Canad. J. Math. Vol. **59** (6), 2007 pp. 1223–1244

Calderón–Zygmund Operators Associated to Ultraspherical Expansions

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Abstract. We define the higher order Riesz transforms and the Littlewood–Paley g-function associated to the differential operator $L_\lambda f(\theta) = -f''(\theta) - 2\lambda \cot\theta f'(\theta) + \lambda^2 f(\theta)$. We prove that these operators are Calderón–Zygmund operators in the homogeneous type space $((0,\pi), (\sin t)^{2\lambda}\,dt)$. Consequently, L^p weighted, $H^1 - L^1$ and L^∞ − BMO inequalities are obtained.

1 Introduction

B. Muckenhoupt and E. Stein [5] defined and studied the versions of some objects to classical Fourier analysis (conjugate functions, maximal functions, g-functions and multipliers) for the system of the ultraspherical polynomials. It seems to us that their approach follows the lines of the classical Fourier analysis in the torus. In particular, the relationships among Fourier series, analytic functions, and harmonic functions play an essential role. For instance, their definition of the conjugate function was via a boundary value limit of certain *conjugate harmonic function* which satisfies the appropriate Cauchy–Riemann equations. The technique involved the definition of the *harmonic extension*, including a careful analysis of its kernel. Then they built a conjugate harmonic function and proved the existence of a boundary value function, the conjugate function. They got L^p boundedness for p in the range $1 < p < \infty$ and some substitutive inequality in the case $p = 1$. This method was followed later by different authors when defining classical operators for orthogonal expansions. In [5] they did not study the kernel of the conjugate function.

Five years later, Stein's [7] celebrated monograph appeared, where maximal functions, g-functions, Riesz transforms, and multipliers were also defined. As far as we understand, he systematically used a point of view based on an analysis of a general Laplacian. He studied the heat and Poisson semigroups associated with that Laplacian and, from them, he derived the rest of the operators by using some spectral formulas.

Arising naturally from [5] is the study of some other classical operators in the context of the system of the ultraspherical polynomials. In particular, our aim is to study higher order Riesz transforms. In order to define those operators, it seems that the natural procedure to follow is that suggested by Stein [7], and we do so in this

Received by the editors March 18, 2005.

The authors were partially supported by RTN Harmonic Analysis and Related Problems contract HPRN-CT-2001-00273-HARP. The first author was supported in part by KBN grant 5P03A02821. The second and third authors were partially supported by BFM grant 2002-04013-C02-02

AMS subject classification: Primary: 42C05; secondary: 42C15.

Keywords: ultraspherical polynomials, Calderón–Zygmund operators.

paper. Later, we realized that the method used to prove the boundedness of higher order Riesz transforms could be applied to a more general class of operators. For instance, we present in this paper its application to the study of boundedness of the Littlewood–Paley g-function, although more operators fit this technique, *e.g.,* multipliers of Laplace transform type. More concretely, we find that all these operators are naturally Calderón–Zygmund operators in a space of homogeneous type. Therefore, we get as a byproduct of the general theory L^p, H^1, BMO boundedness and weighted inequalities for them (see Theorem 1.1).

Using a different method, the (first order) Riesz transform was studied in [1]. It is defined following [7], and, among other results, the L^p-boundedness for $p \in (1, \infty)$ and the weak type $(1, 1)$ were obtained.

We consider the ultraspherical polynomials $P_n^\lambda(x)$, $\lambda > 0$, defined as the coefficients in the expansion of the generating function $(1-2x\omega+\omega^2)^{-\lambda} = \sum_{n=0}^\infty \omega^n P_n^\lambda(x)$ (see [8] for further details). It is known that the set $\{P_n^\lambda(\cos\theta) : n \in \mathbb{N}\}$ is orthogonal and complete in $L^2[0, \pi]$ with respect to the measure $dm_\lambda(\theta) = (\sin\theta)^{2\lambda} d\theta$. The functions $P_n^\lambda(\cos\theta)$ are eigenfunctions of the operator L_λ,

$$L_\lambda f(\theta) = -f''(\theta) - 2\lambda \cot\theta f'(\theta) + \lambda^2 f(\theta), \tag{1.1}$$

with eigenvalues $\mu_n = (n+\lambda)^2$. Every function f in $L^2(dm_\lambda)$ has an ultraspherical expansion $f(\theta) = \sum_{n=0}^\infty a_n P_n^\lambda(\cos\theta) \|P_n^\lambda\|_{L^2(dm_\lambda)}^{-1}$. Following [7], we define its Poisson integral as

$$Pf(e^{-t}, \theta) = e^{-t\sqrt{L_\lambda}} f(\theta) = \sum_{n=0}^\infty a_n e^{-t(n+\lambda)} P_n^\lambda(\cos\theta) \|P_n^\lambda\|_{L^2(dm_\lambda)}^{-1}. \tag{1.2}$$

The calculus formula $s^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty e^{-ts} t^{a-1}\, dt$ can be used to define

$$L_\lambda^{-l/2} f(\theta) = \frac{1}{\Gamma(l)} \int_0^\infty e^{-t\sqrt{L_\lambda}} f(\theta) t^{l-1}\, dt, \quad l \geq 1,\ f \in L^2(dm_\lambda). \tag{1.3}$$

On the other hand, it is easy to check that L_λ is formally self-adjoint on the space $L^2(dm_\lambda)$ and that it factorizes as $L_\lambda f(\theta) = (-\partial_\theta^* \partial_\theta + \lambda^2) f(\theta)$, where $\partial_\theta^* = \partial_\theta + 2\lambda\cot\theta$ is formally adjoint to ∂_θ, *i.e.,* $\langle \partial_\theta^* f, g\rangle_{L^2(m_\lambda)} = -\langle f, \partial_\theta g\rangle_{L^2(m_\lambda)}$. Following [7], the Riesz transform ($l = 1$) and the higher order Riesz transforms ($l > 1$) are defined as $R_\lambda^l f(\theta) = \partial_\theta^l (L_\lambda)^{-l/2} f(\theta)$, $l \geq 1$. Also, the Littlewood–Paley g-function is defined as

$$\mathcal{G} f(\theta) = \Big(\int_0^1 r \log\frac{1}{r} \big(|\partial_r Pf(r,\theta)|^2 + |\partial_\theta Pf(r,\theta)|^2 \big)\, dr \Big)^{1/2}.$$

For these operators, we have the following result.

Theorem 1.1 *The operators $R_\lambda^l f$, for any $l \geq 1$, and $\mathcal{G} f$ are bounded in $L^p(w\, dm_\lambda)$, $1 < p < \infty$ for any weight w in the Muckenhoupt class A_p with respect to the measure dm_λ and of weak type $(1, 1)$ with respect to the measure $w\, dm_\lambda$ for $w \in A_1$. Also, they map L^∞ into* BMO(dm_λ) *and $H^1(dm_\lambda)$ boundedly into $L^1(dm_\lambda)$.*

In fact, this theorem is a consequence of the general theory for Calderón–Zygmund operators in spaces of homogeneous type and the following theorems.

Theorem 1.2 *For any $l \geq 1$, the operators R_λ^l are Calderón–Zygmund operators in the homogeneous type space $\big([0,\pi), dm_\lambda\big)$.*

For the g-function, we study the operators separately

$$\mathcal{G}^1 f(\theta) = \Big(\int_0^1 r\log\frac{1}{r}\big|\partial_r P_r f(\theta)\big|^2\,dr\Big)^{1/2}, \tag{1.4}$$

$$\mathcal{G}^2 f(\theta) = \Big(\int_0^1 r\log\frac{1}{r}\big|\partial_\theta P_r f(\theta)\big|^2\,dr\Big)^{1/2}. \tag{1.5}$$

Theorem 1.3 *There exist vector valued Calderón–Zygmund operators in the homogeneous type space $\big([0,\pi), dm_\lambda\big)$ mapping scalar-valued functions into functions with values in $L^2((0,1),dr)$, $T_{\mathcal{G}^i}$, $i = 1,2$, such that $\mathcal{G}^i f(\theta) = \|T_{\mathcal{G}^i} f(\theta)\|_{L^2((0,1),dr)}$.*

Obviously, the following lemma is needed. The proof of this lemma is a series of easy but tedious calculations that we leave to the reader.

Lemma 1.4 *The measure $dm_\lambda(\theta) = (\sin\theta)^{2\lambda}\,d\theta$ is doubling in $[0,\pi]$.*

The structure of the paper is as follows. In Section 2, we state the technical Theorem 2.2 as an intermediate step in the proofs of Theorems 1.2 and 1.3. In Section 3 we verify the hypothesis of Theorem 2.2 for the higher order Riesz transforms, and in Section 4 we do the corresponding checking for the g-function.

