I, \left|\frac{d_I}{d_S} - d\right| \leqslant d_2.$$

It follows from (3.8) that

$$\begin{aligned}\frac{\kappa}{d_S} &= \frac{1}{|\Omega|}\left[N - \left(1 - \frac{d_I}{d_S}\right)\int_\Omega I\,\mathrm{d}x\right] \\ &\leqslant \frac{N}{|\Omega|}(2 + d + d_2) \ \text{ for all } \ 0 < d_I, \left|\frac{d_I}{d_S} - d\right| \leqslant d_2.\end{aligned}$$

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Up to a subsequence, we assume $\frac{\kappa}{d_S} \to l \geqslant \frac{N}{|\Omega|}$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to d$. For any given $\varepsilon > 0$, set

$$\tilde{I}_1^{\varepsilon}(x) = \begin{cases} \dfrac{(l-\varepsilon)(\beta(x)-\gamma(x))-\gamma(x)m(x)}{l(\beta(x)-\gamma(x))+\frac{l}{d}\gamma(x)}, & \text{if } (l-\varepsilon)(\beta(x)-\gamma(x)) \\ & \quad -\gamma(x)m(x) > 0,\ x \in \bar{\Omega}, \\ 0, & \text{if } (l-\varepsilon)(\beta(x)-\gamma(x)) \\ & \quad -\gamma(x)m(x) \leqslant 0,\ x \in \bar{\Omega}, \end{cases}$$

$$\tilde{I}_2^{\varepsilon}(x) = \begin{cases} \dfrac{(l+\varepsilon)(\beta(x)-\gamma(x))-\gamma(x)m(x)}{l(\beta(x)-\gamma(x))+\frac{l}{d}\gamma(x)}, & \text{if } (l+\varepsilon)(\beta(x)-\gamma(x)) \\ & \quad -\gamma(x)m(x) > 0,\ x \in \bar{\Omega}, \\ \varepsilon, & \text{if } (l+\varepsilon)(\beta(x)-\gamma(x)) \\ & \quad -\gamma(x)m(x) \leqslant 0,\ x \in \bar{\Omega}. \end{cases}$$

It is easy to check that $\tilde{I}_1^{\varepsilon}$ and $\tilde{I}_2^{\varepsilon}$ are respectively lower and upper solutions of (3.7) for all $0 < d_I$, $\left|\frac{d_I}{d_S} - d\right| \leqslant d_2$. Letting $\varepsilon \to 0$ yields that $\tilde{I} \to \hat{I}$ in $C(\bar{\Omega})$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to d$, where

$$\hat{I}(x) = \begin{cases} \dfrac{l(\beta(x)-\gamma(x))-\gamma(x)m(x)}{l(\beta(x)-\gamma(x))+\frac{l}{d}\gamma(x)}, & \text{if } l(\beta(x)-\gamma(x))-\gamma(x)m(x) > 0,\ x \in \bar{\Omega}, \\ 0, & \text{if } l(\beta(x)-\gamma(x))-\gamma(x)m(x) \leqslant 0,\ x \in \bar{\Omega}. \end{cases}$$

Then, $I = \frac{\kappa}{d_S}\frac{d_S}{d_I}\tilde{I} \to \frac{l}{d}\hat{I}$ and $S = \frac{\kappa}{d_S}(1-\tilde{I}) \to l(1-\hat{I})$ in $C(\bar{\Omega})$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to d$.

(ii) Noting that $0 < \tilde{I} < 1$, up to a subsequence, there exists $0 \leqslant \hat{I} \leqslant 1$ such that $\tilde{I}(x) \to \hat{I}(x)$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$ for every $x \in \bar{\Omega}$. Similar to the proof of theorem 2.4, we obtain

$$H^- \subset \left\{ x \in \bar{\Omega} : 0 \leqslant \hat{I}(x) < 1 \right\},$$

which implies that $\int_\Omega (1-\hat{I})\,\mathrm{d}x \geqslant \int_{H^-}(1-\hat{I})\,\mathrm{d}x > 0$. Thus, there exists some constant $d_3 > 0$ such that $\int_\Omega (1-\tilde{I})\,\mathrm{d}x \geqslant \frac{1}{2}\int_\Omega (1-\hat{I})\,\mathrm{d}x > 0$ and

$$\frac{N}{|\Omega|} < \frac{\kappa}{d_S} < \frac{N}{\int_\Omega (1-\tilde{I})\,\mathrm{d}x} \leqslant \frac{2N}{\int_\Omega (1-\hat{I})\,\mathrm{d}x}$$

due to (3.6) for all $0 < d_I$, $\frac{d_S}{d_I} < d_3$. Then, $\frac{\kappa}{d_I} = \frac{\kappa}{d_S}\frac{d_S}{d_I} \to 0$ and $I = \frac{\kappa}{d_I}\tilde{I} \to 0$ uniformly on $\bar{\Omega}$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$. Up to a subsequence, we assume

