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SOME TOPOLOGICAL PROPERTIES OF VECTOR MEASURES AND THEIR INTEGRAL MAPS

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Abstract

In the present paper we prove that every finite dimensional non-atomic measure ν *is open* and monotone (viz. ν^{-1} preserves connected sets) relative to the usual Fréchet–Nikodým topology on its domain and the relative topology on its range. An arbitrary finite dimensional measure is found on the other hand to be biquotient.

Given a vector measure ν, we further investigate the properties of its integral map $T_\nu: \phi \to \int \phi d\nu$ defined on the set of functions ϕ in $L_1(|\nu|)$ for which $\phi(s) \in [0,1]$ $|\nu|$-almost everywhere. When ν is finite dimensional, T_ν is found to be always open. In general, when T_ν is open, the set of extreme points of the closed convex hull of the range of ν is proved to be closed, and when ν and T_ν are both open, the range of ν in itself is closed.

1. Introduction

We assume throughout the paper ν to be a measure defined on a σ-algebra $\mathscr{A}$ of subsets of some set S with values in a real Fréchet space X. As X is metrizable, according to Hoffman–Jorgensen (1971) there exists a finite positive measure λ on $\mathscr{A}$ such that $\nu \equiv \lambda$; such a measure λ is said to be a *control measure* of ν. Unless otherwise stated, we denote by λ a control measure of ν.

The quotient σ-algebra of $\mathscr{A}$ modulo the σ-ideal of λ-null sets is denoted by $\mathscr{A}$ in itself. On this new $\mathscr{A}$ the Fréchet–Nikodým metric induced by λ is defined by $\rho(A,B) = \lambda(A \Delta B)$, $A, B \in \mathscr{A}$, where $A \Delta B$ denotes the symmetric difference of A and B. The topology τ induced by ρ on $\mathscr{A}$ is independent of the choice of λ, and $\nu: \mathscr{A} \to X$ is continuous relative to this topology.

The spaces $L_1(\lambda)$ and $L_\infty(\lambda)$ are denoted briefly by L_1 and L_∞ respectively. As X is a Fréchet space, each $\phi \in L_\infty$ has an integral $\int \phi d\nu \in X$ in the sense of Bartle, Dunford and Schwartz (1955). Following Lindenstrauss (1966) we denote by T the integral map $\phi \to \int \phi d\nu$ from L into X. It follows from the Radon–Nikodým theorem that T is continuous relative to the weak*-topology $\sigma(L_\infty, L_1)$ on L_∞ and the weak topology $\sigma(X, X')$ on X, where X' denotes the continuous dual of X. The set of $\phi \in L_\infty$ for which $0 \leqq \phi(s) \leqq 1$ for λ-almost every $s \in S$ is denoted by P. Since λ is a finite measure, we have $L_\infty \subset L_1$, and unless otherwise stated, P is assumed to have the topology of L_1. The subset of P consisting of characteristic functions χ_A of sets A in $\mathscr{A}$ is denoted by P_0. The set P is convex and it is compact relative to the weak*-topology (see Kingman and Robertson (1968)). The restriction of T to P will be denoted by T_ν and the closed convex hull of the range $\nu(\mathscr{A})$ of ν by K. Then K is weakly compact and $T_\nu(P) = K$ (see e.g. Anantharaman (1973) or Kluvánek and Knowles (1974), Theorem IV.6.1).

For every $a \in \nu(\mathscr{A})$ it is easy to see that every characteristic function in $T_\nu^{-1}(a)$ is an extreme point of this set. A measure ν is said to have *property* (*) (Anantharaman and Garg) if for every $a \in \nu(\mathscr{A})$ each extreme point of $T_\nu^{-1}(a)$ is a characteristic function. The measure is further called *semi-convex* (Halmos (1948)) if for every $A \in \mathscr{A}$ there exists $B \in \mathscr{A}$, $B \subset A$, such that $\nu(B) = \frac{1}{2}\nu(A)$.

As we see in (Anantharaman and Garg), every semi-convex measure has property (*), whereas the converse is false; there further exist non-atomic l_2-valued measures that do not have property (*).

If E and F are two topological spaces, a continuous onto map $f: E \to F$ is said to be *open* if $f(U)$ is open for every open subset U of E, and f is called *biquotient* (Michael (1968)) if for every $y \in F$ and for each open cover $\mathscr{U}$ of $f^{-1}(y)$ there exists a neighborhood of y that is covered by finitely many $f(U)$, $U \in \mathscr{U}$. In case f is not onto, it is said to be open or biquotient if it is so with respect to the relative topology of $f(X)$. The map f is further called *monotone* (Kuratowski (1968)) if f^{-1} preserves connected sets, and *weakly monotone* (Whyburn (1970); Garg (1977)) if $f^{-1}(y)$ is connected for each $y \in F$. These properties of ν and T_ν are defined in terms of the τ- and $L_1(\lambda)$-topologies on $\mathscr{A}$ and P respectively.