Let us mention just a word about notation. Throughout the paper, the letter C will denote a constant whose value may vary from line to line, and let us call any finite linear combination of ultraspherical polynomials a *polynomial function*, that is, any f of the form

$$f = \sum_{n=0}^{N} \frac{a_n P_n^\lambda}{\|P_n^\lambda\|_{L^2(dm_\lambda)}}, \quad a_n = \int_0^\pi f(\theta)\frac{P_n^\lambda(\cos\theta)}{\|P_n^\lambda\|_{L^2(dm_\lambda)}}\,dm_\lambda(\theta). \tag{1.6}$$

2 Preliminaries and Technical Tools

Following [3, 4], a *space of homogeneus type* (X,ρ,μ) is a set X together with a quasimetric ρ and a positive measure μ on X such that for every $\theta \in X$ and $r > 0$, $\mu(B(\theta,r)) < \infty$, and such that there exists $0 < C < \infty$ such that for every $\theta \in X$ and $r > 0$ $\mu(B(\theta,2r)) \leq C\mu(B(\theta,r))$. In our case, $X = [0,\pi]$ with the metric given by the absolute value, and the measure is dm_λ.

We say that a kernel $K\colon X \times X\backslash\{x = y\} \to \mathbb{C}$ is a *standard kernel* if there exist $\varepsilon > 0$ and $C < \infty$ such that for all $x \neq y \in X$ and z with $\rho(x,z) \leq \varepsilon\rho(x,y)$,

$$|K(x,y)| \leq \frac{C}{\mu(B(x,r))}, \quad \text{where } r = \rho(x,y), \tag{2.1}$$

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \leq \frac{\rho(x,z)}{\rho(x,y)}\frac{C}{\mu(B(x,r))}. \tag{2.2}$$

Thus, a *Calderón–Zygmund operator* (with associated kernel K) is a linear operator T bounded in L^2 such that, for every $f \in L^2$ and x outside the support of f,

$$Tf(x) = \int_X K(x,y) f(y)\, d\mu(y).$$

It is known that any Calderón–Zygmund operator as above is bounded in $L^p(w\, d\mu)$, for $1 < p < \infty$ and any weight w in the Muckenhoupt class A_p with respect to the measure $d\mu$. They also map $L^1(w\, d\mu)$ into $L^{1,\infty}(w\, d\mu)$ for any weight w in the Muckenhoupt class A_1 with respect to the measure $d\mu$. They also map $L^\infty(\mu)$ boundedly into $\mathrm{BMO}(d\mu)$ and $H^1(d\mu)$ into $L^1(d\mu)$ (see [2]).

For $\mathbb{B}$ a Banach space, vector-valued Calderón–Zygmund operators T from $L^2(d\mu)$ into $L^2_{\mathbb{B}}(d\mu)$ are defined in the same way as scalar valued ones, but considering $K\colon X \times X \backslash \{x = y\} \to \mathbb{B}$ instead of a scalar valued kernel, and taking $\mathbb{B}$-norms in (2.1) and (2.2) instead of absolute values. The boundedness results mentioned above also hold in the vector valued case (see [6]).

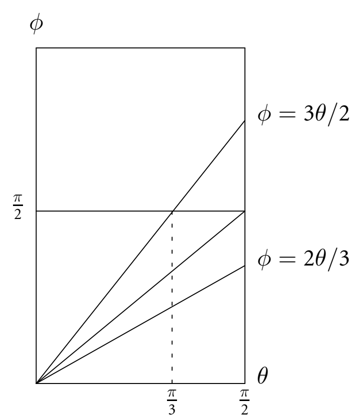

Figure 1

The symmetry with respect to $\pi/2$ in the kernels of our operators will play an important role in the proofs. Also, it will be useful to have in mind the picture of the area where we are placing the variables θ and ϕ (Figure 1). Our first step is studying in detail the behavior of the measure of $B(\theta, |\theta - \phi|)$.

Lemma 2.1 *There exists a constant $C > 0$ such that for any $\theta \in [0, \pi/2]$,*

$$m_\lambda(B(\theta, |\theta-\phi|)) \le C \begin{cases} |\theta-\phi|(\sin(\theta \vee \phi))^{2\lambda} & \text{if } \phi \in [0, \pi/2], \\ \phi & \text{if } \phi \in [\pi/2, \pi],\ \phi > \frac{3}{2}\theta, \\ |\theta-\phi|(\sin(\theta \vee \phi))^{2\lambda} & \text{if } \phi \in [\pi/2, \pi],\ \phi < \frac{3}{2}\theta. \end{cases}$$

Proof Assume that $\phi \in [0, \pi/2]$. There are three possible cases.

Case 1: $B(\theta, |\theta-\phi|) \subset (0, \pi/2)$. In this case, $\theta + |\theta-\phi| = \phi$ for $\phi > \theta$, and $\theta + |\theta-\phi| = 2\theta - \phi \le 2\theta$ for $\phi < \theta$, thus

$$\begin{aligned} m_\lambda(B(\theta, |\theta-\phi|)) &= \int_{\theta-|\theta-\phi|}^{\theta+|\theta-\phi|} (\sin t)^{2\lambda}\, dt \le (\sin(\theta + |\theta-\phi|))^{2\lambda} 2|\theta-\phi| \\ &\le C(\sin(\theta \vee \phi))^{2\lambda} |\theta-\phi|. \end{aligned}$$

Case 2: $B(\theta, |\theta-\phi|) = (0, \phi)$. In this case, $\theta - |\theta-\phi| = 2\theta - \phi \le 0$, thus $\theta \le \phi/2$, and $|\theta-\phi| = \phi - \theta \ge \frac{1}{2}\phi$. Therefore

$$m_\lambda(B(\theta, |\theta-\phi|)) = \int_0^\phi (\sin t)^{2\lambda}\, dt \le (\sin\phi)^{2\lambda}\phi \le C(\sin(\theta \vee \phi))^{2\lambda}|\theta-\phi|.$$

Case 3: $B(\theta, |\theta-\phi|) = (\phi, 2\theta-\phi)$ with $2\theta - \phi > \pi/2$. Clearly $\theta = \theta \vee \phi$ and $\theta \ge \frac{\pi}{4}$, thus $\sin\theta \ge \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and therefore

$$\begin{aligned} m_\lambda(B(\theta, |\theta-\phi|)) &= \int_\phi^{2\theta-\phi} (\sin t)^{2\lambda}\, dt \le 2\theta - \phi - \phi \\ &= 2|\theta-\phi| \le C|\theta-\phi|(\sin(\theta \vee \phi))^{2\lambda}. \end{aligned}$$

In the case $\phi \in [\pi/2, \pi]$ and $\phi < 3\theta/2$, we have $\theta - |\theta-\phi| = 2\theta - \phi \ge 0$ and $\pi/2 \le \phi \le 3\pi/4$. Thus,

$$m_\lambda(B(\theta, |\theta-\phi|)) = \int_{2\theta-\phi}^{\phi} (\sin t)^{2\lambda}\, dt \le \phi - (2\theta-\phi) \le C|\theta-\phi|\big(\sin(\theta \vee \phi)\big)^{2\lambda}.$$

Finally, observe that for $\phi \in [\pi/2, \pi]$ and $\phi > 3\theta/2$, we have $B(\theta, |\theta-\phi|) = (2\theta-\phi, \phi)$ and $m_\lambda(B(\theta, |\theta-\phi|)) \le \int_0^\phi (\sin t)^{2\lambda}\, dt \le \phi$. ∎

Theorem 2.2 *Let $\mathbb{B}$ be a Banach space, and $T\colon L^2(dm_\lambda) \to L^2_{\mathbb{B}}(dm_\lambda)$ be an operator given by integration against a kernel K in the Calderón–Zygmund sense, such that the following hold:*

(i) *If $|\theta-\phi| \ge \pi/6$, then $\|K(\theta,\phi)\|_{\mathbb{B}} \le C$ and $\|\partial_\theta K(\theta,\phi)\|_{\mathbb{B}} + \|\partial_\phi K(\theta,\phi)\|_{\mathbb{B}} \le C$.*

(ii) *For every (θ,ϕ) either belonging to*

$$[0,\pi/2] \times [0,\pi/2] \quad \textit{or to} \quad [0,\pi/2] \times [\pi/2,\pi] \cap \Big\{\phi \le \frac{3}{2}\theta\Big\},$$

we have

$$\|K(\theta,\phi)\|_{\mathbb{B}} \le \frac{C}{|\theta-\phi|}\frac{1}{(\sin(\theta\vee\phi))^{2\lambda}}, \tag{2.3}$$

$$\|\partial_\theta K(\theta,\phi)\|_{\mathbb{B}} + \|\partial_\phi K(\theta,\phi)\|_{\mathbb{B}} \le \frac{C}{|\theta-\phi|^2}\frac{1}{(\sin(\theta\vee\phi))^{2\lambda}}. \tag{2.4}$$

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(iii) *The kernel K is symmetric ($K(\pi-\theta,\pi-\phi) = K(\theta,\phi)$) or antisymmetric ($K(\pi-\theta,\pi-\phi) = -K(\theta,\phi)$).*

Then T is a Calderón–Zygmund operator in $(0,\pi)$ with $\rho(x,y) = |x-y|$.

