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SOME NOTES ON THE THEORY OF HOLOMORPHIC CURVES

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§1. Introduction

In this paper, we shall give some notes on the order functions of holomorphic curves and, applying these facts, we may formulate the theory of holomorphic curves more precisely than those in Ahlfors [2], Cowen and Griffiths [4], Weyl [7] or Wu [8] in several cases.

Let $x: |z| < R \to P^n(C)$ $(n \geqq 1, 0 < R \leqq \infty)$ be a non-degenerate holomorphic curve and x^p be the associated curve of rank p of x $(p = 1, \cdots, n; x^1 = x)$.

Let

$$X = (x_1(z), \cdots, x_{n+1}(z))$$

be a reduced representation of x where $x_1, \cdots, x_{n+1}$ are holomorphic functions in $|z| < R$ without common zero for all and

$$X^p = [X, dX/dz, \cdots, d^{p-1}X/dz^{p-1}] \qquad (p = 1, \cdots, n)$$

the osculating p-element of X $(X^1 = X)$ (see Weyl [7]), which is a representation of x^p. Let $T_p(r)$ be the order function of x^p $(p = 1, \cdots, n)$. Then it is known that

$$(1) \quad V_p(r) + \{T_{p+1}(r) - 2T_p(r) + T_{p-1}(r)\} = \Omega_p(r) - \Omega_p(r_o) \qquad (r_o \leqq r < R)$$

where r_o is a fixed positive number, $V_p(r)$ is the valence in $|z| < r$ of the stationary points of rank p of x and

$$\Omega_p(r) = \frac{1}{2\pi}\int_0^{2\pi} \log \{|X^{p+1}||X^{p-1}|/|X^p|^2\} d\theta \qquad (z = re^{i\theta})$$

(Weyl [7], p. 123).

In the theory of holomorphic curves, it is essential to evaluate Ω_p in terms of T_p. For example,

"When $R = \infty$, for any number $\alpha > 1$, the inequality

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$$2\Omega_p(r) \leqq \alpha \log T_p(r) - 2 \log r + O(1)$$

holds except for r of an open set $E \subset [r_o, \infty)$ such that

$$\int_E r^{-1} dr < \infty \text{ .''}$$

(See Weyl [7], Chapter III.)

There is a similar result for $R < \infty$ (Weyl [7], Chapter IV).

We note that, in this paper, we always use $r = |z|$ as the independent variable instead of $\log r$. (cf. Weyl [7], Wu [8].)

In the proof of the defect relations for holomorphic curves, this estimate of Ω_p plays a fundamental role and necessarily it comes out exceptional sets in many inequalities of which we are in need to prove the defect relations. (See Ahlfors [2], Weyl [7] etc.)

On the other hand, in the Nevanlinna theory of meromorphic functions in $|z| < R$ (Hayman [5], Nevanlinna [6]), the second fundamental theorem tells us that, for a non-constant meromorphic function $f(z)$ in $|z| < R$, the exceptional set does not come out if the order of f is finite and the defect relation holds either

(I) for any non-constant meromorphic function when $R = \infty$, or

(II) for any meromorphic function f such that

$$\limsup_{r \to R} T(r, f)/\log (R - r)^{-1} = \infty$$

when $R < \infty$.

For the systems of holomorphic functions in $|z| < R$, similar results are known (Cartan [3]).

In this paper, after the model of the case of meromorphic functions stated above, we shall remove the exceptional sets in the case of holomorphic curves applying the method used in Ahlfors [1] when $T_1(r)$ is of finite order and weaken "the hypothesis H" (Weyl [7], p. 201) in the case of $R < \infty$.

We will use the notation used in Weyl [7] in the main.

§ 2. Lemmas

We prepare some lemmas for later use.

LEMMA 1. *Let $f(r)$ be a function defined on $[r_o, R)$ with continuous non-negative derivative and $f(r_o) \geqq 1$. Then for any numbers $\alpha > 1$ and* $\mu \geqq 0$,

(I) *when* $R = \infty$, *the inequality*

$$f'(r) \leqq \{f(r)\}^{\alpha} r^{\mu - 1}$$

holds except in an open set $E \subset [r_0, \infty)$ *such that*

$$\int_E r^{\mu-1} dr \leqq (\alpha - 1)^{-1} ;$$

(II) *when* $R < \infty$, *the inequality*

$$f'(r) \leqq \{f(r)\}^{\alpha} (R - r)^{-\mu - 1}$$

holds except in an open set $E \subset [r_0, R)$ *such that*

$$\int_E (R - r)^{-\mu-1} dr \leqq (\alpha - 1)^{-1} .$$

(*See Weyl* (7] p. 155 *and* p. 197.)