We first study in Section 2 the openness of the map T_ν. It is proved in Theorem 2.2 to be open whenever the range of ν is finite dimensional. If on the other hand T_ν is open, where the range of ν may be infinite dimensional, the extreme points of the closed convex hull of the range of ν are proved in Proposition 2.3 to form a closed set. In Proposition 2.6 we compare the openness of T_ν with its openness relative to the weak*-topology on P and the weak topology of its range K.

Section 3 deals with the properties of ν. Every semi-convex measure is proved in Theorem 3.1 to be weakly monotone, and then in Theorem 3.5 a finite dimensional non-atomic measure is proved to be open and monotone.

These results have already appeared in Anantharaman (1974) and Kluvanek and Knowles (1974), and the openness part has been obtained independently by Karafiat (1974) using different methods. An infinite dimensional non-atomic measure is not generally open even if it is semi-convex (see Remark 3.6). In general, if ν and T_ν are both open, it is proved in Proposition 3.8 that the range of ν is closed. Finally, in Theorem 3.11, every finite dimensional measure is proved to be biquotient.

The principal tools employed here are two characterizations of open maps by Sikorski (1955) and Hájek (1967), two results of Michael (1959, 1968) on lower semi-continuous set-valued functions and on biquotient maps, and a theorem of Jerison (1954) on the extreme points of a limit of compact convex sets.

2. Properties of T_ν

We first state the two characterizations of open maps due to Sikorski (1955) and Hájek (1967) that are used repeatedly in the paper.

Let E be any topological space. The space of nonempty closed subsets of E is denoted by 2^E. The *superior* and *inferior* limits of a net (A_α) in 2^E, denoted by $\mathrm{Ls}\,A_\alpha$ and $\mathrm{Li}\,A_\alpha$, are defined to be the set of elements x of E of which every neighborhood intersects the net frequently or eventually respectively (Kuratowski (1966)).

SIKORSKI'S THEOREM. *If E and F are metric spaces, then a continuous onto map $f\colon E \to F$ is open if and only if for every sequence $\{y_n\}$ of elements of F converging to some $y \in F$ we have*

$$f^{-1}(y) = \mathrm{Li}\,f^{-1}(y_n) = \mathrm{Ls}\,f^{-1}(y_n).$$

HÁJEK'S THEOREM. *If E is a topological space and F is a Hausdorff space, then a continuous onto map $f\colon E \to F$ is open if and only if for every net (y_α) of elements of F converging to $y \in F$ we have*

$$f^{-1}(y) = \mathrm{Ls}\,f^{-1}(y_\alpha).$$

Let ν, λ, P and K be as defined in Section 1. For any net (A_α) in 2^P the limits $\mathrm{Ls}\,A_\alpha$ and $\mathrm{Li}\,A_\alpha$ are defined relative to the L_1-norm topology of P. Let σ denote the metric on P induced by the norm of L_1 and d be the *Hausdorff metric* (see Kuratowski (1966)) induced by σ on 2^P, viz. if $A, B \in 2^P$, then $d(A,B)$ is the supremum of $\sigma(\phi, B)$ and $\sigma(\psi, A)$ for $\phi \in A$ and $\psi \in B$.

LEMMA 2.1. *If a finite signed measure ν is absolutely continuous with respect to a finite positive measure λ, then for every sequence $\{x_n\}$ of elements of*

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K converging to some real number x, the sequence $\{T_\nu^{-1}(x_n)\}$ converges to $T_\nu^{-1}(x)$ in the Hausdorff metric on 2^P.

Proof. We shall first assume that $\nu \equiv \lambda$. Since the metric induced by λ on P is easily verified to be equivalent to the one induced by the variation $|\nu|$ of ν, in this case we may further assume $\lambda = |\nu|$. Let S^+, S^- denote a Hahn decomposition of S relative to ν and let $\alpha = \nu(S^-)$, $\beta = \nu(S^+)$. Since

$$T_\nu(P) = K = \overline{\mathrm{co}}\, \nu(\mathscr{A}) = [\alpha, \beta],$$

it clearly suffices to show that $d(T_\nu^{-1}(x), T_\nu^{-1}(y)) \leqq |x-y|$ for every $x, y \in [\alpha, \beta]$.

Let us first prove that for every $\phi \in T_\nu^{-1}(x)$, $\sigma(\phi, T_\nu^{-1}(y)) \leqq |x-y|$, and for this we need to show that there exists $\psi \in T^{-1}(y)$ such that $\|\psi - \phi\|_1 \leqq |x-y|$. When $y = x$, ψ may clearly be taken to be ϕ. In case $y < x$, let

$$\psi = \phi + \frac{x-y}{x-\alpha}(\chi_{S^-} - \phi) = \frac{y-\alpha}{x-\alpha}\phi + \frac{x-y}{x-\alpha}\chi_{S^-}.$$

