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PROPERTY (FA) OF THE GAUSS–PICARD MODULAR GROUP

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Abstract

In this note, we prove that the Gauss–Picard modular group $\mathbf{PU}(2, 1; \Theta_1)$ has Property (FA). Our result gives a positive answer to a question by Stover ['Property (FA) and lattices in SU(2,1)', *Internat. J. Algebra Comput.* **17** (2007), 1335–1347] for the group $\mathbf{PU}(2, 1; \Theta_1)$.

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1. Introduction

Whether a group G has Property (FA) is an important question in the study of lattices in semisimple Lie groups. In the study of Property (FA), there is a fundamental theorem due to Serre [6].

THEOREM 1.1. *A group G has Property (FA) if and only if:*
(1) *G is finitely generated;*
(2) *G does not split as a nontrivial free product with amalgamation;*
(3) *G does not admit a homomorphism onto $\mathbb{Z}$.*

Since the irreducible lattices in $\mathbf{Sp}(n, 1)$ for $n \geq 2$, $\mathbf{F}_{4(-20)}$, and semisimple Lie groups with $\mathbb{R}$-rank at least two always have Property (FA) (see [1]), the remaining interesting cases are the fundamental groups of real and complex hyperbolic manifolds, that is, lattices in $\mathbf{PSO}_0(n, 1)$ and $\mathbf{PU}(n, 1)$.

In [5] there are many cocompact Fuchsian groups, that is, lattices in $\mathbf{PSL}(2, \mathbb{R})$, which split as a free product with amalgamation. It is well known that cocompact Fuchsian triangle groups have Property (FA) and the classical modular group $\mathbf{PSL}(2, \mathbb{Z})$ does not have Property (FA), since $\mathbf{PSL}(2, \mathbb{Z})$ is a free product of two finite cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

Let Θ_d denote the ring of algebra integers in the quadratic number field $\mathbb{Q}(\sqrt{-d})$, where d is a square-free positive integer. In [3] Frohman and Fine proved that the

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Bianchi group $\mathbf{PSL}(2; \Theta_d)$ splits as a nontrivial free product with amalgamation for $d \neq 3$. But in [6] Serre proved that $\mathbf{PSL}(2; \Theta_3)$ has Property (FA).

As the complex hyperbolic analogue of Bianchi group $\mathbf{PSL}(2; \Theta_d)$, the group $\mathbf{PU}(2, 1; \Theta_d)$ is called the Picard modular group, which is a subgroup of $\mathbf{PU}(2, 1)$ with entries in Θ_d.

The study of Property (FA) of Picard modular groups was begun by Stover in [7], where the author proved the following theorem.

THEOREM 1.2. *$\mathbf{PU}(2, 1; \Theta_3)$ and $\mathbf{SU}(2, 1; \Theta_3)$ have Property (FA).*

This theorem indicates that there is a connection between certain real and complex hyperbolic lattices. In the same paper [7], Stover asked the following question.

QUESTION 1.3. Does $\mathbf{PU}(2, 1; \Theta_d)$ or $\mathbf{SU}(2, 1; \Theta_d)$ have Property (FA) for $d \neq 3$?

The aim of this note is to show the following result.

THEOREM 1.4. *$\mathbf{PU}(2, 1; \Theta_1)$ and $\mathbf{SU}(2, 1; \Theta_1)$ have Property (FA).*

2. Preliminaries

2.1. Complex hyperbolic space. In this subsection, we recall some basic material about complex hyperbolic space. More details can be found in [2, 4].

Let $\mathbb{C}^{2,1}$ denote the three-dimensional complex vector space $\mathbb{C}^3$ equipped with the Hermitian form

$$\langle z, w\rangle = z_1\bar{w}_3 + z_2\bar{w}_2 + z_3\bar{w}_1,$$

where $z = (z_1, z_2, z_3)^t$ and $w = (w_1, w_2, w_3)^t$. The vector x^t stands for the transpose of vector x. Consider the subspaces of $\mathbb{C}^{2,1}$:

$$V_- = \{z \in \mathbb{C}^{2,1} \mid \langle z, z\rangle < 0\},$$
$$V_0 = \{z \in \mathbb{C}^{2,1} - \{0\} \mid \langle z, z\rangle = 0\}.$$

Complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ is defined to be the complex projective subspace $\mathbb{P}(V_-)$ equipped with the Bergman metric, where $\mathbb{P} : \mathbb{C}^{2,1} - \{0\} \to \mathbb{C}P^2$ is the canonical projection onto the complex projective space. We consider the complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ as the Siegel domain $\{z = (z_1, z_2) \in \mathbb{C}^2 \mid 2\Re(z_1) + |z_2|^2 < 0\}$. The boundary of complex hyperbolic space is $\partial\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}(V_0)$, which can be identified with the one-point compactification $\bar{\mathfrak{N}}$ of the Heisenberg group $\mathfrak{N}$ by stereographic projection. The point at infinity is $q_\infty = (1, 0, 0)^t$.

The group of biholomorphic transformations of complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ is $\mathbf{PU}(2, 1)$, which is the projectivization of the unitary group $\mathbf{U}(2, 1)$ preserving the Hermitian form. If we consider the special unitary group $\mathbf{SU}(2, 1)$, it is clear that $\mathbf{SU}(2, 1)$ is a threefold cover of $\mathbf{PU}(2, 1)$ by the subgroup $\{I, \omega I, \omega^2 I\}$, where I stands for the identity matrix and ω stands for the primitive cube root of unity.

2.2. Property (FA). Let G be a group, and Υ be a tree with an action by G. Let Υ^G denote the subtree of fixed points of the G-action. We say that G has Property (FA)

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if $\Upsilon^G \neq \emptyset$ for every tree Υ on which G acts without inversions. Although Theorem 1.1 is fundamental, we have the following two propositions which will be crucial in the proof of Theorem 1.4 in the next section.