$\frac{\kappa}{d_S} \to l$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$. For any given $\varepsilon > 0$, set

$$\tilde{I}_1^\varepsilon(x) = \begin{cases} \dfrac{(l-\varepsilon)(\beta(x)-\gamma(x))-\gamma(x)m(x)}{l(\beta(x)-\gamma(x))}, & \text{if } (l-\varepsilon)(\beta(x)-\gamma(x)) \\ & \quad -\gamma(x)m(x) > 0, \ x \in \bar{\Omega}, \\ 0, & \text{if } (l-\varepsilon)(\beta(x)-\gamma(x)) \\ & \quad -\gamma(x)m(x) \leqslant 0, \ x \in \bar{\Omega}, \end{cases}$$

$$\tilde{I}_2^\varepsilon(x) = \begin{cases} \dfrac{(l+\varepsilon)(\beta(x)-\gamma(x))-\gamma(x)m(x)}{l(\beta(x)-\gamma(x))}, & \text{if } (l+\varepsilon)(\beta(x)-\gamma(x)) \\ & \quad -\gamma(x)m(x) > 0, \ x \in \bar{\Omega}, \\ \varepsilon, & \text{if } (l+\varepsilon)(\beta(x)-\gamma(x)) \\ & \quad -\gamma(x)m(x) \leqslant 0, \ x \in \bar{\Omega}. \end{cases}$$

It is easy to check that $\tilde{I}_1^\varepsilon$ and $\tilde{I}_2^\varepsilon$ are respectively lower and upper solutions of (3.7) for all $0 < d_I, \frac{d_S}{d_I} \leqslant d_3$. Letting $\varepsilon \to 0$ yields that $\tilde{I} \to \hat{I}$ in $C(\bar{\Omega})$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$, where

$$\hat{I}(x) = \begin{cases} \dfrac{l(\beta(x)-\gamma(x))-\gamma(x)m(x)}{l(\beta(x)-\gamma(x))}, & \text{if } l(\beta(x)-\gamma(x))-\gamma(x)m(x) > 0, \ x \in \bar{\Omega}, \\ 0, & \text{if } l(\beta(x)-\gamma(x))-\gamma(x)m(x) \leqslant 0, \ x \in \bar{\Omega}. \end{cases}$$

Hence, $S = \frac{\kappa}{d_S}(1-\tilde{I}) \to l(1-\hat{I})$ in $C(\bar{\Omega})$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$. By (3.2), l is determined by $\int_\Omega l(1-\hat{I})\,\mathrm{d}x = N$.

(iii) In view of (3.6), there exists $d_0 > 0$ such that $\frac{\kappa}{d_S} \geqslant \frac{N}{|\Omega|}$ for all $0 < d_I, \frac{d_S}{d_I} < d_0$. Up to a subsequence, we assume $\frac{\kappa}{d_S} \to l \geqslant \frac{N}{|\Omega|}$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$.

If $l < +\infty$, then $\frac{\kappa}{d_I} = \frac{\kappa}{d_S}\frac{d_S}{d_I} \to 0$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$. We derive from (3.7) that

$$\tilde{I}(x) = \frac{-E(x) - \sqrt{E^2(x) - 4D(x)F(x)}}{2D(x)}, \tag{3.21}$$

where

$$\begin{aligned} D(x) &= -d_I\left(-\frac{\kappa}{d_S} + \frac{\kappa}{d_I}\right)\int_\Omega J(x-y)\,\mathrm{d}y - \frac{\kappa}{d_S}\beta(x) - \left(-\frac{\kappa}{d_S} + \frac{\kappa}{d_I}\right)\gamma(x), \\ E(x) &= d_I\left(-\frac{\kappa}{d_S} + \frac{\kappa}{d_I}\right)\int_\Omega J(x-y)\tilde{I}(y)\,\mathrm{d}y - d_I\left(m(x) + \frac{\kappa}{d_S}\right)\int_\Omega J(x-y)\,\mathrm{d}y \\ &\quad + \frac{\kappa}{d_S}\beta(x) - \left(m(x) + \frac{\kappa}{d_S}\right)\gamma(x), \\ F(x) &= d_I\left(m(x) + \frac{\kappa}{d_S}\right)\int_\Omega J(x-y)\tilde{I}(y)\,\mathrm{d}y, \end{aligned}$$