Since $\alpha \leqq y < x$, ψ is then a convex combination of ϕ and χ_{S^-}, whence $\psi \in P$, and we have

$$T_\nu(\psi) = T_\nu(\phi) + \frac{x-y}{x-\alpha} T_\nu(\chi_{S^-} - \phi) = x + \frac{x-y}{x-\alpha}(\alpha - x) = y,$$

whereas

$$\|\psi - \phi\|_1 = \frac{x-y}{x-\alpha}\int |\chi_{S^-} - \phi|\, d|\nu| = \frac{x-y}{x-\alpha}\left\{\int_{S^+} \phi d\nu - \int_{S^-} (1-\phi) d\nu\right\}$$
$$= \frac{x-y}{x-\alpha}\left(\int \phi d\nu - \alpha\right) = \frac{x-y}{x-\alpha}(x-\alpha) = x-y.$$

When $y > x$, we have $\beta \geqq y > x$, and so putting this time

$$\psi = \phi + \frac{y-x}{\beta - x}(\chi_{S^+} - \phi),$$

it follows as above that $\psi \in P$, $T_\nu(\psi) = y$ and $\|\psi - \phi\|_1 = y - x$. Thus $\sigma(\phi, T_\nu^{-1}(y)) \leqq |x-y|$ for every $\phi \in T_\nu^{-1}(x)$. On interchanging x and y it follows that $\sigma(\psi, T_\nu^{-1}(x)) \leqq |x-y|$ for every $\psi \in T^{-1}(y)$, and so we have $d(T_\nu^{-1}(x), T_\nu^{-1}(y)) \leqq |x-y|$, which proves the lemma in case $\nu \equiv \lambda$.

In general there exists a set $S_1 \in \mathscr{A}$ such that ν is equivalent to the restriction of λ to S_1 and the complement S_2 of S_1 is ν-null. Let

$$P_1 = \{\phi \in P : 0 \leqq \phi \leqq \chi_{S_1}\}, \qquad P_2 = \{\phi \in P : 0 \leqq \phi \leqq \chi_{S_2}\},$$

and let T_1 be the restriction of T to P_1. For every x and y in $T_1(P_1) = [\alpha, \beta]$ we have

$$T_\nu^{-1}(x) = T_1^{-1}(x) + P_2, \qquad T_\nu^{-1}(y) = T_1^{-1}(y) + P_2,$$

from which it follows that

$$d(T_\nu^{-1}(x), T_\nu^{-1}(y)) \leqq d(T_1^{-1}(x), T_1^{-1}(y)).$$

If $\{x_n\}$ is a sequence in $[\alpha, \beta]$ converging to x, it follows from the first part of the proof that $d(T_1^{-1}(x_n), T_1^{-1}(x)) \to 0$ as $n \to \infty$, and so $d(T_\nu^{-1}(x_n), T_\nu^{-1}(x)) \to$ as $n \to \infty$. Hence the lemma.

THEOREM 2.2. *If the range of ν is finite dimensional, then T_ν is open.*

PROOF. Let $\nu: \mathcal{A} \to R^k$ and λ be a control measure for ν, e.g. its variation. For each $i = 1, 2, \cdots, k$, let $\nu_i(A)$ be the i-th coordinate of $\nu(A)$ for every $A \in \mathcal{A}$, and $T_i(\phi)$ be the i-th coordinate of $T_\nu(\phi)$ for every $\phi \in P$. It follows that $T_i(\phi) = \int \phi d\nu_i$ for every $\phi \in P$ and $1 \leqq i \leqq k$. The map $T_\nu: P \to K$ is continuous, and to prove its openness it suffices to show by Hájek's theorem that Ls $T_\nu^{-1}(y_\alpha) = T_\nu^{-1}(y)$ for every net (y_α) of elements of K converging to $y \in K$. Since R^k is a metric space, we may replace the net by a sequence $\{y_n\}$, and it further suffices to show that the sequence $\{T_\nu^{-1}(y_n)\}$ converges to $T_\nu^{-1}(y)$ in the Hausdorff metric on 2^P (see Kuratowski (1966)).

Let, for every n, $y_n = (y_{n,i})_{i=1}^k$ and $y = (y_i)_{i=1}^k$. Since the sequence $\{y_n\}$ converges to y, the sequence $\{y_{n,i}\}$ converges to y_i for $1 \leqq i \leqq k$. For each i we have, by Lemma 2.1, $d(T_i^{-1}(y_{n,i}), T_i^{-1}(y_i)) \to 0$. Since the operation $(A, B) \to A \cap B$, $A, B \in 2^P$, is continuous relative to the Hausdorff metric on 2^P (see Kuratowski (1966)), we obtain

$$d\left(\bigcap_{i=1}^{k} T_i^{-1}(y_{n,i}),\ \bigcap_{i=1}^{k} T_i^{-1}(y_i) \right) \to 0,$$

i.e. $d(T_\nu^{-1}(y_n), T^{-1}(y)) \to 0$. Hence T_ν is open. This completes the proof.

