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THE ACTION OF THE DICKSON INVARIANTS ON LENGTH n STEENROD OPERATIONS

BY

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ABSTRACT. The purpose of this paper is to give an explicit formula for the action of the n dimensional Dickson invariants on the admissible mod 2 Steenrod operations of length n.

Let $S_n = H^*((Z/2)^n, Z/2) = Z/2[x_1, \ldots, x_n]$, $GL_n = GL_n(\mathbf{F}_2)$, and let e_n denote the n dimensional Steinberg idempotent. Recall that $e_n \in \mathbf{F}_2GL_n(\mathbf{F}_2)$ and that e_n is given by the formula $e_n = \bar{B}_n\bar{\Sigma}_n$ where $\bar{B}_n$ denotes the sum in $\mathbf{F}_2GL_n(\mathbf{F}_2)$ of all upper triangular matrices and $\bar{\Sigma}_n$ denotes the sum in $\mathbf{F}_2GL_n(\mathbf{F}_2)$ of all permutation matrices. S. Priddy and S. Mitchell have recently shown in [4] that e_n can be used to stably split the classifying space $B(Z/2)^n$. They call the summand corresponding to the Steinberg idempotent $M(n)$.

There are two descriptions of $H^*(M(n))$. (All coefficients are assumed to be $Z/2$.) One is due to Mitchell and Priddy [4] and relies on Nakaoka's calculations involving symmetric products [5]. It shows that $H^*(M(n))$ has a basis given by $\{Sq^I(x_1^{-1} \ldots x_n^{-1}) \mid I \text{ is admissible of length } n\}$; (see 3.5, 5.1 and 5.6 (b) of [4]). The other arises from work of S. Mitchell [3] and relies on early work of L. Dickson who shows in [2] that the invariants in S_n under the action of GL_n, denoted by $S_n^{GL_n}$, form a graded polynomial algebra on generators $\{c_{n,i}\}$ where $0 \leqq i \leqq n$ and $c_{n,i}$ is in dimension $2^n - 2^i$. A good source for a complete description of the Dickson invariants is [7]. Mitchell's work shows that $H^*(M(n))$ is a free $S_n^{GL_n}$ module; (see 3.11 of [3]). The purpose of this paper is to combine these viewpoints to give an explicit formula for the action of the Dickson invariants on the mod 2 Steenrod operations of length n. The main result is the following statement.

PROPOSITION. *Let $I = (i_1, \ldots, i_n)$ with Sq^I admissible. For $J = (j_1, \ldots, j_n)$, let $I + J = (i_1 + j_1, \ldots, i_n + j_n)$. Let $I' = (i_2, \ldots, i_n)$. Let Sq^A and Sq^B be sums of admissible Steenrod operations satisfying*

$$Sq^{A+I'}(x_2^{-1} \ldots x_n^{-1}) = c_{n-1,i-1} \cdot Sq^{I'}(x_2^{-1} \ldots x_n^{-1})$$

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and

$$Sq^{B+I'}(x_2^{-1} \ldots x_n^{-1}) = c_{n-1,i} \cdot Sq^{I'}(x_2^{-1} \ldots x_n^{-1}).$$

Then

$$\begin{aligned} c_{n,i} \cdot Sq^I(x_1^{-1} \ldots x_n^{-1}) &= Sq^{i_1+2^{n-1}-2^{i-1}}[Sq^{A+I'}(x_1^{-1} \ldots x_n^{-1})] \\ &\quad + Sq^{i_1+2^{n-1}}[Sq^{B+I'}(x_1^{-1} \ldots x_n^{-1})] \end{aligned}$$

if $0 < i < n$ *and*

$$\begin{aligned} c_{n,0} \cdot Sq^I(x_1^{-1} \ldots x_n^{-1}) &= Sq^{i_1+2^{n-1}}[Sq^{B+I'}(x_1^{-1} \ldots x_n^{-1})] \\ &= Sq^{I+J}(x_1^{-1} \ldots x_n^{-1}) \end{aligned}$$

where $J = (2^{n-1}, 2^{n-2}, \ldots, 2, 1)$.