which implies that $\tilde{I} \to \frac{[l(\beta-\gamma)-\gamma m]^+}{l(\beta-\gamma)}$ in $C(\bar{\Omega})$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$. We claim that $\liminf_{d_I \to 0, \frac{d_I}{d_S} \to +\infty} \|\tilde{I}\|_{L^\infty(\Omega)} > 0$. Assume on the contrary that it is false.

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Then there exist two sequences $\{d_{I_k}\}$ and $\left\{\frac{d_{I_k}}{d_{S_k}}\right\}$ with $d_{I_k} \to 0$ and $\frac{d_{I_k}}{d_{S_k}} \to +\infty$ as $k \to +\infty$ such that $\tilde{I}_k \to 0$ as $k \to +\infty$. Then, $\frac{[l(\beta-\gamma)-\gamma m]^+}{l(\beta-\gamma)} \equiv 0$; that is, $l(\beta-\gamma) - \gamma m \leqslant 0$. Since $l \geqslant \frac{N}{|\Omega|}$, we have $\frac{N}{|\Omega|}(\beta-\gamma) - \gamma m \leqslant 0$ contradicting that Ω^+ is nonempty. As a result, there exists some positive constant C_1 such that $\|I\|_{L^\infty(\Omega)} = \frac{\kappa}{d_S}\frac{d_S}{d_I}\|\tilde{I}\|_{L^\infty(\Omega)} \geqslant C_1\frac{d_S}{d_I}$ for all $0 < d_I, \frac{d_S}{d_I} < d_0$. Note that $I = \frac{\kappa}{d_I}\tilde{I} = \frac{\kappa}{d_S}\frac{d_S}{d_I}\tilde{I} \leqslant C_2\frac{d_S}{d_I}$ with C_2 being some positive constant for all $0 < d_I, \frac{d_S}{d_I} < d_0$. One can get that $I(x) \to 0$ uniformly on $\bar{\Omega}$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$. We conclude that $S = \frac{\kappa}{d_S}(1-\tilde{I}) \to l[1 - \frac{[l(\beta-\gamma)-\gamma m]^+}{l(\beta-\gamma)}] := S^*$ in $C(\bar{\Omega})$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$. In addition, l is determined by $\int_\Omega S^* \,\mathrm{d}x = N$.

If $l = +\infty$, then (3.6) implies that $\int_\Omega \tilde{I}\,\mathrm{d}x = \frac{|\Omega| - \frac{N}{\frac{\kappa}{d_S}}}{(1-\frac{d_S}{d_I})} \geqslant \frac{|\Omega|}{2}$ for all $0 < d_I, \frac{d_S}{d_I} < d_0$. Then $\frac{\kappa}{d_I} = \frac{\int_\Omega I\,\mathrm{d}x}{\int_\Omega \tilde{I}\,\mathrm{d}x} \leqslant \frac{2N}{|\Omega|}$ and $I = \frac{\kappa}{d_I}\tilde{I} \leqslant \frac{2N}{|\Omega|}$ for all $0 < d_I, \frac{d_S}{d_I} < d_0$. Up to a subsequence, we assume $\frac{\kappa}{d_I} \to \vartheta \leqslant \frac{2N}{|\Omega|}$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$. By (3.21), $\tilde{I} \to 1$ in $C(\bar{\Omega})$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$. Then, $I = \frac{\kappa}{d_I}\tilde{I} \to \vartheta$ in $C(\bar{\Omega})$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$. Note that $S = \frac{\kappa}{d_S}(1-\tilde{I})$ satisfies

$$\frac{d_S}{\kappa}d_I\int_\Omega J(x-y)(S(y)-S(x))\,\mathrm{d}y - \frac{\beta(x)S(x)\tilde{I}(x)}{m(x)+S(x)+\frac{\kappa}{d_I}\tilde{I}(x)} + \gamma(x)\tilde{I}(x) = 0. \tag{3.22}$$