PROPOSITION 2.3. *If ν is a measure such that T_ν is open, then the extreme points of the closed convex hull of its range form a closed set.*

PROOF. Let $K = \overline{\text{co}}\, \nu(\mathcal{A})$, and let $\{x_n\}$ be a sequence of extreme points of K which converges to some $x \in X$. Since K is closed, $x \in K$. As X is a Fréchet space, $K = T_\nu(P)$ (see Section 1). By Proposition 2 of Anantharaman (1973) (or see Kluvánek and Knowles (1974), Corollary VI.1.1), a point x of K is an extreme point of K if and only if $T_\nu^{-1}(x)$ is a singleton and a characteristic function. Thus for each n there exists a unique set $E_n \in \mathcal{A}$ such that $x_n = \nu(E_n)$. Since P and K are metrizable and $T_\nu: P \to K$ is open, it follows from Sikorski's theorem that $T_\nu^{-1}(x)$ is the superior and inferior limit of

$\{T_\nu^{-1}(x_n)\}$, i.e. of the sequence $\{\{\chi_{E_n}\}\}$, and so $T_\nu^{-1}(x)$ must be a singleton in P. Since $\chi_{E_n} \in P_0$ for every n, and P_0 is closed in P, we have $T_\nu^{-1}(x) \in P_0$, and so x is an extreme point of K. This completes the proof.

The above proposition yields, with the help of Theorem 2.2,

COROLLARY 2.4. *The extreme points of the closed convex hull of the range of every finite dimensional measure form a closed set.*

REMARK 2.5. The above corollary does not hold in infinite dimensions in general. The following example was suggested by Professor J. L. B. Gamlen. *Let λ be the restriction of the Lebesgue measure on R to $[-1, 1]$, $\mathscr{A}$ be the* domain of λ, and define the measure $\nu: \mathscr{A} \to L_1[0,1]$ by

$$\nu(E) = \chi_{E \cap [0,1]} + \lambda(E \cap [-1, 0])\chi_{[0,1]}$$

for each $E \in \mathscr{A}$. It may be verified that

$$K = \overline{\text{co}}\, \nu(\mathscr{A}) = \{\phi + \alpha\chi_{[0,1]}: \phi \in L_1[0,1], 0 \leqq \phi \leqq 1, 0 \leqq \alpha \leqq 1\}$$

and

$$\text{ext}\, K = \{\chi_A : A \in \mathscr{A}, A \subset [0,1], 0 \leqq \lambda(A) < 1\} \cup$$
$$\{\chi_A + \chi_{[0,1]}: A \in \mathscr{A}, A \subset [0,1], \lambda(A) > 0\}.$$

Then $\text{ext}\, K$ contains the convergent sequence $\{\chi_{[0,1-\frac{1}{n}]}\}$ whose limit $\chi_{[0,1]}$ is not in $\text{ext}\, K$. It also shows, according to Proposition 2.3, that T_ν is not open in general.

In the next proposition we compare the openness of T_ν relative to two topologies on its domain and range. For any net (A_α) in 2^P, its superior limit relative to the weak*-topology on P will be denoted by $\text{Ls}^* A_\alpha$.

PROPOSITION 2.6. *If the range of ν is relatively compact and T_ν is open, then the map $T_\nu: (P, w^*) \to (K, w)$ is open.*

Conversely, if ν is semi-convex and the map $T_\nu: (P, w^*) \to (K, w)$ is open, then the range of ν is compact and T_ν is open.

PROOF. Suppose that $\nu(\mathscr{A})$ is relatively compact and T_ν is open. Then $K = \overline{\text{co}}\, \nu(\mathscr{A})$ is compct. Let (x_α) be a net of elements of K converging weakly to some $x \in K$. Then (x_α) converges to x relative to the given topology on X, and as T_ν is open, according to Hajek's theorem we have

$$T_\nu^{-1}(x) = \text{Ls}\, T_\nu^{-1}(x_\alpha).$$

Since the weak*-topology of P is coarser than the L_1-norm topology, we obtain

$$T_\nu^{-1}(x) = \mathrm{Ls}\, T_\nu^{-1}(x_\alpha) \subset \mathrm{Ls}^*\, T_\nu^{-1}(x_\alpha) \subset T_\nu^{-1}(x),$$

the last inclusion being a consequence of the continuity of $T_\nu: (P, w^*) \to (K, w)$. Now it follows from Hájek's theorem that $T_\nu: (P, w^*) \to (K, w)$ is open.

Conversely, suppose that ν is semi-convex and $T_\nu: (P, w^*) \to (K, w)$ is open. According to Kingman and Robertson (1968) we have $\nu(\mathscr{A}) = K$. The map $\nu: (\mathscr{A}, \tau) \to (K, w)$ is clearly continuous, and we claim that it is open.

For let (x_α) be a net of elements of K converging weakly to some $x \in K$. Since $K = T_\nu(P)$ and $T_\nu: (P, w^*) \to (K, w)$ is assumed to be open, by Hájek's theorem we have

$$T_\nu^{-1}(x) = \mathrm{Ls}^*\, T_\nu^{-1}(x_\alpha).$$

According to a theorem of Jerison (1954) we have

$$\mathrm{Ls}^*\, T_\nu^{-1}(x_\alpha) = \overline{\mathrm{co}}\,(\mathrm{Ls}^*\, \mathrm{ext}\, T_\nu^{-1}(x_\alpha)).$$