Proof Lemma 2.1 and (2.3) clearly imply (2.1) in the region stated in (ii). On the other hand, for $(\theta,\phi) \in [0,\pi/2]\times[\pi/2,\pi]\cap\{\phi \geq \frac{3}{2}\theta\}$, we have $\pi/6 \leq |\theta-\phi| \leq \pi$, and therefore by (i), $\|K(\theta,\phi)\|_{\mathbb{B}} \leq C \leq \frac{C\pi}{\phi}$. By using again Lemma 2.1, (2.3) and the symmetry condition (iii), we easily obtain (2.1).

By using Lemma 2.1 and (2.4), we get

$$\|\partial_\theta K(\theta,\phi)\|_{\mathbb{B}} + \|\partial_\phi K(\theta,\phi)\|_{\mathbb{B}} \leq \frac{C}{|\theta-\phi|}\frac{1}{m_\lambda(B(\theta,|\theta-\phi|))} \tag{2.5}$$

for θ and ϕ in the region stated in (ii). For $(\theta,\phi)\in[0,\pi/2]\times[\pi/2,\pi]\cap\{\phi\geq\frac{3}{2}\theta\}$, we have $\pi/6 \leq |\theta-\phi| \leq \pi$ and therefore by (i), $\|\partial_\theta K(\theta,\phi)\|_{\mathbb{B}} + \|\partial_\phi K(\theta,\phi)\|_{\mathbb{B}} \leq C \leq \frac{C}{|\theta-\phi|\phi}$. By Lemma 2.1, we have (2.5) in $(\theta,\phi)\in[0,\pi/2]\times[\pi/2,\pi]\cap\{\phi\geq\frac{3}{2}\theta\}$. The symmetry condition (iii) implies that (2.5) holds for any $(\theta,\phi)\in[0,\pi]\times[0,\pi]$. By standard calculations, this inequality implies condition (2.2) for $\varepsilon = 1/2$ ■

3 Proof of Theorem 1.2

Given a polynomial function f, we have

$$\begin{aligned} R_\lambda^l f(\theta) = \partial_\theta^l (L_\lambda)^{-l/2} f(\theta) &= \partial_\theta^l\Big(\sum_{n=0}^N \frac{a_n}{(n+\lambda)^k}\frac{P_n^\lambda(\cos\theta)}{\|P_n^\lambda\|_{L^2(dm_\lambda)}}\Big) \\ &= \sum_{n=0}^N \frac{a_n}{(n+\lambda)^l}\frac{\partial_\theta^l P_n^\lambda(\cos\theta)}{\|P_n^\lambda\|_{L^2(dm_\lambda)}}. \end{aligned} \tag{3.1}$$

The case $l = 1$ was extensively studied in [1], where the following were proved: its boundedness in L^p for $p \in (1,\infty)$, its weak type $(1,1)$, the fact that it is a principal value, the boundedness of the maximal operator, etc. In order to prove Theorem 1.2, we first prove the boundedness in L^2 of the higher order Riesz transforms.

3.1 L^2 Boundedness of Higher Order Riesz Transforms

We shall see that for any polynomial function (1.6), we have

$$\|R_\lambda^l f\|_{L^2(dm_\lambda)} \leq C\,\|f\|_{L^2(dm_\lambda)},$$

with a constant independent of N. First of all, let us observe that without loss of generality, we can consider only polynomial functions f such that there only appear $P_n^\lambda(\cos\theta)$ for $n \geq l+1$ in their expansion. Since any linear operator is bounded on finite dimensional spaces, we have that the restriction of R_λ^l to the span of $\{P_0^\lambda,\dots,P_l^\lambda\}$ is bounded, and it only remains to check the case when $n \geq l+1$.

The case $l = 1$ was treated in [1], and it was seen to be bounded in $L^2(dm_\lambda)$. Let us explore the case $l = 2$. In this case, for any $n \geq 3$

$$R_\lambda^2 P_n^\lambda(\cos\theta) = \frac{1}{(n+\lambda)^2}\partial_\theta^2 P_n^\lambda(\cos\theta) = \frac{-2\lambda}{(n+\lambda)^2}\cos\theta P_{n-1}^{\lambda+1}(\cos\theta) + \frac{4\lambda(\lambda+1)}{(n+\lambda)^2}(\sin\theta)^2 P_{n-2}^{\lambda+2}(\cos\theta). \tag{3.2}$$

On the other hand, from (1.1) we obtain $\partial_\theta^2 = -L_\lambda - 2\lambda\cot\theta\,\partial_\theta + \lambda^2$ and therefore

$$R_\lambda^2 = \partial_\theta^2 L_\lambda^{-1} = -\mathrm{Id} - 2\lambda\cot\theta\,\partial_\theta L_\lambda^{-1} + \lambda^2 L_\lambda^{-1}. \tag{3.3}$$

With this expression,

$$R_\lambda^2 P_n^\lambda = -P_n^\lambda + \frac{4\lambda^2}{(n+\lambda)^2}\cos\theta P_{n-1}^{\lambda+1} + \frac{\lambda^2}{(n+\lambda)^2}P_n^\lambda. \tag{3.4}$$

Mixing (3.2) and (3.4) one has

$$\frac{2\lambda(2\lambda+1)}{(n+\lambda)^2}\cos\theta P_{n-1}^{\lambda+1}(\cos\theta) = P_n^\lambda(\cos\theta) - \frac{\lambda^2}{(n+\lambda)^2}P_n^\lambda(\cos\theta) + \frac{4\lambda(\lambda+1)}{(n+\lambda)^2}(\sin\theta)^2 P_{n-2}^{\lambda+2}(\cos\theta). \tag{3.5}$$

In particular, with this calculation, from (3.4) we get

$$R_\lambda^2 P_n^\lambda(\cos\theta) = \frac{-1}{2\lambda+1}P_n^\lambda(\cos\theta) + \frac{1}{2\lambda+1}\frac{1}{(n+\lambda)^2}P_n^\lambda(\cos\theta) + \frac{8\lambda^2(\lambda+1)}{2\lambda+1}\frac{1}{(n+\lambda)^2}(\sin\theta)^2 P_{n-2}^{\lambda+2}(\cos\theta),$$

and therefore

$$\begin{aligned} R_\lambda^2 f(\theta) = \frac{-1}{2\lambda+1}\sum_{n=3}^{N}\frac{a_n}{\|P_n^\lambda\|_{L^2(dm_\lambda)}}P_n^\lambda(\cos\theta) &+ \frac{1}{2\lambda+1}\sum_{n=3}^{N}\frac{a_n}{\|P_n^\lambda\|_{L^2(dm_\lambda)}}\frac{1}{(n+\lambda)^2}P_n^\lambda(\cos\theta) \\ &+ \frac{8\lambda^2(\lambda+1)}{2\lambda+1}\sum_{n=3}^{N}\frac{a_n}{\|P_n^\lambda\|_{L^2(dm_\lambda)}}\frac{1}{(n+\lambda)^2}(\sin\theta)^2 P_{n-2}^{\lambda+2}(\cos\theta). \end{aligned}$$

Clearly, the first two sums are operators bounded in $L^2(dm_\lambda)$. For the third one, we will use the following lemma.

Lemma 3.1 *For every $k \geq 0$, the operator T acting on polynomial functions*

$$Tf(\theta) = \sum_{n=k+1}^{N} \frac{a_n}{\|P_n^\lambda\|_{L^2(dm_\lambda)}} \frac{1}{(n+\lambda)^l} (\sin\theta)^k P_{n-k}^{\lambda+k}(\cos\theta)$$

is bounded in $L^2(dm_\lambda)$.