Let $\max_{x\in\bar{\Omega}} S(x) = S(x_0)$. By virtue of (3.22), we have $\frac{\beta(x_0)S(x_0)}{m(x_0)+S(x_0)+\frac{\kappa}{d_I}\tilde{I}(x_0)} \leqslant \gamma(x_0)$ implying $S(x) \leqslant S(x_0) \leqslant \max_{x\in\bar{\Omega}}\frac{\gamma(x)m(x)}{\beta(x)-\gamma(x)} + \frac{2N}{|\Omega|}\max_{x\in\bar{\Omega}}\frac{\gamma(x)}{\beta(x)-\gamma(x)}$ for all $0 < d_I, \frac{d_S}{d_I} < d_0$. We derive from (3.22) that

$$\begin{aligned} S(x) = &\frac{\frac{d_S}{\kappa}d_I\int_\Omega J(x-y)(S(y)-S(x))\,\mathrm{d}y(m(x)+S(x)+\frac{\kappa}{d_I}\tilde{I}(x))}{(\beta(x)-\gamma(x))\tilde{I}(x)} \\ &+ \frac{\gamma(x)\tilde{I}(x)(m(x)+\frac{\kappa}{d_I}\tilde{I}(x))}{(\beta(x)-\gamma(x))\tilde{I}(x)}, \end{aligned}$$

which implies that $S \to \frac{\gamma(m+\vartheta)}{\beta-\gamma}$ in $C(\bar{\Omega})$ as $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$. If $\vartheta = 0$, then $N = \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x$ due to (3.2). If $\vartheta > 0$, then $\vartheta = \frac{N - \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x}{\int_\Omega \frac{\beta}{\beta-\gamma}\,\mathrm{d}x}$ due to (3.2).

To sum up, up to subsequences of $d_I \to 0$ and $\frac{d_I}{d_S} \to +\infty$, one of the following statements hold:

(A1) $(S, I) \to (S^*, 0) = (l - \frac{[l(\beta-\gamma)-\gamma m]^+}{\beta-\gamma}, 0)$, where the positive constant l is determined by $\int_\Omega S^*\,\mathrm{d}x = N$. In addition, there exists positive constants $0 < d_0 \ll 1$, C_1 and C_2 such that

$$C_1\frac{d_S}{d_I} \leqslant \|I\|_{L^\infty(\Omega)} \leqslant C_2\frac{d_S}{d_I} \quad \text{for all} \quad 0 < d_I, \frac{d_S}{d_I} < d_0.$$

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(A2) $(S, I) \to (\frac{\gamma m}{\beta-\gamma}, 0)$ and $N = \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x$.

(A3) $(S, I) \to (\frac{\gamma(m+I^*)}{\beta-\gamma}, I^*)$, where $I^* = \frac{N-\int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x}{\int_\Omega \frac{\beta}{\beta-\gamma}\,\mathrm{d}x} > 0$.

In the following, we aim to prove the conclusions item by item.

(a) If $N < \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x$, then (A1) must hold.

(b) If $N = \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x$, then (A1) or (A2) holds. If (A2) holds, there is nothing to prove. Assume that (A1) holds. By the positivity of I, we know 0 is the principal eigenvalue of

$$d_I \int_\Omega J(x-y)(\varphi(y)-\varphi(x))\,\mathrm{d}y + \left(\frac{\beta(x)S(x)}{m(x)+S(x)+I(x)} - \gamma(x)\right)\varphi(x) = -\lambda\varphi(x), \quad x \in \Omega. \tag{3.23}$$

In view of [39, Theorem 2.2], we have $\min_{x\in\bar\Omega}\left\{\gamma(x) - \frac{\beta(x)S^*(x)}{m(x)+S^*(x)}\right\} = 0$ implying $S^*(x) \leqslant \frac{\gamma(x)m(x)}{\beta(x)-\gamma(x)}$ for all $x \in \bar\Omega$. Set $x \in \Omega_l := \{x \in \bar\Omega : l(\beta(x) - \gamma(x)) - \gamma(x)m(x) \leqslant 0\}$. If Ω_l is empty, then $S^*(x) = \frac{\gamma(x)m(x)}{\beta(x)-\gamma(x)}$ for all $x \in \bar\Omega$. If Ω_l is nonempty, then $S^*(x) = l \leqslant \frac{\gamma(x)m(x)}{\beta(x)-\gamma(x)}$ for all $x \in \Omega_l$. Note that $\int_{\Omega_l} l\,\mathrm{d}x + \int_{\Omega\backslash\Omega_l} \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x = N = \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x$. We have $\int_{\Omega_l}(l - \frac{\gamma m}{\beta-\gamma})\,\mathrm{d}x = 0$. Hence, $\frac{\gamma(x)m(x)}{\beta(x)-\gamma(x)} = l$ for all $x \in \Omega_l$. We derive that $S^*(x) = \frac{\gamma(x)m(x)}{\beta(x)-\gamma(x)}$ for all $x \in \bar\Omega$.

