DERIVATIVE OF SINGULAR SET-FUNCTIONS

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The purpose of this paper is to prove that the general derivative of a completely additive singular set-function defined on certain measurable subsets of an abstract measure space is zero almost everywhere. As a corollary the celebrated Lebesgue decomposition theorem has been sharpened.

This result is well known for set-functions defined on measurable subsets of an n-dimensional Euclidean space (**2**, p. 119). The proof in this setting depends on two things: Vitali's covering theorem and the fact that for every measurable set A there exists an open set O which contains A and the images of O and A under the set-function can be made arbitrarily close. Here the covering theorem is due to Trjitzinsky and the open set is replaced by an envelope, an entirely measure-theoretic concept.

Let ϕ be a measure defined on a σ-field Σ of subsets of an abstract space S. A subset $A \subset S$ is said to be indefinitely covered by a family H of measurable sets if for every $x \in A$ there exists a sequence $\{\gamma_n\}$ contained in H, containing x for each n, and $\phi(\gamma_n) \to 0$ as $n \to \infty$. A family $G \subset \Sigma$ of measurable sets is said to be regular in the sense of the measure ϕ if the following conditions are satisfied:

(i) $\phi_e(D) < \infty$, where $D = \bigcup_{\gamma \epsilon G} \gamma$ and ϕ_e denotes the outer measure.

(ii) Denote by $\rho(\gamma)$ the set of points outside γ and indefinitely covered by those $\gamma' \in G$ which have points in common with γ, and let $\alpha(u)$ be a real-valued function with the following property: given an $\epsilon > 0$, there exists a sequence $\{\eta_n\}$ of positive numbers converging to zero such that whenever $0 < u_{i,n} < \eta_n$ and

$$\sum_{i=1}^{n} u_{i,n} \leqslant \phi_e(D) \qquad \text{for } n = 1, 2, \ldots,$$

then

$$\sum_{n,\, i=1}^{\infty} \alpha(u_{i,n}) < \epsilon;$$

we postulate that $\phi_e(\rho(\gamma)) < \alpha(\phi(\gamma))$.

(iii) Let $\Omega_a(\gamma)$ denote the union of the sets γ' which have points in common with γ and $\phi(\gamma') < a\phi(\gamma)$. We postulate that there exist two numbers a and b $(b > a > 1)$ such that $\phi_e[\Omega_a(\gamma)] < b\phi(\gamma)$ for each γ.

As an example of the function α mentioned in (ii) we may take $\alpha(u) = u^c$ $(c > 1)$. In this case

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$$\sum_{n,i} \alpha(u_{i,n}) = \sum_{n,i} u_{i,n}^{c} = \sum_{n,i} u_{i,n}^{c-1} u_{i,n}$$
$$< \sum_{n,i} \eta_n^{c-1} u_{i,n} = \sum_n \eta_n^{c-1} \sum_i u_{i,n} \leqslant \phi_e(D) \sum_n \eta_n^{c-1}.$$

It suffices to choose

$$\eta_n^{c-1} = \epsilon \frac{2^{-n}}{\phi_e(D)}.$$

The following theorem has been proved by W. J. Trjitzinsky (**3**, p. 16).

THEOREM 1. *Let* $\Delta(G)$ (*or simply* Δ) *denote the set of points indefinitely covered* (*in the sense of measure* ϕ) *by the regular family G. Then*

(a) Δ *is* ϕ*-measurable*;

(b) *there exists a sequence* $\{\gamma_i\}$ *of disjoint sets in G such that* $\phi(\Delta - \Delta \cap \Gamma) \leqslant s$, *where*

$$s = \sum_{i=1}^{\infty} \alpha(\phi(\gamma_i)), \qquad \Gamma = \bigcup_{i=1}^{\infty} \gamma_i,$$

and s can be made less than any positive number ϵ;

(c) *given* $\epsilon > 0$, *the sequence* $\{\gamma_i\}$ *can be chosen in such a way that* $s < \epsilon$ *and*

$$\phi(\Delta) - \epsilon < \phi(\Gamma) < \phi(\Delta) + \epsilon.$$

A set-function ψ, finite and real valued, defined on subsets of Δ is said to be completely additive if

(i) $$\psi\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \psi(E_n)$$

for every sequence $\{E_n\}$ of disjoint subsets of Δ for which

$$\sum_{n=1}^{\infty} |\psi(E_n)| < +\infty$$

and

(ii) $\psi(E_1 - E_2) = \psi(E_1) - \psi(E_2)$ for all $E_2 \subset E_1 \subset \Delta$.

In the following ψ shall denote a non-negative completely additive set-function. For, if it is not non-negative, it can be expressed as the difference of two non-negative completely additive set-functions (**2**, p. 11). Let $G(E)$ denote the subfamily of G whose elements intersect the set E and $\Delta[G(E)]$ denote the set indefinitely covered by $G(E)$. E is said to be a kernel if $\phi(\Delta[G(E)] - E) = 0$. It is said to be an envelope if its complement with respect to $\Delta(G)$ is a kernel.

Kernel and envelope are roughly equivalent to closed and open set, respectively. $\Delta[G(E)] - E$ represents the boundary of a kernel E; hence the assumption that it has measure zero.