PROPOSITION 2.1 [7, Proposition 2.4]. *Suppose that G is a finitely presented group and $N \trianglelefteq G$ a normal subgroup such that N and G/N have Property (FA). Then G also has Property (FA).*

PROPOSITION 2.2 [7, Proposition 2.5]. *Suppose that G is a group with subgroups $A = \langle a_i \rangle$ and $B = \langle b_j \rangle$ with $G = \langle A, B\rangle$ and that G acts on a tree Υ. If $\Upsilon^A, \Upsilon^B \neq \emptyset$ and every $a_i b_j$ has a fixed point on Υ, then $\Upsilon^G \neq \emptyset$.*

3. Proof of Theorem 1.4

In this section we give a proof of Theorem 1.4 which is similar to the proof of Theorem 1.2 in [7].

Let $\mathscr{D}(\Theta_1)$ denote the diagonal subgroup of $\mathbf{SU}(2, 1; \Theta_1)$ and $\mathscr{N}(\Theta_1)$ denote the subgroup of strictly upper triangular matrices. The Borel subgroup of upper triangular matrices is

$$\mathscr{B}(\Theta_1) = \mathscr{N}(\Theta_1) \rtimes \mathscr{D}(\Theta_1).$$

It is clear that the Borel subgroup of $\mathbf{PU}(2, 1; \Theta_1)$, which is the projectivization of the Borel subgroup in $\mathbf{SU}(2, 1; \Theta_1)$, equals the subgroup Γ_∞, the stabilizer of q_∞ in $\mathbf{PU}(2, 1; \Theta_1)$. The following theorem, proved by Falbel *et al.* in [2], is crucial in the proof of Theorem 1.4.

THEOREM 3.1. *The Gauss–Picard modular group* $\mathbf{PU}(2, 1; \Theta_1)$ *has a presentation*

$$\langle I_0, Q, T : I_0^2 = Q^2 = (I_0 Q)^3 = (I_0 T)^{12} = (I_0 Q T)^8 = [(I_0 T)^3, T] \\ = [Q, T] = \mathit{Identity}\rangle.$$

We use the same notation as in [2]. Furthermore, Falbel *et al.* [2] proved that the Gauss–Picard modular group can be generated by R, Q, T, I_0 and that the Borel subgroup Γ_∞ has the presentation

$$\Gamma_\infty = \langle R, Q, T : Q^2 = R^4 = (R^{-1} Q T)^4 = [R, T] = [Q, T] = \mathit{Identity}\rangle.$$

PROOF OF THEOREM 1.4. It is clear that the groups $\mathbf{SU}(2, 1; \Theta_1)$ and $\mathbf{PU}(2, 1; \Theta_1)$ are isomorphic, since there is a unique cube root of unity in Θ_1. Hence it is enough to prove that the Gauss–Picard modular group $\mathbf{PU}(2, 1; \Theta_1)$ has Property (FA).

Firstly, we prove that the Borel subgroup Γ_∞ has Property (FA). To do this, according to Theorem 1.1, we need to show that Γ_∞ cannot map onto $\mathbb{Z}$ and cannot split as a free product with amalgamation. Assume that the Borel subgroup can map onto $\mathbb{Z}$; then we get a contradiction by considering the presentation of the group. Therefore, the Borel subgroup cannot map onto $\mathbb{Z}$.

To show that the Borel subgroup cannot split as a nontrivial product with amalgamation, we consider the short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_\infty \longrightarrow \Delta \longrightarrow 1,$$

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described in [2, Proposition 2, Section 3]. The subgroup $\Delta \subset \mathrm{Isom}(\mathbb{Z}[i])$ is of index two, and generated by a rotation $\widehat{Q}$ of order two and another rotation $\widehat{R}$ of order four. We also have $(\widehat{Q}\,\widehat{R})^4 = 1$. According to Proposition 2.2 the group Δ has Property (FA), so it cannot split as a free product with amalgamation. Now suppose that Γ_∞ can split as a free nontrivial product with amalgamation. Since the $\mathbb{Z}$ factor is central in Γ_∞, the subgroup $\mathbb{Z}$ must be contained in the amalgamation subgroup. It follows from the short exact sequence that the group Δ can split as a nontrivial free product with amalgamation. This is a contradiction. Hence the Borel subgroup has Property (FA).

Finally, we show that the group $\mathbf{PU}(2, 1; \Theta_1)$ has Property (FA) by applying Proposition 2.2. We know that

$$\mathbf{PU}(2, 1; \Theta_1) = \langle I_0, \Gamma_\infty \rangle = \langle I_0, \langle R, T, Q \rangle \rangle.$$

Since $\langle I_0 \rangle = \mathbf{Z}/2\mathbb{Z}$ is a finite group, clearly it has Property (FA). We have shown that Γ_∞ has Property (FA). Now let us consider an action of $\mathbf{PU}(2, 1; \Theta_1)$ on a tree Υ. We know that $\Upsilon^{\langle I_0 \rangle}, \Upsilon^{\langle R,T,Q \rangle} \neq \emptyset$. In order to prove that the products $I_0 R$, $I_0 T$ and $I_0 Q$ have fixed points on Υ, we just need to show that these elements have finite order. This follows from the presentation of $\mathbf{PU}(2, 1; \Theta_1)$, which is $(I_0 Q)^3 = (I_0 R)^4 = (I_0 T)^{12} = \mathit{Identity}$. So we have shown that $\mathbf{PU}(2, 1; \Theta_1)$ has Property (FA) and this completes the proof. □

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