Proof By using the orthogonality of $(\sin\theta)^k P_{n-k}^{\lambda+k}$ in $L^2(dm_\lambda)$ and

$$\|P_n^\lambda\|^2_{L^2(dm_\lambda)} = 2^{1-2\lambda}\pi\Gamma(\lambda)^{-2}\frac{\Gamma(n+2\lambda)}{(n+\lambda)n!}$$

(see [8]), we obtain

$$\begin{aligned}\Big| \sum_{n=k+1}^{N} \frac{a_n}{\|P_n^\lambda\|_{L^2(dm_\lambda)}} \frac{1}{(n+\lambda)^k} (\sin\theta)^k P_{n-k}^{\lambda+k}\Big|^2_{L^2(dm_\lambda)} \\ = \sum_{n=k+1}^{N} \frac{a_n^2}{\|P_n^\lambda\|^2_{L^2(dm_\lambda)}} \frac{\|P_{n-k}^{\lambda+k}\|^2_{L^2(dm_{\lambda+k})}}{(n+\lambda)^{2k}} \leq C \sum_{n=k+1}^{N} a_n^2 = C\,\|f\|^2_{L^2(dm_\lambda)}.\end{aligned}$$

■

We use induction in l to prove the boundedness of R_λ^l for any $l \geq 1$. Let us assume that R_λ^k is bounded in $L^2(dm_\lambda)$ for $k \leq l-1$, and let us see that it also holds for $k = l$. From (3.3), for $l \geq 3$, we have

$$\begin{aligned}R_\lambda^l &= \partial_\theta^l L_\lambda^{-l/2} = \partial_\theta^{l-2} R_\lambda^2 L_\lambda^{-(l-2)/2} \\ &= -\partial_\theta^{l-2} L_\lambda^{-(l-2)/2} - 2\lambda\partial_\theta^{l-2}\cot\theta\,\partial_\theta L_\lambda^{-l/2} + \lambda^2\partial_\theta^{l-2} L_\lambda^{-l/2} \\ &= -R_\lambda^{l-2} - 2\lambda\partial_\theta^{l-2}\cot\theta\,\partial_\theta L_\lambda^{-l/2} + \lambda^2 R_\lambda^{l-2} L_\lambda^{-1}.\end{aligned} \tag{3.6}$$

By the induction hypothesis, first and last operators are bounded in $L^2(dm_\lambda)$. It remains to show that the second term is also bounded.

Lemma 3.2 *For any $k \geq 0$,*

$$\partial_\theta^k\Big(\cos\theta P_{n-1}^{\lambda+1}(\cos\theta)\Big) = \sum_{a=1}^{k+1} \sum_{\substack{A,\,B:\\ A+B=a}} C_{AB}(\cos\theta)^A(\sin\theta)^B P_{n-a}^{\lambda+a}(\cos\theta),$$

where $C_{AB} \in \mathbb{R}$ may be zero.

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Proof We will prove the result by induction in k. For $k = 0$, it is clearly true with $C_{10} = 1$ and $C_{01} = 0$. Let us suppose that the formula holds until $k - 1$. For k, we have

$$\partial_\theta^k(\cos\theta P_{n-1}^{\lambda+1}(\cos\theta)) = \partial_\theta \sum_{a=1}^{k} \sum_{\substack{A,B:\\ A+B=a}} C_{AB}(\cos\theta)^A(\sin\theta)^B P_{n-a}^{\lambda+a}(\cos\theta),$$

and it is enough to see that any term $\partial_\theta((\cos\theta)^A(\sin\theta)^B P_{n-a}^{\lambda+a}(\cos\theta))$ is a sum of terms of the form $(\cos\theta)^{\tilde{A}}(\sin\theta)^{\tilde{B}} P_{n-\tilde{a}}^{\lambda+\tilde{a}}(\cos\theta)$ with $1 \le \tilde{a} \le k+1$, $\tilde{A} + \tilde{B} = \tilde{a}$. And this holds, since

$$\begin{aligned}\partial_\theta\big((\cos\theta)^A(\sin\theta)^B P_{n-a}^{\lambda+a}(\cos\theta)\big) &= A(\cos\theta)^{A-1}(\sin\theta)^{B+1} P_{n-a}^{\lambda+a}(\cos\theta)\\ &\quad + B(\cos\theta)^{A+1}(\sin\theta)^{B-1} P_{n-a}^{\lambda+a}(\cos\theta)\\ &\quad - 2(\lambda+a)(\cos\theta)^A(\sin\theta)^{B+1} P_{n-(a+1)}^{\lambda+(a+1)}(\cos\theta).\end{aligned}$$

∎

Thus, for a polynomial function $f = \sum_{n=l+1}^{N} a_n \|P_n^\lambda\|_{L^2(dm_\lambda)}^{-1} P_n^\lambda$, we can write

$$\begin{aligned}(3.7)\quad &\partial_\theta^{l-2}\cot\theta\,\partial_\theta L_\lambda^{-l/2} f(\theta)\\ &= \sum_{n=l+1}^{N} \frac{a_n}{\|P_n^\lambda\|_{L^2(dm_\lambda)}} \frac{1}{(n+\lambda)^l}\partial_\theta^{l-2}\big(-2\lambda\cos\theta P_{n-1}^{\lambda+1}(\cos\theta)\big)\\ &= \sum_{a=1}^{l-1}\sum_{\substack{A,B:\\ A+B=a}} C_{AB}\sum_{n=l+1}^{N} \frac{a_n}{\|P_n^\lambda\|_{L^2(dm_\lambda)}} \frac{1}{(n+\lambda)^l}(\cos\theta)^A(\sin\theta)^B P_{n-a}^{\lambda+a}(\cos\theta)\\ &= \sum_{a=1}^{l-1}\sum_{\substack{A,B:\\ A+B=a}} C_{AB} T_{A,B,a}^{l-1} f(\theta).\end{aligned}$$

Thus, the boundedness of this operator follows from the next lemma.

Lemma 3.3 *For any $l \ge 2$, the operators $T_{A,B,a}^{l-1}$ appearing in (3.7) with non zero coefficients are bounded in $L^2(dm_\lambda)$.*

Proof We will proceed by induction in A: in the case $A = 0$, we must prove that for every $a \le l - 1$,

$$T_{0,aa}^{l-1} f(\theta) = \sum_{n=l+1}^{N} \frac{a_n}{\|P_n^\lambda\|_{L^2(dm_\lambda)}} \frac{1}{(n+\lambda)^l}\cos\theta(\sin\theta)^a P_{n-a}^{\lambda+a}(\cos\theta)$$

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is bounded in $L^2(dm_\lambda)$. This holds by Lemma 3.1. Now assume that the induction hypothesis is true up to $A-1$, that is, for any $\tilde{A} \le A-1$, we have that for every $a \le l-1$, the operator $T^{l-1}_{\tilde{A},a-\tilde{A},a} f(\theta)$ is bounded in $L^2(dm_\lambda)$. For $A \ge 1$, by (3.5) with $n-(a-1)$ and $\lambda+(a-1)$ instead of n and λ, one can write

$$\begin{aligned}
&\frac{1}{(n+\lambda)^l}(\cos\theta)^A(\sin\theta)^B P^{\lambda+a}_{n-a}\\
&\quad= \frac{C}{(n+\lambda)^{l-2}}(\cos\theta)^{A-1}(\sin\theta)^B \times\\
&\qquad\times \left(P^{\lambda+(a-1)}_{n-(a-1)} + \frac{C}{(n+\lambda)^2} P^{\lambda+(a-1)}_{n-(a-1)} + \frac{C}{(n+\lambda)^2}(\sin\theta)^2 P^{\lambda+(a+1)}_{n-(a+1)} \right)\\
&\quad= \frac{C}{(n+\lambda)^{l-2}}(\cos\theta)^{A-1}(\sin\theta)^B P^{\lambda+(a-1)}_{n-(a-1)}\\
&\qquad+ \frac{C}{(n+\lambda)^{l}}(\cos\theta)^{A-1}(\sin\theta)^B P^{\lambda+(a-1)}_{n-(a-1)}\\
&\qquad+ \frac{C}{(n+\lambda)^{l}}(\cos\theta)^{A-1}(\sin\theta)^{B+2} P^{\lambda+(a+1)}_{n-(a+1)}.
\end{aligned}$$

By the induction hypothesis, the first two terms give rise to operators bounded in $L^2(dm_\lambda)$. For the third term, if $a+1 \le l-1$, it also gives rise to a bounded operator, by the same reason. But we want to prove the boundedness for any $a \le l-1$, and thus it remains to prove the case $a = l-1$. Since

$$\partial_\theta^{l-2} \cot\theta \partial_\theta L_\lambda^{-l/2} P_n^\lambda = -2\lambda \partial_\theta^{l-2}(\cos\theta P^{\lambda+1}_{n-1}),$$

an operator $T^{l-1}_{A,B,l-1}$ in which there appear terms with $P^{\lambda+(l-1)}_{n-(l-1)}$ in the left-hand side of expression (3.7) necessarily comes from the terms in which the derivatives ∂_θ^{l-2} act on the polynomial $P^{\lambda+1}_{n-1}$ and not in any other term (otherwise we would obtain $P^{\lambda+a}_{n-a}$ with $a < l-1$). Therefore, the only terms with non-zero coefficients coming by applying (3.5) in the right-hand side of (3.7) must sum to a constant times the operator

$$T^{l-1}_{1,l-2,l-1} f(\theta) = \sum_{n=l+1}^{N} \frac{a_n}{\|P_n^\lambda\|_{L^2(dm_\lambda)}} \frac{1}{(n+\lambda)^l} \cos\theta(\sin\theta)^{l-2} P^{\lambda+(l+1)}_{n-(l-1)}(\cos\theta).$$