As an additional property of the set-function ψ we assume that

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HYPOTHESIS H. *For every measurable set $A \subset \Delta$ and every $\epsilon > 0$, there exists an envelope O such that $A \subset O \subset \Delta$ and $\psi(O - A) < \epsilon$; also $x \in A, x \in \gamma \in G$, and $\phi(\gamma)$ sufficiently small imply $\gamma \subset O$.*

By the upper derivative of ψ at a point $M \in \Delta$ with respect to a regular family G we shall mean the least upper bound of the ratio $\psi(\gamma \cap \Delta)/\phi(\gamma)$ over all γ which contain M and whose measures tend to zero. The upper derivative is denoted by $\bar{D}(\psi, M, G)$. The lower derivative is defined dually. When these two derivatives are equal, the common value is denoted by $D(\psi, M, G)$ and is called the general derivative.

THEOREM 2. *For every non-negative ψ that satisfies Hypothesis H, if $\bar{D}(\psi, M, G) \geqslant \mu$ for every $M \in A \subset \Delta$, then $\psi(A) \geqslant \mu\phi(A)$.*

Proof. Let X denote a subfamily of G whose elements γ are contained in an envelope $O \supset A$ and satisfy the inequality $\psi(\gamma \cap \Delta)/\phi(\gamma) > \lambda$ for some $\lambda < \mu$. If

$$\sup_{\substack{M \in \gamma \\ \phi(\gamma) \to 0}} \frac{\psi(\gamma \cap \Delta)}{\phi(\gamma)} \geqslant \mu$$

for every $M \in A$, then X covers A indefinitely. Hence, by Theorem 1(c), there exists a sequence $\{\gamma_i\}$ of disjoint sets belonging to X for which

$$\phi\left(A - A \cap \left(\bigcup_{i=1}^{\infty} \gamma_i\right)\right) < \xi \quad \text{or} \quad \phi\left(A \cap \left(\bigcup_{i=1}^{\infty} \gamma_i\right)\right) > \phi(A) - \xi$$

for any preassigned $\xi > 0$. Because of Hypothesis H, O can be chosen such that for any $\epsilon > 0$, $\psi(A) > \psi(O) - \epsilon$; and since the γ_i are disjoint and all contained in O,

$$\psi(O) \geqslant \sum_{i=1}^{\infty} \psi(\gamma_i).$$

Therefore

$$\psi(A) > \psi(O) - \epsilon \geqslant \sum_{i=1}^{\infty} \psi(\gamma_i) - \epsilon \geqslant \lambda \sum_{i=1}^{\infty} \phi(\gamma_i) - \epsilon$$
$$= \lambda\phi\left(\bigcup_{i=1}^{\infty} \gamma_i\right) - \epsilon \geqslant \lambda\phi\left(A \cap \left(\bigcup_{i=1}^{\infty} \gamma_i\right)\right) - \epsilon > \lambda\phi(A) - \lambda\xi - \epsilon.$$

Since ξ and ϵ can be made arbitrarily small, by letting $\lambda \to \mu$, one finds $\psi(A) \geqslant \mu\phi(A)$.

ψ is said to be singular on Δ if there exists a null-set $E_0 \subset \Delta$ such that for all $A \subset \Delta$, $\psi(A) = \psi(E_0 \cap A)$.

THEOREM 3. *If ψ is singular on Δ and satisfies Hypothesis H, then $D(\psi, M, G) = 0$ almost everywhere.*

Proof. Since ψ is singular, there exists a set of measure zero $E_0 \subset \Delta$ such that for every measurable $A \subset \Delta$, $\psi(A \cap (\Delta - E_0)) = 0$. Let

$$F = \{M \in \Delta | D(\psi, M, G) > 0\}.$$

We shall prove that $\phi(F) = 0$. Otherwise there exists a positive integer N for which the set

$$E_N = \{M \in \Delta - E_0 | D(\psi, M, G) > 1/N\}$$

will have a positive measure and $\phi(E_N \cap \gamma) > 0$ for some γ. Since $E_N \subset \Delta - E_0$ and $D(\psi, M, G) > 1/N$ on $E_N \cap \gamma$, Theorem 2 yields

$$\psi[(\Delta - E_0)] \geqslant \psi(E_N \cap \gamma) \geqslant \phi(E_N \cap \gamma)1/N > 0.$$

But this contradicts the assumption that ψ is singular, for it shows that ψ is positive on a subset of $\Delta - E_0$.

COROLLARY. *Let ψ be a completely additive set-function defined on subsets of Δ and satisfying the Hypothesis H. Then there exists a unique decomposition of ψ in the form*

$$\psi(E) = \int_E D(\psi, M, G) d\phi + S(E),$$

where $E \subset \Delta$ is measurable and $S(E)$ is singular.

Proof. Every completely additive set-function defined on measurable subsets of Δ can be expressed as the sum of an absolutely continuous and a singular set-function (**2**, p. 33), denoted by $A(E)$ and $S(E)$, respectively. Trjitzinsky has proved (**3**, p. 26) that there exists an integrable function f such that $A(E) = \int_E f d\phi$ and $D(A, M, G) = f$ almost everywhere. Therefore

$$D(\psi, M, G) = f + D(S, M, G) \text{ almost everywhere.}$$

By the previous theorem the second term on the right-hand side is zero almost everywhere, and the proof is complete.

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