LEMMA 2. *Let two functions* $f(r)$ *and* $F(r)$ *be given on* $[r_0, R)$ $(r_0 > 0)$ *such that* $f(r)$ *is continuous and*

$$1 + \log (r/r_0) + \int_{r_0}^{r} \{(\log r - \log t) \exp (f(t))\}/t dt \leqq F(r) \qquad (r_0 \leqq r < R) .$$

Then, for any numbers $\alpha > 1$ *and* $\mu \geqq 0$,
(I) *when* $R = \infty$,

$$f(r) \leqq \alpha^2 \log F(r) + \mu(\alpha + 1) \log r$$

for all $r \geqq r_0$ *except in an open set* $E \subset [r_0, \infty)$ *such that*

$$\int_E r^{\mu-1} dr \leqq 2(\alpha - 1)^{-1} ;$$

(II) *when* $R < \infty$,

$$f(r) \leqq \alpha^2 \log F(r) + (\mu + 1)(\alpha + 1) \log (R - r)^{-1} + (\alpha + 1) \log r$$

for all r *in* $[r_0, R)$ *except in an open set* $E \subset [r_0, R)$ *such that*

$$\int_E (R - r)^{-\mu-1} dr \leqq 2(\alpha - 1)^{-1} .$$

We may prove this lemma as in Weyl [7], p. 156 or p. 197 using Lemma 1.

DEFINITION. We say that a holomorphic curve x in $|z| < R$ is admissible if either $R = \infty$ and x is non-degenerate, or if $R < \infty$, x is non-degenerate and the following condition holds:

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(2) $$\lim_{r\to R} T_p(r) = \infty \qquad (p = 1, \cdots, n).$$

Note that, if $R = \infty$ and x is non-degenerate, the condition (2) holds. (See Weyl [7].)

LEMMA 3. *For any admissible curve x in $|z| < R$, the following inequality holds for all r in $[R_o, R)$ $(r_o \leqq R_o)$:*

$$1 + \log(r/r_o) + \int_{r_o}^{r} \{(\log r - \log t)\exp(2\tilde{\Omega}_p(t))\}/t dt \leqq 2T_p(r)$$

where $\tilde{\Omega}_p(t) = \Omega_p(t) + \log(t/r_o)$ and R_o depends on the curve. (See Weyl [7], p. 154 and p. 196.)

From now on in § 2 and § 3, we assume that all curves in our mind are admissible.

Applying Lemmas 1 and 2 to our curves, we obtain the following by Lemma 3.

PROPOSITION 1. *For any numbers $\alpha > 1$ and $\mu \geqq 0$,*
(I) *when $R = \infty$, the inequality*

$$2\tilde{\Omega}_p(r) \leqq \alpha^2 \log T_p(r) + \mu(\alpha + 1)\log r + O(1)$$

holds for all $r \geqq R_o$ except in an open set $E \subset [R_o, \infty)$ such that

$$\int_E r^{\mu-1} dr < \infty;$$

(II) *when $R < \infty$, the inequality*

$$2\tilde{\Omega}_p(r) \leqq \alpha^2 \log T_p(r) + (\mu + 1)(\alpha + 1)\log(R - r)^{-1} + O(1)$$

holds for all r in $[R_o, R)$ except in an open set $E \subset [R_o, R)$ such that

$$\int_E (R - r)^{-\mu-1} dr < \infty.$$

(*See Weyl* [7], p. 198.)

Using this proposition, we obtain the following as in Weyl [7].

PROPOSITION 2. *For any numbers $\varepsilon > 0$ and $\mu \geqq 0$, there exists an r_1 such that*
(I) *when $R = \infty$,*

(3) $$T_{p+1}(r) < (1 + 1/p + \varepsilon)T_p(r) + O(\log r),$$

(4) $$T_{p-1}(r) < (1 + 1/(n + 1 - p) + \varepsilon)T_p(r) + O(\log r)$$

for all $r \geqq r_1$ *except in an open set* $E \subset [r_1, \infty)$ *such that*

$$\int_E r^{\mu-1} dr < \infty\,;$$

(II) *when* $R < \infty$,

(5) $$T_{p+1}(r) < (1 + 1/p + \varepsilon)T_p(r) + O(\log(R - r)^{-1})\,,$$

(6) $$T_{p-1}(r) < (1 + 1/(n + 1 - p) + \varepsilon)T_p(r) + O(\log(R - r)^{-1})$$

for all $r \in [r_1, R)$ *except in an open set* $E \subset [r_1, R)$ *such that*

$$\int_E (R - r)^{-\mu-1} dr < \infty\,.$$

§ 3. Theorems and applications

In this section, we are going to investigate the orders of $T_p(r)$ and improve Propositions 1 and 2.