Since ν is semi-convex, it has property (*) and so

$$\mathrm{ext}\, T_\nu^{-1}(x_\alpha) = \{\chi_E : E \in \mathscr{A}, \nu(E) = x_\alpha\}$$

for each α. Hence we obtain

$$T_\nu^{-1}(x) = \overline{\mathrm{co}}\,(\mathrm{Ls}^*\{\chi_E : E \in \mathscr{A}, \nu(E) = x_\alpha\}).$$

According to Milman's theorem we have

$$\mathrm{ext}\, T_\nu^{-1}(x) \subset \mathrm{Ls}^*\{\chi_E : E \in \mathscr{A}, \nu(E) = x_\alpha\},$$

and as $x \in \nu(\mathscr{A})$ and ν has property (*), we get

$$\{\chi_E : E \in \mathscr{A}, \nu(E) = x\} \subset \mathrm{Ls}^*\{\chi_E : E \in \mathscr{A}, \nu(E) = x_\alpha\}.$$

Since the weak*-topology coincides with the L_1-norm topology on the set P_0, we have

$$\{\chi_E : E \in \mathscr{A}, \nu(E) = x\} \subset \mathrm{Ls}\{\chi_E : E \in \mathscr{A}, \nu(E) = x_\alpha\},$$

and as $(P_0, \|\ \|_1)$ can be identified with $(\mathscr{A}, \tau)$, we get

$$\nu^{-1}(x) \subset \mathrm{Ls}\, \nu^{-1}(x_\alpha).$$

The reverse inclusion follows from the continuity of ν, whence $\nu: (\mathscr{A}, \tau) \to (K, w)$ is open by Hájek's theorem.

Now K is weakly compact, and to show that it is compact let (x_α) be a net of elements of K converging weakly to some $x \in K$. Since $x \in \nu(\mathscr{A})$, $\mathrm{Ls}\, \nu^{-1}(x_\alpha)$ is not empty by above. Hence there exists a subnet $(\nu^{-1}(x_\beta))$ of

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$(\nu^{-1}(x_\alpha))$ and, for each β, a set $E_\beta \in \nu^{-1}(x_\beta)$ such that (E_β) converges to some $E \in \nu^{-1}(x)$. As ν is continuous relative to the given topology of K, (x_β) converges to x in the latter topology as well, and so the two topologies on K coincide. Hence K is compact.

Finally we need to prove that T_ν is open. The spaces P and K are metrizable. Thus let $\{x_n\}$ be a sequence in $K(=\nu(\mathscr{A}))$ converging to some $x \in K$. Then $\{x_n\}$ converges weakly to x, and as $\nu:(\mathscr{A},\tau)\rightarrow(K,w)$ is open, by Hájek's theorem we have $\nu^{-1}(x)=\operatorname{Ls}\nu^{-1}(x_n)$. Further, for any subsequence $\{x_m\}$ of $\{x_n\}$ we have $\nu^{-1}(x)=\operatorname{Ls}\nu^{-1}(x_m)$, and so it follows that $\nu^{-1}(x)=\operatorname{Li}\nu^{-1}(x_n)$ (see Kuratowski (1966)). Hence we obtain

$$\{\chi_E : E \in \mathscr{A}, \nu(E)=x\} \subset \operatorname{Li} T_\nu^{-1}(x_n).$$

As ν has the property (*) and $x \in \nu(\mathscr{A})$, we have

$$T_\nu^{-1}(x) \subset \overline{\operatorname{co}}\{\chi_E : E \in \mathscr{A}, \nu(E)=x\},$$

whence

$$T_\nu^{-1}(x) \subset \operatorname{Li} T_\nu^{-1}(x_n) \subset \operatorname{Ls} T_\nu^{-1}(x_n),$$

and so T_ν is open by Sikorski's theorem. This completes the proof of the proposition.

With the help of Theorem 2.2 we obtain, from the first part of Proposition 2.6,

COROLLARY 2.7. *If ν is finite dimensional, then the map $T_\nu:(P,w^*)\rightarrow K$ is open.*

3. Properties of ν

THEOREM 3.1. *Every semi-convex measure is weakly monotone.*

PROOF. Let λ be a control measure of ν. For every pair of sets A, B in $\mathscr{A}$, let $A \leqq B$ if $\lambda(A/B)=0$. Then $\leqq$ is a partial order on $\mathscr{A}$ and $(\mathscr{A},\leqq)$ is a complete lattice (see Halmos (1950), p. 169). It is easy to see that this order is independent of the choice of λ. If $\mathscr{C}$ is a chain for $\leqq$, then the order topology (see Birkhoff (1967)) of C coincides with the one induced by τ on $\mathscr{C}$. Indeed, if $C \in \mathscr{C}$ and (C_1, C_2) is an open order interval containing C, then on putting

$$\begin{aligned} r &= \tfrac{1}{2}\lambda(C_1) && \text{if } C=\phi \\ &= \tfrac{1}{2}\min\{\lambda(C\backslash C_1), \lambda(C_2\backslash C)\} && \text{if } \phi < C < S \\ &= \tfrac{1}{2}\lambda(S\backslash C_1) && \text{if } C=S \end{aligned}$$

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we have

$$B(C, r) \equiv \{E \in \mathscr{C}, \lambda(E \Delta C) < r\} \subset (C_1, C_2).$$

On the other hand, for every $C \in \mathscr{C}$ and $r > 0$ one may easily find an order interval (C_1, C_2) containing C ($[C, C_2)$ if $C = \phi$ and $(C_1, S]$ if $C = S$) such that $(C_1, C_2) \subset B(C, r)$.