(c) If $N > \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x$, then (A1) or (A3) holds. Assume that (A1) holds. By the proof of (b), we know $S^*(x) \leqslant \frac{\gamma(x)m(x)}{\beta(x)-\gamma(x)}$ for all $x \in \bar\Omega$. Then, $N = \int_\Omega S^*\,\mathrm{d}x \leqslant \int_\Omega \frac{\gamma m}{\beta-\gamma}\,\mathrm{d}x$, which is a contradiction. Hence (A3) holds. The proof is completed.

□

Proof of theorem 2.8. In view of [45, Corollaries 2.12 and 2.13] and theorem 2.3, there exists an endemic steady state (S, I). We first give the proof of (i). Since $\int_\Omega I\,\mathrm{d}x \leqslant N$, up to a subsequence, assume $\int_\Omega I\,\mathrm{d}x \to r$ for some constant $r \geqslant 0$ as $d_S \to +\infty$. In view of (3.8), we have

$$\frac{\kappa}{d_S} = \frac{1}{|\Omega|}\left[N - \left(1 - \frac{d_I}{d_S}\right)\int_\Omega I\,\mathrm{d}x\right] \to \frac{1}{|\Omega|}(N-r) \quad \text{as} \quad d_S \to +\infty.$$

Then there exists some positive constant d_S^* such that $\frac{\kappa}{d_S}$ is uniformly bounded with respect to $d_S > d_S^*$. Note that $S = \frac{\kappa}{d_S} - \frac{d_I}{d_S}I$. We derive that S is uniformly bounded with respect to $d_S > d_S^*$. In view of (3.19), I is uniformly bounded with respect to $d_S > d_S^*$. Hence, $S = \frac{\kappa}{d_S} - \frac{d_I}{d_S}I \to \frac{1}{|\Omega|}(N-r) := S^*$ as $d_S \to +\infty$. It follows from

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the second equation of (3.1) that

$$I(x) = \frac{-B(x) - \sqrt{B^2(x) - 4A(x)C(x)}}{2A(x)},$$

where

$$\begin{aligned}
A(x) &= -d_I \int_\Omega J(x-y)\,\mathrm{d}y - \gamma(x),\\
B(x) &= -\left(d_I \int_\Omega J(x-y)\,\mathrm{d}y + \gamma(x)\right)(m(x)+S(x))\\
&\quad + d_I \int_\Omega J(x-y)I(y)\,\mathrm{d}y + \beta(x)S(x),\\
C(x) &= d_I \int_\Omega J(x-y)I(y)\,\mathrm{d}y(m(x)+S(x)),
\end{aligned}$$

which implies that $I \to I^*$ in $C(\bar{\Omega})$ as $d_S \to +\infty$ and I^* satisfies (2.3). We know either $I^* \equiv 0$ or $I^* > 0$. Otherwise, there exists some $x_0 \in \bar{\Omega}$ such that $I^*(x_0) = \min_{x\in\bar{\Omega}} I^*(x) = 0$, then we derive from (2.3) that $\int_\Omega J(x_0-y)I^*(y)\,\mathrm{d}y = 0$. Thus, $I^*(x) = 0$ for all $x \in \bar{\Omega}$, which is a contradiction. By the positivity of I, 0 is the principal eigenvalue of (3.23). If $r = 0$, then $I^* \equiv 0$ and $S^* \equiv \frac{N}{|\Omega|}$. Note that $w = \frac{I}{\|I\|_{L^\infty(\Omega)}}$ satisfies

$$d_I \int_\Omega J(x-y)(w(y)-w(x))\,\mathrm{d}y + \frac{\beta(x)Sw}{m(x)+S+I} - \gamma(x)w = 0.$$

There exists some $\hat{w}(x) > 0$ such that $w(x) \to \hat{w}(x)$ as $d_S \to +\infty$ and $\hat{w}$ satisfies

$$d_I \int_\Omega J(x-y)(\hat{w}(y)-\hat{w}(x))\,\mathrm{d}y + \frac{\frac{N}{|\Omega|}\beta(x)}{m(x)+\frac{N}{|\Omega|}}\hat{w} - \gamma(x)\hat{w} = 0.$$

As a result, $\lambda_v(d_I) = 0$ contradicting $R_0 > 1$. Hence, $r > 0$ and $I^* > 0$.