The invariant $c_{k,k}$ equals 1 *in the above formulae.*

REMARKS.

a) These formulae should be interpreted as saying that the action of an invariant in dimension k on an operation of length n is determined by the actions of the invariants in dimension $k/2$ and the dimension immediately preceding $k/2$ on operations of length $n - 1$.

b) The Adem relations must be used in certain cases to express the Steenrod operations on the right hand side of these formulae in admissible form. Mitchell's work guarantees that only Steenrod operations of the correct length will be present in the admissible form of these operations.

The proof of this proposition requires four lemmas.
The first two are proved by Clarence Wilkerson.

Lemma 1 follows from 1.1 and 1.3 (b) of [7] with x_n changed to x_1. Lemma 2 is 2.3 (a) of [7].

LEMMA 1. $c_{n,i} = c_{n-1,i-1}^2 + x_1^{2^k} \sum_{k=0}^{n-1} c_{n-1,i} \cdot c_{n-1,k}$.

LEMMA 2. $Sq^k c_{n,i} = Sq^{k-2^{i-1}} c_{n,i-1} + (Sq^{k-2^{n-1}} c_{n,n-1}) \cdot c_{n,i}$ *for* $0 < k < 2^n$. *Here*, $Sq^t = 0$ if $t < 0$.

The third lemma is proved by Mitchell and Priddy. It is 2.3 (a) of [4].

LEMMA 3. *Let* $I = (i_1, \ldots, i_n)$ *and let* $k > n$.

Then $Sq^I(x_1^{-1} \ldots x_k^{-1}) = (x_1^{-1} Sq^I(x_2^{-1} \ldots x_k^{-1}))e_k$.

The fourth lemma is a consequence of lemmas 1 and 2.

LEMMA 4. *Let* $N = 2^{n-1} - 2^{i-1}$ *and let* $M = 2^{n-1} - 2^i$. *Then*

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$$c_{n,i} = \sum_{j=0}^{N} x_1^{N-j} Sq^j c_{n-1,i-1} + \sum_{j=0}^{M} x_1^{2^{n-1}-j} Sq^j c_{n-1,i}.$$

PROOF: Rearranging the sums shows that it suffices to prove

$$c_{n,i} = \sum_{j=0}^{2^{n-1}} x_1^j [Sq^{N-j} c_{n-1,i-1} + Sq^{2^{n-1}-j} c_{n-1,i}].$$

Lemma 1 then reduces the problem to proving

$$Sq^{N-j} c_{n-1,i-1} + Sq^{2^{n-1}-j} c_{n-1,i}$$

$$= \begin{cases} c_{n-1,i-1}^2 & \text{if } j = 0 \\ c_{n-1,i} \cdot c_{n-1,n-k} & \text{if } j = 2^{n-k},\ k = 1, \ldots, n \\ 0 \text{ otherwise} & \end{cases}$$

If $j = 0$, this is clear for $c_{n-1,i-1}$ is in dimension N and $2^{n-1} > \dim(c_{n-1,i})$.

If $j = 2^{n-1}$, then $N - j < 0$ and $2^{n-1} - j = 0$ giving the result.

Suppose $0 < j < 2^{n-1}$. Apply lemma 2 to $Sq^{2^{n-1}-j} c_{n-1,i}$ to obtain

$$Sq^{2^{n-1}-j} c_{n-1,i} = Sq^{2^{n-1}-j-2^{i-1}} c_{n-1,i-1} + [Sq^{2^{n-1}-j-2^{n-2}} c_{n-1,n-2}] \cdot c_{n-1,i}.$$

Now $2^{n-1} - j - 2^{i-1} = N - j$ so, the above formula gives

$$\begin{aligned} Sq^{N-j} c_{n-1,i-1} + Sq^{2^{n-1}-j} c_{n-1,i} &= [Sq^{2^{n-1}-j-2^{n-2}} c_{n-1,n-2}] \cdot c_{n-1,i} \\ &= [Sq^{2^{n-2}-j} c_{n-1,n-2}] \cdot c_{n-1,i}. \end{aligned}$$

Thus, it suffices to show that

$$Sq^{2^{n-2}-j} c_{n-1,n-2} = \begin{cases} c_{n-1,n-k} & \text{if } j = 2^{n-k},\ k = 2, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

If $2^{n-2} < j < 2^{n-1}$, the result holds since $2^{n-2} - j < 0$. If $j = 2^{n-2}$, the result is clear. If $j < 2^{n-2}$, the process can be repeated several times to reduce the problem to showing

$$Sq^{2^{n-k}-j} c_{n-1,n-k} = \begin{cases} c_{n-1,n-k} & \text{if } j = 2^{n-k} \\ 0 & \text{if } 2^{n-k} < j < 2^{n-k+1}. \end{cases}$$

This result is clear and lemma 4 follows.