Now assume ν to be semi-convex and A, B to be any two distinct elements of $\mathscr{A}$ such that $\nu(A) = \nu(B)$. It is sufficient to show that A and B are members of a connected subset $\mathscr{E}$ of $\nu^{-1}(\nu(A))$. We have $\nu(A \backslash B) = \nu(B \backslash A) = x$, say. Since ν is semi-convex, one can find, (see Schmets (1966), p. 185), chains $\mathscr{C} = \{C_\theta : \theta \in [0,1]\}$ and $\mathscr{D} = \{D_\theta : \theta \in [0,1]\}$ of subsets of $A \backslash B$ and $B \backslash A$ respectively with the following properties:

$$C_0 = D_0 = \phi, \qquad C_1 = A \backslash B, \qquad D_1 = B \backslash A,$$

$$C_{\theta_1} < C_{\theta_2} \text{ if and only if } \theta_1 < \theta_2,$$

$$D_{\theta_1} < D_{\theta_2} \text{ if and only if } \theta_1 < \theta_2,$$

and $\nu(C_\theta) = \theta x = \nu(D_\theta)$ for every $\theta \in [0,1]$.

Since the maps $\theta \to C_\theta$ and $\theta \to D_\theta$ are order-isomorphisms of the chain $[0,1]$ onto the chains $\mathscr{C}$ and $\mathscr{D}$ respectively, they are also homeomorphisms relative to their chain topologies. Since these chain topologies coincide with their induced τ-topologies, the maps are homeomorphisms relative to the latter topologies as well. As each of the operations of union, intersection and complementation is continuous on $\mathscr{A} \times \mathscr{A}$ relative to the product topology $\tau \times \tau$ (see Halmos (1950), p. 168), the map

$$\theta \to E_\theta \equiv (A \backslash C_\theta) \cup D_\theta, \ \theta \in [0,1]$$

is continuous. Hence $\mathscr{E} = \{E_\theta : \theta \in [0,1]\}$ is a connected subset of $\mathscr{A}$. Further, for every $\theta \in [0,1]$ we have

$$\nu(E_\theta) = \nu(A) - \nu(C_\theta) + \nu(D_\theta) = \nu(A),$$

while $E_0 = A$, $E_1 = (A \backslash (A \backslash B)) \cup (B \backslash A) = B$. Thus A and B belong to the connected subset $\mathscr{E}$ of $\nu^{-1}(A)$, which completes the proof.

As a consequence of Theorem 3.1 we obtain the following well-known result of Halmos (1948):

Corollary 3.2 (Halmos). *Every finite dimensional non-atomic measure is semi-convex.*

Proof. It will clearly suffice to prove the following: Let $\nu : A \to X$ be semi-convex, and μ be a finite signed measure on $\mathscr{A}$ that is absolutely continuous with respect to ν. Then the measure $\pi : \mathscr{A} \to X \times R$ defined by $\pi(E) = (\nu(E), \mu(E))$, $E \in \mathscr{A}$ is also semi-convex.

Let λ be a control measure of ν, $A \in \mathscr{A}$, and $\pi(A) = a = (a_1, a_2)$. Since the restriction ν_A of ν to A is semi-convex, the set $\mathscr{E} = \{E \in \mathscr{A}: E \subset A, \nu(E) = \frac{1}{2}a_1\}$ is nonempty, and is by Theorem 3.1 connected in $(\mathscr{A}, \tau)$. As μ is continuous on $\mathscr{A}$, $\mu(\mathscr{E})$ is an interval. For each $E \in \mathscr{E}$, $A \setminus E \in \mathscr{E}$, and so $\frac{1}{2}a_2 = \frac{1}{2}\mu(A) = \frac{1}{2}(\mu(E) + \mu(A \setminus E)) \in \mu(\mathscr{E})$, i.e. $\pi^{-1}(\frac{1}{2}a)$ is nonempty.

REMARK 3.3. Theorem 3.1 does not hold in general for non-atomic measures in infinite dimensions. For if ν is the measure as defined in Remark 2.5, it may be easily verified that $\nu^{-1}\{\chi_{[0,1]}\} = \{\chi_{[-1,0]}, \chi_{[0,1]}\}$, which is not connected.

The converse of Theorem 3.1 does not hold even in finite dimensions, as is evident from any atomic measure with a single atom. In infinite dimensions there exist non-atomic monotone measures that are not semi-convex. For let λ be the restriction of the Lebesgue measure on R to $[0, 1]$, $\mathscr{A}$ be the domain of λ, and define the measure $\nu: \mathscr{A} \to L_1[0, 1]$ by $\nu(E) = \chi_E$ for every $E \in \mathscr{A}$. As $\nu \equiv \lambda$, it is non-atomic. As $\|\nu(E) - \nu(F)\| = \lambda(E \Delta F)$ for every $E, F \in \mathscr{A}$, ν is a homeomorphism, and so it is monotone. However, ν is obviously not semi-convex.