By using (3.5) with $n-(l-2)$ and $\lambda+(l-2)$ instead of n and λ, we can write

$$\begin{aligned}
&\frac{1}{(n+\lambda)^l}\cos\theta(\sin\theta)^{l-2} P^{\lambda+(l+1)}_{n-(l-1)}\\
&\quad= \frac{C(\sin\theta)^{l-2}}{(n+\lambda)^{l-2}}\left(P^{\lambda+(l-2)}_{n-(l-2)} + \frac{C}{(n+\lambda)^2} P^{\lambda+(l-2)}_{n-(l-2)} + \frac{C}{(n+\lambda)^2}(\sin\theta)^2 P^{\lambda+l}_{n-l} \right)\\
&\quad= \frac{C(\sin\theta)^{l-2}}{(n+\lambda)^{l-2}} P^{\lambda+(l-2)}_{n-(l-2)} + \frac{C(\sin\theta)^{l-2}}{(n+\lambda)^{l}} P^{\lambda+(l-2)}_{n-(l-2)} + \frac{C(\sin\theta)^{l}}{(n+\lambda)^{l}} P^{\lambda+l}_{n-l}.
\end{aligned}$$

By Lemma 3.1, the operators to which these terms give rise are bounded in $L^2(dm_\lambda)$. ■

3.2 Kernel of the Riesz Transform

Observe that by (1.2), $Pf(e^{-t},\theta) = e^{-t\lambda} f(e^{-t},\theta)$, where

$$f(r,\theta) = \sum_{n=0}^{\infty} a_n r^n P_n^{\lambda}(\cos\theta) \|P_n^{\lambda}\|_{L^2(dm_\lambda)}^{-1}$$

is defined by Muckenhoupt and Stein [5]. They compute explicitly the kernel $P(r,\theta,\phi)$ of $f(r,\theta)$, and therefore

$$Pf(r,\theta) = r^{\lambda} \int_0^{\pi} P(r,\theta,\phi) f(\phi)\, dm_\lambda(\phi), \quad \theta \in [0,\pi], \tag{3.8}$$

where $r = e^{-t}$ and

$$P(r,\theta,\phi) = \frac{\lambda}{\pi} \int_0^{\pi} \frac{(1-r^2)\sin^{2\lambda-1} t}{\big(1 - 2r(\cos\theta\cos\phi + \sin\theta\sin\phi\cos t) + r^2\big)^{\lambda+1}} dt.$$

Before continuing further, let us state some useful notation.

$$\begin{gathered} \sigma = \sin\theta\sin\phi, \quad a = \cos\theta\cos\phi + \sigma\cos t = \cos(\theta-\phi) - \sigma(1-\cos t), \\ \Delta_r = 1 - 2r\cos(\theta-\phi) + r^2 = (1-r)^2 + 2r(1-\cos(\theta-\phi)), \\ \Delta = \Delta_1, \quad D_r = 1 - 2ra + r^2 = \Delta_r + 2r\sigma(1-\cos t). \end{gathered} \tag{3.9}$$

Lemma 3.4 ([1, Lemma 2]) *Given $f \in L^1(dm_\lambda)$ and $l \geq 1$, for almost every $\theta \in [0,\pi]$, we have that*

$$(L_\lambda)^{-\frac{l}{2}} f(\theta) = \int_0^{\pi} W_\lambda^l(\theta,\phi) f(\phi)\, dm_\lambda(\phi), \tag{3.10}$$

where $W_\lambda^l(\theta,\phi) = \frac{1}{\Gamma(l)} \int_0^1 r^{\lambda-1} (\log\frac{1}{r})^{l-1} P(r,\theta,\phi)\, dr$. Given $f \in L^1(dm_\lambda)$, $l \geq 1$ and θ outside the support of f, we have that

$$R_\lambda^l f(\theta) = \int_0^{\pi} R_\lambda^l(\theta,\phi) f(\phi)\, dm_\lambda(\phi), \tag{3.11}$$

where

$$R_\lambda^l(\theta,\phi) = \frac{1}{\Gamma(l)} \int_0^1 r^{\lambda-1} \Big(\log\frac{1}{r}\Big)^{l-1} \frac{\lambda}{\pi}(1-r^2) \int_0^{\pi} (\sin t)^{2\lambda-1} \partial_\theta^l \Big(\frac{1}{D_r^{\lambda+1}}\Big)\, dt.$$

Proof $L_\lambda^{-1/2}$ is defined in (1.3). Then to get (3.10), it is enough to apply Fubini's theorem. To prove (3.11), it is enough to justify the differentiation inside the integral sign. See [1] for the details. ■

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Now we shall see that the kernel $R^l_\lambda(\theta,\phi)$ satisfies the hypothesis in Theorem 2.2. Since $P(r,\pi-\theta,\pi-\phi)=P(r,\theta,\phi)$, we have $R^l_\lambda(\pi-\theta,\pi-\phi)=(-1)^l R^l_\lambda(\theta,\phi)$. Therefore $R^l_\lambda(\theta,\phi)$ satisfies condition (iii) in Theorem 2.2. Observe that $\partial_\theta^{l+1}(D_r^{-\lambda-1})$ and $\partial_\phi\partial_\theta^l(D_r^{-\lambda-1})$ are quotients with a bounded function in the numerator and a certain power of D_r in the denominator. Since $D_r \geq C$ for $r\in(0,1/2)$ and $D_r\geq C$ for $r\in(1/2,1)$ and $|\theta-\phi|>\pi/6$, we get

$$|R^l_\lambda(\theta,\phi)|+|\partial_\theta R^l_\lambda(\theta,\phi)|+|\partial_\phi R^l_\lambda(\theta,\phi)| \\ \leq C\int_0^{1/2} r^{\lambda-1}\Big(\log\frac{1}{r}\Big)^{l-1}dr + C\int_{1/2}^1 r^{\lambda-1}\Big(\log\frac{1}{r}\Big)^{l-1}dr \leq C.$$

With this we obtain condition (i) in Theorem 2.2. In order to prove condition (ii), we need a careful analysis of $\partial_\theta^l\big(\frac{1}{D_r^{\lambda+1}}\big)$. Recalling our notation (3.9), let

$$b=\partial_\theta a = -\sin(\theta-\phi)-\cos\theta\sin\phi(1-\cos t),$$

and observe $\partial_\theta b = a$. We have the following lemma to describe $\partial_\theta^l\big(\frac{1}{D_r^{\lambda+1}}\big)$ more precisely.

Lemma 3.5

$$\partial_\theta^l\Big(\frac{1}{D_r^{\lambda+1}}\Big)=\sum c_{l,k,i,j}\frac{r^{i+j}a^ib^j}{D_r^{\lambda+1+k}},$$

where $c_{l,k,i,j}\neq 0$ *only if*

$$k=1,\dots,l,\quad i+j=k,\quad j\geq 2k-l. \tag{3.12}$$

Proof For $l=1$, $\partial_\theta^l\big(\frac{1}{D_r^{\lambda+1}}\big)=\frac{Crb}{D_r^{\lambda+2}}$ and therefore only $c_{1,1,0,1}$ is nonzero. Assume that the lemma is true for l. Since

$$\partial_\theta\Big(\frac{r^{i+j}a^ib^j}{D_r^{\lambda+1+k}}\Big)=c_1\frac{r^{i+j}a^{i-1}b^{j+1}}{D_r^{\lambda+1+k}}+c_2\frac{r^{i+j}a^{i+1}b^{j-1}}{D_r^{\lambda+1+k}}+c_3\frac{r^{i+j+1}a^ib^{j+1}}{D_r^{\lambda+1+(k+1)}} \tag{3.13}$$

(we assume $c_1=0$ if $i=0$ and $c_2=0$ if $j=0$), we must check that all these expressions satisfy (3.12) for $l+1$. The first two are obvious. For the third one:

$$(j+1)\geq 2k-l+1>2k-(l+1),\quad (j-1)\geq 2k-l-1=2k-(l+1),\\ (j+1)\geq 2k-l+1=2(k+1)-(l+1). \qquad\blacksquare$$

As a consequence we may write $R^l_\lambda(\theta,\phi)=\sum c_{l,k,i,j}M_{n,k,i,j}(\theta,\phi)$, where $c_{l,k,i,j}$ are as in the last lemma and

$$M_{l,k,i,j}(\theta,\phi)=\int_0^1\int_0^\pi r^{\lambda+i+j-1}\Big(\log\frac{1}{r}\Big)^{l-1}(1-r^2)\frac{a^ib^j\sin^{2\lambda-1}t}{D_r^{\lambda+1+k}}\,dt\,dr.$$

The next step is checking that each of these terms verifies condition (ii) of Theorem 2.2. This will be done in two lemmas.