Now prove the existence and uniqueness of positive solution of (2.3). Let $\tau \geqslant 0$ be a real number. Consider

$$d_I \int_\Omega J(x-y)(I(y)-I(x))\,\mathrm{d}y + \frac{\beta(x)\frac{1}{|\Omega|}(N-\tau)I}{m(x)+\frac{1}{|\Omega|}(N-\tau)+I} - \gamma(x)I = 0. \tag{3.24}$$

It follows from [**9**, Theorem 1.6] that (3.24) admits a unique positive solution if and only if $\lambda_\tau < 0$, where λ_τ is defined by

$$\lambda_\tau := \inf_{\substack{\varphi\in L^2(\Omega)\\ \varphi\neq 0}} \frac{\frac{d_I}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))^2\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega\left(\gamma(x) - \frac{\beta(x)\frac{N-\tau}{|\Omega|}}{m(x)+\frac{N-\tau}{|\Omega|}}\right)\varphi^2(x)\,\mathrm{d}x}{\int_\Omega \varphi^2(x)\,\mathrm{d}x}.$$

Since $R_0 > 1$, we have $\lambda_0 < 0$. Noting that λ_τ is increasing in τ, there exists $\tau^* > 0$ such that $\lambda_\tau < 0$ for all $0 \leqslant \tau < \tau^*$ and $\lambda_{\tau^*} = 0$. As a result, (3.24) admits a unique

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positive solution I_τ for all $0 \leqslant \tau < \tau^*$ and $I_\tau \equiv 0$ is the only nonnegative solution of (3.24) for all $\tau \geqslant \tau^*$. And it is verified that I_τ is decreasing with respect to $\tau \in [0, \tau^*)$. Set $g(\tau) = \tau - \int_\Omega I_\tau \,\mathrm{d}x$. Then $g(\tau)$ is increasing with respect to $\tau \in [0, \tau^*]$. Since $g(0) < 0$ and $g(\tau^*) > 0$, there exists a unique $\tau_0 > 0$ such that $g(\tau_0) = 0$; that is, $\tau_0 = \int_\Omega I_{\tau_0} \,\mathrm{d}x$. As a consequence, the positive solution of (2.3) uniquely exists. Hence, the limits of S and I are independent of any chosen subsequence.

Now we are devoted to the proof of (ii). Since $\int_\Omega I \,\mathrm{d}x \leqslant N$, up to a subsequence, assume $\int_\Omega I \,\mathrm{d}x \to r$ for some constant $r \geqslant 0$ as $d_I \to +\infty$. In view of (3.8), we have

$$\frac{\kappa}{d_I} = \frac{1}{|\Omega|}\left[\frac{d_S}{d_I}N - \left(\frac{d_S}{d_I} - 1\right)\int_\Omega I \,\mathrm{d}x\right] \to \frac{1}{|\Omega|}r \quad \text{as} \quad d_I \to +\infty.$$

Noting that $\frac{d_S}{d_I}S + I = \frac{\kappa}{d_I}$, there exists some constant $d_I^* > 0$ such that I is uniformly bounded with respect to $d_I > d_I^*$. Then $\frac{\beta SI}{m+S+I} - \gamma I$ is uniformly bounded with respect to $d_I > d_I^*$. We derive from the second equation of (3.1) that

$$I(x) = \frac{\int_\Omega J(x-y)I(y)\,\mathrm{d}y}{\int_\Omega J(x-y)\,\mathrm{d}y} + \frac{\frac{\beta(x)S(x)I(x)}{m(x)+S(x)+I(x)} - \gamma(x)I(x)}{d_I\int_\Omega J(x-y)\,\mathrm{d}y},$$

which implies that $I \to I^*$ in $C(\bar{\Omega})$ as $d_I \to +\infty$ and I^* satisfies $\int_\Omega J(x-y)(I^*(y) - I^*(x))\,\mathrm{d}y = 0$. Thus, I^* is a constant and $I^* = \frac{r}{|\Omega|}$.