LEMMA 3.4. *If ν has property (*) and T_ν is open, then ν is open.*

PROOF. Similar to the second part of Proposition 2.6.

THEOREM 3.5. *Every finite dimensional non-atomic measure is open and monotone.*

PROOF. Since a finite dimensional non-atomic measure ν is semi-convex (Corollary 3.2), it has property (*), and so ν is open by Theorem 2.2 and Lemma 3.4. Moreover, since an open map from a Hausdorff space to a locally compact space is monotone whenever it is weakly monotone (see Whyburn (1970), p. 558), it follows from Thoerem 3.1 that ν is monotone.

REMARK 3.6. The converse of Lemma 3.4 holds for semi-convex measures. The proof is similar to that of Proposition 2.6. The first conclusion of Theorem 3.5 does not hold in infinite dimensions in general even for semi-convex measures. For let ν be the measure defined in Remark 2.5. According to Theorem V.5.1 of Kluvánek and Knowles (1974) there exists a semi-convex measure μ (defined possibly on a different σ-algebra) whose range is equal to $\overline{\mathrm{co}}\, \nu(\mathscr{A})$. As we saw in Remark 2.5, $\mathrm{ext}\, \overline{\mathrm{co}}\, \nu(\mathscr{A})$ is not

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norm-closed, whence T_μ is not open by Proposition 2.3. Since μ is semi-convex, μ is not open either, by Lemma 3.4.

PROBLEM 3.7. To investigate conditions on an infinite dimensional semi-convex measure ν under which ν (or equivalently T_ν) is open or otherwise biquotient (see Theorem 3.11).

PROPOSITION 3.8. *If ν and T_ν are both open, then the range of ν is closed.*

PROOF. Let us identify $\mathscr{A}$ with $P_0 \equiv \{\chi_A : A \in \mathscr{A}\}$. The maps $T_\nu^{-1}: K \to 2^P$ and $\nu^{-1}: \nu(\mathscr{A}) \to 2^{P_0}$ are then lower semi-continuous. Since K and $\nu(\mathscr{A})$ have the relative topology of X, they are metrizable, and so are paracompact. Since ν and T_ν are continuous, and their domains are complete metric spaces, $\nu^{-1}(\nu(A))$ and $T_\nu^{-1}(x)$ are complete for every $A \in \mathscr{A}$ and $x \in K$. Hence, according to a theorem of Michael (1959), there exist two lower semi-continuous maps $f: K \to 2^P$ and $g: \nu(\mathscr{A}) \to 2^{P_0}$ such that $f(x)$ is a compact subset of $T_\nu^{-1}(x)$ for every $x \in K$, while $g(\nu(A))$ is a compact subset of $\nu^{-1}(\nu(A))$ for every $A \in \mathscr{A}$. Define $h: K \to 2^P$ by

$$\begin{aligned} h(x) &= f(x) \cup g(x) \quad \text{when} \quad x \in \nu(\mathscr{A}) \\ &= f(x) \qquad\qquad \text{when} \quad x \in K \backslash \nu(\mathscr{A}). \end{aligned}$$

Then $h(x)$ is compact for each $x \in K$.

To prove that $\nu(\mathscr{A})$ is closed, suppose that there exists a sequence $\{x_n\}$ in $\nu(\mathscr{A})$ that converges to some element $x_0 \notin \nu(\mathscr{A})$. We claim that $\operatorname{Ls} g(x_n) = \phi$. For, if not, then there exists a subsequence $\{g(x_m)\}$ of $\{g(x_n)\}$ and for each m, a function $\chi_{E_m} \in g(x_m) (\subset \nu^{-1}(x_m))$ such that the sequence $\{\chi_{E_m}\}$ converges to some element ϕ of P. Since P_0 is closed in P, $\phi \in P_0$, say $\phi = \chi_E$ $(E \in \mathscr{A})$, and as $T_\nu: P \to K$ is continuous by Lemma 2 of Anantharaman (1973), we have

$$x_0 = \lim x_m = \lim T_\nu(\chi_{E_m}) = T_\nu(\phi) = \nu(E),$$

which contradicts the fact that $x_0 \notin \nu(\mathscr{A})$. Thus we have (see Kuratowski (1966), p. 337)

$$\operatorname{Ls} h(x_n) = \operatorname{Ls}\{f(x_n) \cup g(x_n)\} = \operatorname{Ls} f(x_n) \cup \operatorname{Ls} g(x_n) = \operatorname{Ls} f(x_n).$$

Since f is lower semi-continuous, according to a theorem of Sikorski (1955) we have $f(x_0) = \operatorname{Li} f(x_n) = \operatorname{Ls} f(x_n)$. As $h(x_0) = f(x_0)$, we have

$$h(x_0) = \operatorname{Li} f(x_n) \subset \operatorname{Li} h(x_n) \subset \operatorname{Ls} h(x_n) = \operatorname{Ls} f(x_n) = f(x_0) = h(x_0),$$

so that $h(x_0) = \operatorname{Li} h(x_n) = \operatorname{Ls} h(x_n)$.