Lemma 3.6 *There exists a constant $C > 0$ such that for every $\theta \in [0, \pi/2]$ and $2\theta/3 \le \phi \le 3\theta/2$, we have*

$$|M_{l,k,i,j}(\theta,\phi)| \le \frac{C}{|\theta-\phi|}\frac{1}{(\sin\phi)^{2\lambda}}, \tag{3.14}$$

$$|\partial_\theta M_{l,k,i,j}(\theta,\phi)| + |\partial_\phi M_{l,k,i,j}(\theta,\phi)| \le \frac{C}{|\theta-\phi|^2}\frac{1}{(\sin\phi)^{2\lambda}}. \tag{3.15}$$

Proof Let us start with the first inequality. Since $D_r \ge C$ for $r \in (0, 1/2)$, and $\log\frac{1}{r} \le C(1-r)$ for $r \in (1/2, 1)$, and also by using that $|a| \le C$ and $|b| \le \sin|\theta-\phi| + \sin\phi(1-\cos t)$, we have

$$
\begin{aligned}
(3.16)\quad & |M_{l,k,i,j}(\theta,\phi)| \\
&\le C\int_0^{1/2} r^{\lambda+k-1}\Big(\log\frac{1}{r}\Big)^{l-1} dr \\
&\quad + C\int_{1/2}^1\int_0^\pi (1-r)^l \frac{(\sin|\theta-\phi| + \sin\phi(1-\cos t))^j}{D_r^{\lambda+k+1}}(\sin t)^{2\lambda-1}\,dt\,dr \\
&\le C + C\int_{1/2}^1\int_0^\pi (1-r)^l \frac{(\sin|\theta-\phi|)^j(\sin t)^{2\lambda-1}}{D_r^{\lambda+k+1}}\,dt\,dr \\
&\quad + C\int_{1/2}^1\int_0^{\pi/2} (1-r)^l \frac{(\sin\phi(1-\cos t))^j(\sin t)^{2\lambda-1}}{D_r^{\lambda+k+1}}\,dt\,dr \\
&\quad + C\int_{1/2}^1\int_{\pi/2}^{\pi} (1-r)^l \frac{(\sin\phi)^j(\sin t)^{2\lambda-1}}{D_r^{\lambda+k+1}}\,dt\,dr \\
&= C + (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}).
\end{aligned}
$$

For the first term, we use that $j \ge 2k - l$, and thus $l \ge 2k - j$, also that $\sin t \sim t$, and then we perform successively the changes of variables $x = \sqrt{\frac{\sigma}{\Delta_r}}t$ and $u = \frac{1-r}{\sqrt{\Delta}}$.

$$
\begin{aligned}
(\mathrm{I}) &\le C\int_{1/2}^1 (1-r)^{2k-j}(\sin|\theta-\phi|)^j\int_0^{\pi/2}\frac{t^{2\lambda-1}}{(\Delta_r+\sigma t^2)^{\lambda+k+1}}\,dt\,dr \\
&= C\int_{1/2}^1 (1-r)^{2k-j}(\sin|\theta-\phi|)^j\int_0^{\frac{\pi}{2}\sqrt{\frac{\sigma}{\Delta_r}}}\frac{\Big(\sqrt{\frac{\Delta_r}{\sigma}}x\Big)^{2\lambda-1}\sqrt{\frac{\Delta_r}{\sigma}}\,dx}{\Delta_r^{\lambda+k+1}(1+x^2)^{\lambda+k+1}}\,dr \\
&\le (\sin|\theta-\phi|)^j C\int_{1/2}^1\frac{(1-r)^{2k-j}\,dr}{\sigma^\lambda\Delta_r^{k+1}} \\
&\le C\frac{(\sin|\theta-\phi|)^j}{\sigma^\lambda}\int_0^{\frac{1}{2\sqrt{\Delta}}}\frac{(\sqrt{\Delta}u)^{2k-j}\sqrt{\Delta}\,du}{\Delta^{k+1}(1+u^2)^{k+1}}\,dr \\
&\le C\frac{(\sin|\theta-\phi|)^j}{\sigma^\lambda\Delta^{j/2+1/2}} \le \frac{C}{\sigma^\lambda\Delta^{1/2}} \le \frac{C}{|\theta-\phi|}\frac{1}{(\sin(\theta\vee\phi))^{2\lambda}},
\end{aligned}
$$

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where in the last two inequalities we have used that for $2\theta/3 \leq \phi \leq 3\theta/2$,

$$\begin{aligned} 1-\cos(\theta-\phi) \sim |\theta-\phi|^2 \sim (\sin|\theta-\phi|)^2, \quad \sin\theta \sim \sin\phi, \\ |\sin(\theta-\phi)| \leq C\sin\phi. \end{aligned} \tag{3.17}$$

The proof of these inequalities is trivial, although the argument differs when $\phi \in [0,\pi/2]$ from when $\phi \in [\pi/2,\pi]$. Analogously, we have

$$\begin{aligned} \text{(II)} &\leq C\int_{1/2}^{1}(1-r)^{2k-j}(\sin\phi)^j\int_0^{\pi/2}\frac{t^j t^{2\lambda-1}}{(\Delta_r+\sigma t^2)^{\lambda+k+1}}\,dtdr \\ &= C\int_{1/2}^{1}(1-r)^{2k-j}(\sin\phi)^j\int_0^{\frac{\pi}{2}\sqrt{\frac{\sigma}{\Delta_r}}}\frac{\left(\sqrt{\frac{\Delta_r}{\sigma}}x\right)^{2\lambda+j-1}\sqrt{\frac{\Delta_r}{\sigma}}\,dx}{\Delta_r^{\lambda+k+1}(1+x^2)^{\lambda+k+1}}\,dr \\ &\leq \frac{C}{|\theta-\phi|}\frac{1}{(\sin(\theta\vee\phi))^{2\lambda}}. \end{aligned}$$

For $r \in (1/2,1)$, $D_r \geq C\sigma$ and we obtain

$$\begin{aligned} \text{(III)} &\leq C\int_{1/2}^{1}(1-r)^{2k-j}(\sin\phi)^j\int_{\pi/2}^{\pi}\frac{(\sin t)^{2\lambda-1}}{\sigma^{j/2}(\Delta_r+\sigma(1-\cos t)^2)^{\lambda+k-j/2+1}}\,dtdr \\ &= C\int_{1/2}^{1}(1-r)^{2k-j}(\sin\phi)^j\int_0^{\pi/2}\frac{(\sin t)^{2\lambda-1}}{\sigma^{j/2}(\Delta_r+\sigma(1+\cos t)^2)^{\lambda+k-j/2+1}}\,dtdr \\ &\leq C\int_{1/2}^{1}(1-r)^{2k-j}\frac{(\sin\phi)^j}{\sigma^{j/2}}\int_0^{\frac{\pi}{2}\sqrt{\frac{\sigma}{\Delta_r}}}\frac{\left(\sqrt{\frac{\Delta_r}{\sigma}}x\right)^{2\lambda-1}\sqrt{\frac{\Delta_r}{\sigma}}\,dx}{\Delta_r^{\lambda+k-j/2+1}(1+x^2)^{\lambda+k-j/2+1}}\,dr \\ &\leq \frac{C}{|\theta-\phi|}\frac{1}{(\sin(\theta\vee\phi))^{2\lambda}}, \end{aligned}$$

where the penultimate inequality follows the lines of terms (I) and (II). This ends the proof of inequality (3.14).

By (3.13) and analogous arguments as in (3.16), we have

$$\begin{aligned} |\partial_\theta M_{l,k,i,j}(\theta,\phi)| &\leq C + C\int_{1/2}^{1}\int_0^{\pi}(1-r)^l\frac{|b|^{j+1}}{D_r^{\lambda+k+1}}(\sin t)^{2\lambda-1}\,dtdr \\ &\quad + C\int_{1/2}^{1}\int_0^{\pi}(1-r)^l\frac{|b|^{j-1}}{D_r^{\lambda+k+1}}(\sin t)^{2\lambda-1}\,dtdr \\ &\quad + C\int_{1/2}^{1}\int_0^{\pi}(1-r)^l\frac{|b|^{j+1}}{D_r^{\lambda+k+2}}(\sin t)^{2\lambda-1}\,dtdr \\ &= C + \text{(A)} + \text{(B)} + \text{(C)}. \end{aligned}$$