Claim that S is uniformly bounded with respect to $d_I > d_I^*$. Assume on the contrary that it is false. Then there exists a sequence $\{d_{I_n}\}$ with $d_{I_n} \to +\infty$ as $n \to +\infty$ such that $\|S_n\|_{L^\infty(\Omega)} = \|\frac{\kappa_n}{d_S}(1-\tilde{I}_n)\|_{L^\infty(\Omega)} \to +\infty$ as $n \to +\infty$. Set $w_n = \frac{1-\tilde{I}_n}{\|(1-\tilde{I}_n)\|_{L^\infty(\Omega)}}$. One can get that w_n satisfies

$$\begin{aligned}&\frac{d_{I_n}}{\kappa_n}\kappa_n\|(1-\tilde{I}_n)\|_{L^\infty(\Omega)}\int_\Omega J(x-y)(w_n(y) - w_n(x))\,\mathrm{d}y\\&\quad - \frac{\beta(x)\frac{\kappa_n}{d_S}(1-\tilde{I}_n)\tilde{I}_n}{m(x) + \frac{\kappa_n}{d_S}(1-\tilde{I}_n) + \frac{\kappa_n}{d_{I_n}}\tilde{I}_n} + \gamma(x)\tilde{I}_n = 0.\end{aligned}$$

Noting that

$$\left\|-\frac{\beta\frac{\kappa_n}{d_S}(1-\tilde{I}_n)\tilde{I}_n}{m + \frac{\kappa_n}{d_S}(1-\tilde{I}_n) + \frac{\kappa_n}{d_{I_n}}\tilde{I}_n} + \gamma\tilde{I}_n\right\|_{L^\infty(\Omega)} \leqslant \max_{x\in\bar{\Omega}}(\beta(x) + \gamma(x)),$$

we get that $w_n \to 1$ in $C(\bar{\Omega})$ as $n \to +\infty$. Then, $S_n = \frac{\kappa_n}{d_S}(1-\tilde{I}_n) = \frac{1}{d_S}\kappa_n\|(1-\tilde{I}_n)\|_{L^\infty(\Omega)}w_n \to +\infty$ contradicting $\int_\Omega S_n\,\mathrm{d}x \leqslant N$. Hence, the claim must hold. It follows from the first equation of (3.1) that

$$S(x) = \frac{-E(x) - \sqrt{E^2(x) - 4D(x)F(x)}}{2D(x)},$$

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where

$$D(x) = -d_S \int_\Omega J(x-y)\,\mathrm{d}y, \quad F(x)$$
$$= \left(d_S \int_\Omega J(x-y)S(y)\,\mathrm{d}y + \gamma(x) + I(x)\right)(m(x)+I(x)),$$
$$E(x) = d_S \int_\Omega J(x-y)S(y)\,\mathrm{d}y - d_S \int_\Omega J(x-y)\,\mathrm{d}y(m(x)+I(x))$$
$$- \beta(x)I(x) + \gamma(x)I(x),$$

which implies that $S \to S^*$ in $C(\bar{\Omega})$ as $d_I \to +\infty$ and S^* satisfies (2.4). If $I^* = 0$, we conclude from (2.4) that $S^* = \frac{N}{|\Omega|}$. By the positivity of I, 0 is the principal eigenvalue of the eigenvalue problem (3.23). By virtue of [**39**, Theorem 2.2], we have $\int_\Omega (\gamma - \theta)\,\mathrm{d}x = 0$. A contradiction occurs. Thus, $I^* > 0$.

Finally we prove (iii). We derive from (3.2) and (3.3) that

$$S = \frac{N}{|\Omega|} - \left(1 - \frac{d_I}{d_S}\right)\frac{1}{|\Omega|}\int_\Omega I\,\mathrm{d}x - \frac{d_I}{d_S}I,$$

which implies that $\|S\|_{L^\infty(\Omega)} \leqslant \frac{N}{|\Omega|}(1 + \frac{d_I}{d_S})$. If $\frac{d_I}{d_S} \to C$ for some nonnegative constant C, then S is uniformly bounded with respect to d_S, $d_I > d_*$ with d_* some positive constant. In view of (3.19), I is uniformly bounded with respect to d_S, $d_I > d_*$. If $\frac{d_I}{d_S} \to +\infty$, then $I = \frac{\kappa}{d_I}\tilde{I} = \frac{1}{|\Omega|}[\frac{d_S}{d_I}N - (\frac{d_S}{d_I} - 1)\int_\Omega I\,\mathrm{d}x]\tilde{I}$ is uniformly bounded with respect to d_S, $d_I > d_*$. Similar to the proof of (ii), we can prove S is uniformly bounded with respect to d_S, $d_I > d_*$. The remaining proof is the same as the proof of [**45**, Theorem 4.1]. So we omit it. The proof is completed. □