The order ρ_p of $T_p(r)$ is defined by

$$\limsup_{r\to\infty} \log T_p(r)/\log r = \rho_p \qquad (R = \infty)$$

or

$$\limsup_{r\to R} \log T_p(r)/\log (R - r)^{-1} = \rho_p \qquad (R < \infty)$$

and the lower order λ_p of $T_p(r)$ by

$$\liminf_{r\to\infty} \log T_p(r)/\log r = \lambda_p \qquad (R = \infty)$$

or

$$\liminf_{r\to R} \log T_p(r)/\log (R - r)^{-1} = \lambda_p \qquad (R < \infty)\,.$$

THEOREM 1. *All* $T_p(r)$ *are of the same order.*

Proof. When $R = \infty$, this was proved by Ahlfors [2]. We prove this theorem when $R < \infty$ applying the method used in Ahlfors [1].

Now, let ρ_p be finite, then for any $\rho > \rho_p$, there is an $r_2(\geqq r_1)$ such that, for all $r \in [r_2, R)$,

$$T_p(r) \leqq O((R - r)^{-\rho})\,.$$

Here, we apply Proposition 2, (II), (5).

(i) When $r \notin E$ and $r_2 \leqq r < R$, we have

$$T_{p+1}(r) \leqq O((R-r)^{-\rho}) .$$

(ii) When $r \in E$ and $r_2 \leqq r < R$, let r' be the right hand end point of the maximal interval included in E and containing r. Then, putting $\mu = \rho$, we have

$$(R-r')^{-\rho} - (R-r)^{-\rho} \leqq \rho \int_E (R-r)^{-\rho-1} dr = O(1)$$

so that

$$(R-r')^{-\rho} \leqq (R-r)^{-\rho} + O(1)$$

and

$$\log (R-r')^{-1} = \log (R-r)^{-1} + O(1) .$$

$T_{p+1}(r)$ being increasing and $r' \notin E$,

$$T_{p+1}(r) \leqq T_{p+1}(r') \leqq O((R-r')^{-\rho}) \leqq O((R-r)^{-\rho}) + O(1)$$
$$= O((R-r)^{-\rho}) .$$

By (i) and (ii), for all r sufficiently near R

$$T_{p+1}(r) < O((R-r)^{-\rho}) .$$

This means $\rho_{p+1} \leqq \rho$. As ρ is arbitrarily greater than ρ_p, we obtain $\rho_{p+1} \leqq \rho_p$. Similarly, we obtain $\rho_{p-1} \leqq \rho_p$ applying (6). It follows that, if one of ρ_p is finite, then all ρ_p are finite and same. This means also that if one of ρ_p is infinite, then all ρ_p are infinite. That is, all $T_p(r)$ are of the same order.

THEOREM 2. *All $T_p(r)$ are of the same lower order.*

Proof. (I) $R = \infty$. We have only to prove the case when one of $T_p(r)$ is of positive or infinite lower order. Otherwise, all $T_p(r)$ are of lower order zero.

Now, let λ_p be positive or infinite. Then, for any $0 < \lambda < \lambda_p$, there is an $r_3(\geqq r_1)$ such that

$$T_p(r) \geqq r^{\lambda} \qquad (r \geqq r_3) .$$

We apply Proposition 2, (I), (3).

(i) For $r \notin E$ and $r \geqq r_3$,

$$r^\lambda \leqq (1 + 1/(p-1) + \varepsilon)T_{p-1}(r) + O(\log r) .$$

(ii) For $r \in E$ and $r \geqq r_3$, let $\tilde{r}$ be the left hand end point of the maximal interval included in E and containing r. Then, putting $\mu = \lambda$, we have

$$r^\lambda - \tilde{r}^\lambda = \lambda \int_{\tilde{r}}^{r} t^{\lambda-1} dt \leqq \lambda \int_E t^{\lambda-1} dt = O(1) ,$$

so that

$$r^\lambda \leqq \tilde{r}^\lambda + O(1) .$$

As $\tilde{r} \notin E$ for sufficiently large r and $T_{p-1}(r)$ is increasing,

$$\begin{aligned} \tilde{r}^\lambda &\leqq (1 + 1/(p-1) + \varepsilon)T_{p-1}(\tilde{r}) + O(\log \tilde{r}) \\ &\leqq (1 + 1/(p-1) + \varepsilon)T_{p-1}(r) + O(\log r) \end{aligned}$$