For the first integral, we have that

$$(\mathrm{A}) \le C \int_{1/2}^{1} \int_0^{\pi} (1-r)^l \frac{|b|^{j+1}}{D_r^{\lambda+k+1}} (\sin t)^{2\lambda-1}\, dt dr \le \frac{C}{|\theta-\phi|} \frac{1}{(\sin(\theta \vee \phi))^{2\lambda}},$$

as was proved by the estimates of (I), (II) and (III) above. Observe that if $j = 0$, the second term would not appear. So we may proceed to estimate (B) assuming $j \ge 1$, and obtaining

$$\begin{aligned}
(\mathrm{B}) &\le C \int_{1/2}^{1} (1-r)^l \int_0^{\pi} \frac{(\sin|\theta-\phi| + \sin\phi(1-\cos t))^{j-1}}{D_r^{\lambda+k+1}} (\sin t)^{2\lambda-1}\, dt dr \\
&\le C \int_{1/2}^{1} \int_0^{\pi/2} (1-r)^l \frac{(\sin|\theta-\phi|)^{j-1} (\sin t)^{2\lambda-1}}{D_r^{\lambda+k+1}}\, dt dr \\
&\quad + C \int_{1/2}^{1} \int_0^{\pi/2} (1-r)^l \frac{(\sin\phi(1-\cos t))^{j-1} (\sin t)^{2\lambda-1}}{D_r^{\lambda+k+1}}\, dt dr \\
&\quad + C \int_{1/2}^{1} \int_{\pi/2}^{\pi} (1-r)^l \frac{(\sin\phi)^{j-1} (\sin t)^{2\lambda-1}}{D_r^{\lambda+k+1}}\, dt dr \\
&\le \frac{C}{\sigma^\lambda \Delta} \le \frac{C}{|\theta-\phi|^2} \frac{1}{(\sin(\theta\vee\phi))^{2\lambda}},
\end{aligned}$$

where the last two inequalities follows as in (I), (II), and (III) above. Observe that

$$(\mathrm{C}) \le C \int_{1/2}^{1} (1-r)^l \int_0^{\pi} \frac{(\sin|\theta-\phi| + \sin\phi(1-\cos t))^{j+1}}{D_r^{\lambda+k+2}} (\sin t)^{2\lambda-1}\, dt dr.$$

The same arguments drive to the bound

$$(\mathrm{C}) \le \frac{C}{|\theta-\phi|^2} \frac{1}{(\sin(\theta\vee\phi))^{2\lambda}}.$$

The proof of (3.15) for ∂_ϕ follows the same lines. ■

Lemma 3.7 *For every $\theta, \phi \in [0, \pi/2]$ and ϕ outside the region $2\theta/3 \le \phi \le 3\theta/2$, we also have* (3.14) *and* (3.15).

Proof In the following calculations we will use that for $\theta, \phi \in [0, \pi/2]$,

$$1 - \cos(\theta-\phi) \sim |\theta-\phi|^2 \sim (\sin|\theta-\phi|)^2.$$

Also, for ϕ outside the region $2\theta/3 \le \phi \le 3\theta/2$, $\sin|\theta-\phi| \sim \sin(\theta\vee\phi)$ and $|b| \le C\sin|\theta-\phi|$. Therefore, by the same techniques applied in (3.16), for $M_{l,k,i,j}(\theta, \phi)$

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we have

$$
\begin{aligned}
|M_{l,k,i,j}(\theta,\phi)| &\le C + C\int_{1/2}^{1}(1-r)^{2k-j}\frac{|b|^j}{\Delta_r^{\lambda+k+1}}\,dr\\
&\le C + C(\sin|\theta-\phi|)^j\int_{1/2}^{1}\frac{(1-r)^{2k-j}}{\Delta_r^{\lambda+k+1}}\,dr\\
&\le C + C(\sin|\theta-\phi|)^j\int_{0}^{\frac{1}{2\sqrt{\Delta}}}\frac{(\sqrt{\Delta}u)^{2k-j}\sqrt{\Delta}\,du}{\Delta^{\lambda+k+1}(1+u^2)^{\lambda+k+1}}\,dr\\
&\le C + C\frac{(\sin|\theta-\phi|)^j}{\Delta^{\lambda+j/2+1/2}} \le \frac{C}{|\theta-\phi|}\frac{1}{(\sin(\theta\vee\phi))^{2\lambda}}.
\end{aligned}
$$

We proceed analogously for the derivative

$$
\begin{aligned}
|\partial_\theta M_{l,k,i,j}(\theta,\phi)| \le C &+ C\int_{1/2}^{1}\int_0^\pi (1-r)^l\frac{|b|^{j+1}}{D_r^{\lambda+k+1}}(\sin t)^{2\lambda-1}\,dt\,dr\\
&+ C\int_{1/2}^{1}\int_0^\pi (1-r)^l\frac{|b|^{j-1}}{D_r^{\lambda+k+1}}(\sin t)^{2\lambda-1}\,dt\,dr\\
&+ C\int_{1/2}^{1}\int_0^\pi (1-r)^l\frac{|b|^{j+1}}{D_r^{\lambda+k+2}}(\sin t)^{2\lambda-1}\,dt\,dr\\
\le \frac{C}{|\theta-\phi|^2}&\frac{1}{(\sin(\theta\vee\phi))^{2\lambda}}.
\end{aligned}
$$

■

4 Proof of Theorem 1.3

Let us write $\mathcal{G}^i f(\theta) = \|T_{\mathcal{G}^i} f(\theta)\|_{L^2((0,1),dr)}$, $i = 1,2$ where $T_{\mathcal{G}^i}$ is the operator mapping escalar valued functions into $L^2(dr)$-valued functions given by

$$
T_{\mathcal{G}^1} f(\theta) = \sqrt{r\log\frac{1}{r}}\,\partial_r Pf(r,\theta), \qquad T_{\mathcal{G}^2} f(\theta) = \sqrt{r\log\frac{1}{r}}\,\partial_\theta Pf(r,\theta). \tag{4.1}
$$

4.1 Boundedness in $L^2(dm_\lambda)$ of the g-Functions

For any polynomial function $f \in L^2(dm_\lambda)$, the operator

$$
\partial_r Pf = \sum_{n=0}^{N}(n+\lambda)r^{n+\lambda-1}a_n P_n^\lambda\|P_n^\lambda\|^{-1}
$$

gives a well-defined function in $L^2(dm_\lambda)$, since for each fixed $r \in (0,1)$ and $n \ge 0$, $|(n+\lambda)r^{n+\lambda}| \le C$. Thus, we can write

$$
T_{\mathcal{G}^1} f = \sum_{n=0}^{\infty}\sqrt{r\log\frac{1}{r}}\,(n+\lambda)r^{n+\lambda-1}a_n P_n^\lambda\|P_n^\lambda\|^{-1} = \sum_{n=0}^{\infty} g^1(n)a_n P_n^\lambda\|P_n^\lambda\|^{-1},
$$

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where $g^1(n) = \sqrt{r \log \frac{1}{r}}\,(n+\lambda)r^{n+\lambda-1}$ belongs to $L^2(dr)$ uniformly in n. In particular, this implies that $\mathcal{G}^1$ is bounded in L^2. To get the boundedness of $\mathcal{G}^2$ in L^2 we proceed similarly. Observe that for any polynomial $f \in L^2(dm_\lambda)$,

$$\partial_\theta P f = \sum_{n=1}^{N} r^{n+\lambda} a_n (-2\lambda) \sin\theta P_{n-1}^{\lambda+1} \|P_n^\lambda\|^{-1}$$

(where we have used that $\partial_x P_n^\lambda(x) = 2\lambda P_{n-1}^{\lambda+1}(x)$; see [8] for the details). We can write

$$\begin{aligned} T_{\mathcal{G}^2} f &= \sum_{n=1}^{N} \sqrt{r \log \frac{1}{r}}\; r^{n+\lambda} \frac{a_n}{n+\lambda} (-2\lambda) \sin\theta P_{n-1}^{\lambda+1} \|P_n^\lambda\|^{-1} \\ &= \sum_{n=0}^{\infty} g^2(n) \frac{a_n}{n+\lambda} (-2\lambda) \sin\theta P_{n-1}^{\lambda+1} \|P_n^\lambda\|^{-1} = T_{g^2}(R_\lambda f), \end{aligned}$$

where R_λ is the Riesz transform operator (see [1]) and T_{g^2} is the multiplier associated with the orthogonal system in $L^2(dm_\lambda)$ given by the functions

$$h_n(\theta) = \sin\theta P_{n-1}^{\lambda+1}(\cos\theta)$$

with coefficients $g^2(n) = \sqrt{r \log(1/r)}\,(n+\lambda) r^{n+\lambda-1}$. The coefficients are uniformly bounded in $L^2(dr)$. This, together with the L^2-boundedness of the Riesz transform, give that $\mathcal{G}^2$ is bounded in $L^2(dm_\lambda)$; see [1] for the details.

4.2 Kernel of the g-Function

In the next lemma, we find the vector-valued kernel in the Calderón–Zygmund sense of $T_{\mathcal{G}^i}$, $i = 1, 2$.