Proof of proposition 2.10. Similar to the proof of theorem 2.4, we can prove (i). Here we only prove (ii). By the arguments in the proof of theorem 2.5, we know $I \to I^*$ in $C(\bar{\Omega})$ as $d_S \to 0$ and I^* is a nonnegative constant. Assume $I^* = 0$ and set $(\check{S}, \check{I}) = (\frac{S}{d_S}, \frac{I}{d_S})$. Then $(\check{S}, \check{I})$ satisfies (3.1) with $m \equiv 0$. In view of the proof of theorem 2.5, there exists positive constants $\hat{d}$, C_* and C^* such that $C_* \leqslant \check{I}(x) \leqslant C^*$ for all $x \in \bar{\Omega}$ and $0 < d_S < \hat{d}$. Let $\check{S}(x_0) = \min_{x\in\bar{\Omega}} \check{S}(x)$ and $\check{S}(y_0) = \max_{x\in\bar{\Omega}} \check{S}(x)$. We obtain from the first equation of (3.1) that $-\frac{\beta(x_0)\check{S}(x_0)}{\check{S}(x_0)+\check{I}(x_0)} + \gamma(x_0) \leqslant 0$ and $-\frac{\beta(y_0)\check{S}(y_0)}{\check{S}(y_0)+\check{I}(y_0)} + \gamma(y_0) \geqslant 0$, which implies that $\min_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)-\gamma(x)}C_* \leqslant \check{S}(x) \leqslant \max_{x\in\bar{\Omega}} \frac{\gamma(x)}{\beta(x)-\gamma(x)}C^*$. Adding up two equations of (3.1) with $m \equiv 0$ yields

$$\check{I}(x) = \frac{d_S \int_\Omega J(x-y)(\check{S}(y) - \check{S}(x))\,\mathrm{d}y + d_I \int_\Omega J(x-y)\check{I}(y)\,\mathrm{d}y}{d_I \int_\Omega J(x-y)\,\mathrm{d}y},$$

which implies that $\check{I} \to \check{I}^*$ in $C(\bar{\Omega})$ as $d_S \to 0$ and $\check{I}^*$ satisfies $\int_\Omega J(x-y)(\check{I}^*(y) - \check{I}^*(x))\,\mathrm{d}y = 0$. It follows from Andreu-Vaillo *et al.* [**3**, Proposition 3.3] that $\check{I}^*$ is a

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positive constant. The first equation of (3.1) with $m \equiv 0$ gives

$$\check{S}(x) = \frac{d_S \int_\Omega J(x-y)(\check{S}(y) - \check{S}(x))\,\mathrm{d}y(\check{S}(x) + \check{I}(x)) + \gamma(x)\check{I}(x)\check{I}(x)}{(\beta(x) - \gamma(x))\check{I}(x)},$$

which implies that $\check{S} \to \frac{\gamma \check{I}^*}{\beta - \gamma}$ in $C(\bar{\Omega})$ as $d_S \to 0$. Then, $S = d_S \check{S} \to 0$ in $C(\bar{\Omega})$ as $d_S \to 0$. Hence, $\int_\Omega (S + I)\,\mathrm{d}x \to 0$ as $d_S \to 0$ contradicting (3.2). Now we have $I^* > 0$. By the proof of theorem 2.5, $S \to \frac{\gamma I^*}{\beta - \gamma}$ in $C(\bar{\Omega})$ as $d_S \to 0$ and $\int_\Omega \frac{\gamma I^*}{\beta - \gamma} + I^*\,\mathrm{d}x = N$. Thus, $I^* = \frac{N}{\int_\Omega \frac{\beta}{\beta-\gamma}\,\mathrm{d}x}$. The proof is completed. □

The proof of proposition 2.11 is similar to that of theorem 2.7. So we omit it.

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