and

$$r^\lambda \leqq (1 + 1/(p-1) + \varepsilon)T_{p-1}(r) + O(\log r) + O(1) .$$

By (i) and (ii), for all sufficiently large r,

$$r^\lambda \leqq (1 + 1/(p-1) + \varepsilon)T_{p-1}(r) + O(\log r) + O(1) .$$

This means $\lambda \leqq \lambda_{p-1}$. As $\lambda < \lambda_p$ and λ is arbitrary, we have $\lambda_p \leqq \lambda_{p-1}$. Similarly, using (4) instead of (3), we have $\lambda_p \leqq \lambda_{p+1}$. It follows that, if one of λ_p is not zero, then all λ_p are not zero and same.
(II) $R < \infty$. We can prove this theorem by using Proposition 2, (II) as in the case $R = \infty$.

THEOREM 3. *If $T_1(r)$ is of finite order, then, for any number $\alpha > 1$,*
(I) *when $R = \infty$, the inequality*

$$2\tilde{\Omega}_p(r) \leqq \alpha^2 \log T_p(r) + O(\log r) + O(1)$$

holds for all sufficiently large values r;
(II) *when $R < \infty$, the inequality*

$$2\tilde{\Omega}_p(r) \leqq \alpha^2 \log T_p(r) + O(\log (R-r)^{-1}) + O(1)$$

holds for all r sufficiently near R.

Proof. (I) $R = \infty$. Let $T_1(r)$ be of finite order ρ_1. Then, by Theorem 1, all $T_p(r)$ are of order ρ_1. Thus, for any $\rho > \rho_1$, there is an $r_4 (\geqq R_0)$ such that

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$$T_p(r) \leqq r^\rho \qquad (r \geqq r_4)$$

and for any p-ad A^p (Weyl [7]), as

$$N_p(r, A^p) \leqq T_p(r) + C_p ,$$

C_p being independent of A^p (Ahlfors [2], p. 7 or Wu [8], p. 105),

$$(7) \qquad n_p(r, A^p) \log 2 \leqq \int_r^{2r} n_p(t, A^p)/t dt \leqq N_p(2r, A^p) = O(r^\rho)$$

for $r \geqq r_4$. Now, putting $\mu = \rho$ in Proposition 1,

$$(8) \qquad 2\tilde{\Omega}_p(r) \leqq \alpha^2 \log T_p(r) + \rho(\alpha + 1) \log r + O(1)$$

for all $r \geqq R_o$ except in an open set $E \subset [R_o, \infty)$ such that

$$\int_E r^{\rho-1} dr < \infty .$$

For $r \in E$ and $r \geqq r_4$, let $\tilde{r}$ be the right hand end point of the maximal interval included in E and containing r. Then,

$$(9) \qquad \tilde{r}^\rho - r^\rho = \rho \int_r^{\tilde{r}} t^{\rho-1} dt \leqq \rho \int_E t^{\rho-1} dt = O(1)$$

and

$$(10) \qquad \log \tilde{r} = \log r + O(1) .$$

Further, by (7) and (9),

$$N_p(\tilde{r}, A^p) - N_p(r, A^p) = \int_r^{\tilde{r}} n_p(t, A^p)/t dt \leqq O\left(\int_r^{\tilde{r}} t^{\rho-1} dt\right) = O(1) ,$$

that is,

$$N_p(\tilde{r}, A^p) \leqq N_p(r, A^p) + O(1) .$$

As

$$\underset{A^p}{\mathfrak{M}} N_p(r, A^p) = T_p(r)$$

(Ahlfors [2], p. 8 or Wu [8], p. 107) and $O(1)$ is independent of A^p,

$$(11) \qquad T_p(\tilde{r}) \leqq T_p(r) + O(1) .$$

On the other hand, by (1), we have the equality:

$$V_p(r) + T_{p-1}(r) + T_{p+1}(r) + \log r/r_o = \tilde{\Omega}_p(r) - \tilde{\Omega}_p(r_o) + 2T_p(r)$$

and this shows that

$$2T_p(r) + \tilde{\Omega}_p(r)$$

is increasing. Thus, we have

$$2T_p(r) + \tilde{\Omega}_p(r) \leqq 2T_p(\tilde{r}) + \tilde{\Omega}_p(\tilde{r}) ,$$

so that by (11),

$$\tilde{\Omega}_p(r) \leqq \tilde{\Omega}_p(\tilde{r}) + O(1) .$$

As $\tilde{r} \notin E$, by (8), (10) and (11), we have

$$2\tilde{\Omega}_p(r) \leqq \alpha^2 \log T_p(r) + \rho(\alpha + 1) \log r + O(1) .$$

Thus, for any $r \geqq r_4$, the inequality

$$2\tilde{\Omega}_p(r) \leqq \alpha^2 \log T_p(r) + O(\log r) + O(1)$$

holds.