PROOF OF THE MAIN PROPOSITION. We will use lemmas 3 and 4 and the fact that e_n commutes with the action of the Steenrod algebra. The proof proceeds by induction on n.

The case $n = 1$ is clear as $c_{1,0} = x_1$ and $x_1 Sq^i(x_1^{-1}) = Sq^{i+1}(x_1^{-1})$.

Assume the result holds for the $n - 1$ case. Now

(1) $$Sq^{i_1}\dots Sq^{i_n}(x_1^{-1}\dots x_n^{-1})$$
$$= [Sq^{i_1}(x_1^{-1}Sq^{i_2}\dots Sq^{i_n}(x_2^{-1}\dots x_n^{-1}))]e_n$$

by lemma 3

$$= \left[\sum_{k=0}^{i_2+\dots+i_n-(n-1)} Sq^{i_1-k}(x_1^{-1})Sq^kSq^{i_2}\dots Sq^{i_n}(x_2^{-1}\dots x_n^{-1})\right]e_n.$$

The admissibility of I guarantees that $i_1 - k > 0$ for all k and dimensional considerations preclude using larger values of k. Similarly, for $i > 0$, the induction hypothesis gives

(2) $$Sq^{i_1+2^{n-1}-2^{i-1}}Sq^{A+I'}(x_1^{-1}\dots x_n^{-1}) + Sq^{i_1+2^{n-1}}Sq^{B+I'}(x_1^{-1}\dots x_n^{-1})$$
$$= \left[\sum_{k=0}^{i_2+\dots+i_n-(n-1)} [Sq^{i_1+N-k}(x_1^{-1}c_{n-1,i-1})Sq^kSq^{I'}(x_1^{-1}\dots x_n^{-1})\right.$$
$$\left. + Sq^{i_1+2^{n-1}-k}(x_1^{-1}c_{n-1,i})Sq^kSq^{I'}(x_1^{-1}\dots x_n^{-1})]\right]e_n$$

Now

$$Sq^{i_1+N-k}(x_1^{-1}c_{n-1,i-1}) + Sq^{i_1+2^{n-1}-k}(x_1^{-1}c_{n-1,i})$$
$$= \sum_{j=0}^{N} Sq^{i_1+N-k-j}(x_1^{-1})Sq^jc_{n-1,i-1} + \sum_{j=0}^{2^{n-1}-2^i} Sq^{i_1+2^{n-1}-k-j}(x_1^{-1})Sq^jc_{n-1,i}$$
$$= x_1^{i_1-k-1}\left[\sum_{j=0}^{N} x_1^{N-j}Sq^jc_{n-1,i-1} + \sum_{j=0}^{2^{n-1}-2^i} x_1^{2^{n-1}-j}Sq^jc_{n-1,i}\right]$$
$$= x_1^{i_1-k-1}\cdot c_{n,i} \text{ by lemma 4.}$$

Thus (2) becomes

$$\left[\sum_{k=0}^{i_2+\dots+i_n-(n-1)} x_1^{i_1-k-1}\cdot c_{n,i}Sq^kSq^{i_2}\dots Sq^{i_n}(x_1^{-1}\dots x_n^{-1})\right]e_n$$
$$= c_{n,i}\left[\sum_{k=0}^{i_2+\dots+i_n-(n-1)} x_1^{i_1-k-1}Sq^kSq^{i_2}\dots Sq^{i_n}(x_1^{-1}\dots x_n^{-1})\right]e_n$$

since $c_{n,i}$ is GL_n invariant,

$$= c_{n,i}\cdot Sq^{i_1}\dots Sq^{i_n}(x_1^{-1}\dots x_n^{-1}) \text{ by (1).}$$

The case $i = 0$ is completely analagous and the proposition follows.

Finally, it should be noted that invariant theory also plays an important role in the study of both the Lambda algebra and the Dyer-Lashof algebra. The reader is referred to work of W. Singer [6] and to work of H. E. A. Campbell [1] for recent results involving these connections.

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