Thus $\{h(x_n)\}$ is a sequence in 2^P converging to $h(x_0)$ in the Vietoris topology (see Kuratowski (1966)), and so the set $\{h(x_n): n \geqq 0\}$ is a compact subset of 2^P. Since $h(x_n)$ is compact for every n, according to a theorem of Michael (1951) the set $C = \cup\{h(x_n): n \geqq 0\}\}$ is a compact subset of P. Choosing $\chi_{E_n} \in g(x_n)(\subset h(x_n))$ for every $n \geqq 1$, the sequence $\{\chi_{E_n}\}$ is contained in C, and so it contains a subsequence $\{\chi_{E_m}\}$ which converges to some element ϕ of C. It follows as in the above proof of $\mathrm{Ls}\, g(x_n) = \phi$ that $x_0 = T_\nu(\phi) \in \nu(\mathscr{A})$, and since $x_0 \notin \nu(\mathscr{A})$, this is a contradiction. This completes the proof of the proposition.

With the help of Theorems 2.2 and 3.5 we obtain the following well-known theorem of Liapounoff (1940):

COROLLARY 3.9 (Liapounoff). *The range of every finite dimensional non-atomic measure is compact.*

REMARK 3.10. The range of an infinite dimensional measure ν, with ν and T_ν both open, is not in general weakly closed. For let ν be the measure defined in Remark 3.3. Then the map $T_\nu: P \to K$ is an isometry. As $\nu(\mathscr{A}) = \{\chi_E : E \in \mathscr{A}\} = P_0$, we have $K = \overline{\mathrm{co}}\, P_0 = P$. Since λ is non-atomic, it is clear that P_0 is weak*-dense in P. Further it may be verified that the topologies $\sigma(L_\infty, L_1)$ and $\sigma(L_1, L_\infty)$ coincide on P, and so P_0 is weakly dense in P relative to the latter topology as well. As $P_0 \neq P$, P_0 cannot be weakly closed.

As an extension of the Liapounoff's compactness and convexity theorems it has been proved by Knowles (1973) (see also Anantharaman (1974)) that the range of each restriction of every semi-convex measure is convex and weakly compact. For other equivalent conditions in this direction, see Kingman and Robertson (1968) and Anantharaman and Garg.

THEOREM 3.11. *Every finite dimensional measure is biquotient.*

PROOF. Let ν be a finite dimensional measure and λ be a control measure of ν. Then ν and λ have the same atoms. Let $\mathscr{A}_a$ and $\mathscr{A}_n$ denote the atomic and non-atomic parts (see Halmos (1948)) of $\mathscr{A}$ respectively, ν_a and ν_n be the restrictions of ν to $\mathscr{A}_a$ and $\mathscr{A}_n$, and R_a and R_n be the ranges of ν_a and ν_n respectively. Further, for every $A \in \mathscr{A}$, let A_a and A_n denote the atomic and non-atomic parts of A respectively.

Define the map $f: \mathscr{A} \to \mathscr{A}_a \times \mathscr{A}_a$ by $f(A) = (A_a, A_n)$ for every $A \in \mathscr{A}$, $g: \mathscr{A}_a \times \mathscr{A}_n \to R_a \times R_n$ by $g = \nu_a \times \nu_n$, and $h: R_a \times R_n \to X$ by $h(x, y) = x + y$ for $x \in R_a$, $y \in R_n$. Then $\nu = h \circ g \circ f$, and according to a theorem of Michael (1968) it is sufficient to prove that the maps f, g and h are biquotient.

The product topology on $\mathscr{A}_a \times \mathscr{A}_n$ is induced by the metric

$$\pi((A, B), (C, D)) = \lambda(A \Delta C) + \lambda(B \Delta D), \qquad (A, B), (C, D) \in \mathscr{A}_a \times \mathscr{A}_n,$$

and the map f is in fact an isometry relative to the metrics ρ and π respectively. For f is clearly one-to-one, and if $A, B \in \mathscr{A}$, we have

$$\pi(f(A), f(B)) = \pi((A_a, A_n), (B_a, B_n)) = \lambda(A_a \Delta B_a) + \lambda(A_n \Delta B_n)$$
$$= \lambda((A_a \Delta B_a) \cup (A_n \Delta B_n)) = \lambda(A \Delta B) = \rho(A, B).$$

As ν_a is continuous and $\mathscr{A}_a$ is compact (see Halmos (1947)), ν_a is biquotient. Since ν_n is open by Thoerem 3.2, it is also biquotient, and it follows from Thoerem 1.2 of Michael (1968) that the map $g = \nu_a \times \nu_n$ is biquotient.

Finally, the set R_a is clearly compact, and since ν_n is finite dimensional, R_n is also compact by Liapounoff's theorem (see Cor. 3.9). Thus $R_a \times R_n$ is compact, and since h is continuous on it, it is equally biquotient. Hence the theorem.

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