(II) $R < \infty$. We can carry out the proof parallel to the case $R = \infty$. Corresponding to Proposition 2, we have

COROLLARY 1. *If $T_1(r)$ is of finite order, for any $\varepsilon > 0$, there is an r_5 such that*

(I) *when $R = \infty$,*

$$T_{p+1}(r) < (1 + 1/p + \varepsilon)T_p(r) + O(\log r) ,$$
$$T_{p-1}(r) < (1 + 1/(n + 1 - p) + \varepsilon)T_p(r) + O(\log r)$$

for all $r \geqq r_5$;

(II) *when $R < \infty$,*

$$T_{p+1}(r) < (1 + 1/p + \varepsilon)T_p(r) + O(\log (R - r)^{-1}) ,$$
$$T_{p-1}(r) < (1 + 1/(n + 1 - p) + \varepsilon)T_p(r) + O(\log (R - r)^{-1})$$

for all $r \in [r_5, R)$.

When $R = \infty$, it is well-known (see Wu [8]) that if the original curve x is transcendental:

$$\lim_{r\to\infty} T_1(r)/\log r = \infty ,$$

then all the associated curves x^p are also transcendental.

Similarly to this fact, we give the following for $R < \infty$.

THEOREM 4. *When R is finite, if*

$$\limsup_{r\to R} T_1(r)/\log(R-r)^{-1} = \infty\,, \tag{12}$$

then there exists a sequence $\{s_n\}$ outside an exceptional set E such that

$$\lim_{n\to\infty} s_n = R$$

and

$$\lim_{n\to\infty} T_p(s_n)/\log(R-s_n)^{-1} = \infty \qquad (p = 1, 2, \cdots, n)\,.$$

Proof. When the order of $T_1(r)$ $(=\rho_1)$ is finite, we have the result easily by Corollary 1. When the order of $T_1(r)$ is infinite, then $T_2(r), \cdots, T_n(r)$ are also of order infinite by Theorem 1. In this case, we apply Proposition 2, (II), (6) for $\mu = 0$. First of all, we note that

$$\rho_1' = \limsup_{\substack{r\to R\\ r\notin E}} \log T_1(r)/\log(R-r)^{-1} = \infty\,.$$

In fact, suppose that ρ_1' is finite. For any $\tilde r \in E$ sufficiently near R, let t_1 be the left hand end point of the maximal interval I included in E and containing $\tilde r$ and t_2 the right hand end point of I. Then, $t_1, t_2 \notin E$ and

$$\log(R-t_2)^{-1} - \log(R-t_1)^{-1} = \int_{t_1}^{t_2} (R-r)^{-1}dr \leqq \int_E (R-r)^{-1}dr = O(1)\,,$$
$$\log(R-t_1)^{-1} < \log(R-\tilde r)^{-1} < \log(R-t_2)^{-1}\,.$$

From this and $T_1(r)$ being increasing, we have the following:

$$\log T_1(t_1)/\log(R-t_2)^{-1} \leqq \log T_1(\tilde r)/\log(R-\tilde r)^{-1}$$
$$\leqq \log T_1(t_2)/\log(R-t_1)^{-1}$$

and

$$\lim_{\tilde r\to R} \log(R-t_1)^{-1}/\log(R-t_2)^{-1} = 1$$

so that

$$\limsup_{\substack{\tilde r\to R\\ \tilde r\in E}} \log T_1(\tilde r)/\log(R-\tilde r)^{-1} \leqq \limsup_{\substack{r\to R\\ r\notin E}} \log T_1(r)/\log(R-r)^{-1} = \rho_1'\,.$$

This means that the order of $T_1(r)$ is finite. This is a contradiction. ρ_1' must be ∞. "$\rho_1' = \infty$" means that there is a sequence $\{s_n\} \subset [r_1, R) - E$ such that $s_n \to R$ $(n\to\infty)$ and

$$\lim_{n\to\infty} T_1(s_n)/\log (R - s_n)^{-1} = \infty \ .$$

Applying this fact to Proposition 2, (II), (6) for $p = 2, \cdots, n$, we have the desired result.

COROLLARY 2. *"The hypothesis H" in Weyl* [7], p. 201 *may be changed by the following*:

"When R is finite,

$$x \text{ is admissible and } \limsup_{r\to R} T_1(r)/\log (R - r)^{-1} = \infty \ ."$$

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