Lemma 4.1 *For every $f \in L^1(dm_\lambda)$ and θ outside the support of f, we have*

$$T_{\mathcal{G}^i} f(\theta) = \int_0^\pi \tau_r^i(\theta,\phi) f(\phi)\, dm_\lambda(\phi), \qquad i = 1, 2, \tag{4.2}$$

where

$$\tau_r^1(\theta,\phi) = \sqrt{r \log \frac{1}{r}}\, \frac{\lambda}{\pi} \Big[\lambda r^{\lambda-1} P(r,\theta,\phi) - 2r^{\lambda+1} \int_0^\pi \frac{(\sin t)^{2\lambda-1}}{D_r^{\lambda+1}}\, dt - (\lambda+1) r^\lambda (1-r^2) \int_0^\pi \frac{(\sin t)^{2\lambda-1} \partial_r D_r}{D_r^{\lambda+2}}\, dt \Big], \tag{4.3}$$

$$\tau_r^2(\theta,\phi) = \sqrt{r \log \frac{1}{r}}\, \frac{\lambda(\lambda+1)}{\pi} r^\lambda (1-r^2) \int_0^\pi \frac{(\sin t)^{2\lambda-1} \partial_\theta D_r}{D_r^{\lambda+2}}\, dt \tag{4.4}$$

are kernels taking values in $L^2(dr)$.

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Proof By the definition of $T_{\mathcal{G}^i}$, in order to get (4.2) we only need to put the derivative inside the integrals in the expressions (4.1). With our usual notation (3.9), we write

$$\partial_r P(r,\theta,\phi) = -\frac{\lambda}{\pi} 2r \int_0^\pi \frac{(\sin t)^{2\lambda-1}}{D_r^{\lambda+1}}\,dt + \frac{\lambda}{\pi}(1-r^2)\partial_r \int_0^\pi \frac{(\sin t)^{2\lambda-1}}{D_r^{\lambda+1}}\,dt. \tag{4.5}$$

For negative a, $D_r \geq 1$, and for positive a, $a \leq \cos(\theta-\phi)$. Therefore, if θ does not belong to the support of f, we have that $D_r \geq 1 - \cos(\theta-\phi)^2 \geq C$. This implies that for each r, the remaining integrands in the right-hand side of the equations in (4.1) belong to $L^1(dm_\lambda \times dt)$ for $f \in L^1(dm_\lambda)$. This shows (4.3). Also, if θ does not belong to the support of f, we have that $|\tau_r^1(\theta,\phi)| \leq C\sqrt{r\log(1/r)}\, r^{\lambda-1}$ and $|\tau_r^2(\theta,\phi)| \leq C$, which are functions in $L^2(dr)$. ■

Our aim is to prove that the kernels of $T_{\mathcal{G}^1}$ verify the hypothesis of Theorem 2.2. We have already seen that they are bounded in $L^2(dm_\lambda)$. Observe that since $P(r,\pi-\theta,\pi-\phi) = P(r,\theta,\phi)$, the symmetry condition (iii) holds.

On the other hand, for $|\theta-\phi| > \pi/6$ and $r \in (1/2,1)$, we have $D_r \geq 1 - (\cos(\theta-\phi))^2 \geq C$. For $r \in (0,1/2)$, clearly $D_r \geq C$. Then

$$\|\tau_r^1(\theta,\phi)\|_{L^2(dr)} \leq C\Big\|\sqrt{r\log\frac{1}{r}}(r^{\lambda-1}+1)\Big\|_{L^2(dr)} = C,$$
$$\|\tau_r^2(\theta,\phi)\|_{L^2(dr)} \leq C,$$

as can be easily seen from (4.3) and (4.4). Thus condition (i) in Theorem 2.2 also holds.

Checking condition (ii) requires a bit more work.

Lemma 4.2 *There exists a constant $C > 0$ such that for every $\theta \in [0,\pi/2]$ and $2\theta/3 \leq \phi \leq 3\theta/2$, we have (for $i = 1,2$)*

$$\|\tau_r^i(\theta,\phi)\|_{L^2(dr)} \leq \frac{C}{|\theta-\phi|}\frac{1}{(\sin(\theta\vee\phi))^{2\lambda}}, \tag{4.6}$$

$$\|\partial_\theta \tau_r^i(\theta,\phi)\|_{L^2(dr)} + \|\partial_\phi \tau_r^i(\theta,\phi)]\|_{L^2(dr)} \leq \frac{C}{|\theta-\phi|^2}\frac{1}{(\sin(\theta\vee\phi))^{2\lambda}}. \tag{4.7}$$

Proof When $r \in (0,1/2)$, $D_r \geq (1-r)^2 \geq C$ and therefore

$$|\tau_r^1(\theta,\phi)| \leq C\sqrt{r\log\frac{1}{r}}\,(r^{\lambda-1}+C) \quad\text{and}\quad |\tau_r^2(\theta,\phi)| \leq C.$$

For $r \in (1/2,1)$, let us observe that by (4.3), we can split $|\tau_r^1(\theta,\phi)|$ according to the following sum $|\partial_r(r^\lambda P(r,\theta,\phi))| \leq \sum_{i=1}^3 N_i(r,\theta,\phi)$, where

$$N_1(r,\theta,\phi) = C\int_0^{\pi/2} \frac{(\sin t)^{2\lambda-1}}{D_r^{\lambda+1}}\,dt, \quad N_2(r,\theta,\phi) = C\int_{\pi/2}^{\pi} \frac{(\sin t)^{2\lambda-1}}{D_r^{\lambda+1}}\,dt,$$
$$N_3(r,\theta,\phi) = C(1-r)\int_0^\pi \frac{(\sin t)^{2\lambda-1}|\partial_r D_r|}{D_r^{\lambda+2}}\,dt.$$

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We will use the following estimate

$$I_{\lambda+1}^{2\lambda-1} = \int_0^{\frac{\pi}{2}} \frac{t^{2\lambda-1}}{(\Delta_r + r\sigma t^2)^{\lambda+1}}\,dt \le \frac{C}{\Delta_r r^\lambda \sigma^\lambda}, \tag{4.8}$$

which can be easily obtained by the change of variables $t = \Delta_r^{1/2}(r\sigma)^{-1/2}u$. By using $\sin x \sim x$, $1 - \cos x \sim x^2$ and $1 + \cos x \ge Cx^2$ for $x \in [0, \pi/2]$, it is not difficult to obtain the estimate $N_1(r,\theta,\phi) + N_2(r,\theta,\phi) \le CI_{\lambda+1}^{2\lambda-1}$, where in the case of N_2 we have first made the change of variables $\pi - x = t$. For the term N_3, observe that for $r \in [1/2, 1]$,

$$(1-r)|\partial_r D_r| \le C(1-r)[|1-r| + |1-\cos(\theta-\phi)| + |\sigma(1-\cos t)|] \le C\,D_r. \tag{4.9}$$

Thus, after applying the same change of variables $\pi - x = t$ used above for N_2, we get that $N_3(r,\theta,\phi) \le CI_{\lambda+1}^{2\lambda-1}$. This gives (with the only restriction being $r \in [1/2, 1]$)

$$|\partial_r(r^\lambda P(r,\theta,\phi))| \le C\,I_{\lambda+1}^{2\lambda-1}. \tag{4.10}$$

The next step is integrating in r. By using (4.8), $\Delta_r \ge C\,((1-r)^2 + \Delta)$, $r\log\frac{1}{r} \sim 1-r$ for $r \in (1/2, 1)$, the change of variables $u = 1 - r/\sqrt{\Delta}$ and (3.17), we obtain

$$\int_{1/2}^1 |\tau_r^1(\theta,\phi)|^2\,dr \le \frac{C}{\sigma^{2\lambda}} \int_{1/2}^1 \frac{r\log\frac{1}{r}\,dr}{((1-r)^2+\Delta)^2} \le \frac{C}{\sigma^{2\lambda}\Delta} \le \frac{C}{|\theta-\phi|^2(\sin(\theta\vee\phi))^{4\lambda}}.$$

The case of τ_r^2 is treated similarly. For the estimates concerning the derivative, let us observe first that the same arguments as in Lemma 4.1 allow us to put the derivatives inside the integrals, and then

$$\begin{aligned} \partial_\theta\tau_r^1(\theta,\phi) = \sqrt{r\log\frac{1}{r}}\,\frac{\lambda}{\pi}\Bigg[r^{\lambda-1}(\lambda(1-r^2)-r^2)\int_0^\pi \frac{(\sin t)^{2\lambda-1}\partial_\theta D_r}{D_r^{\lambda+2}}\,dt \\ -(\lambda+1)r^\lambda(1-r^2)\int_0^\pi \frac{(\sin t)^{2\lambda-1}}{D_r^{\lambda+1}}\,\frac{\partial_\theta\partial_r D_r\,D_r - (\lambda+2)\partial_\theta D_r\partial_r D_r}{D_r^2}\,dt\Bigg]\,dr, \end{aligned} \tag{4.11}$$

By using (4.9), the analogous $|\frac{(1-r)\partial_\theta D_r}{D_r}| \le C$, that $|\partial_\theta^2 D_r| \le C$ and also the following estimates for $1/2 \le r \le 1$,

$$\begin{aligned} \Big|\frac{\partial_\theta D_r}{D_r}\Big| &\le \frac{2r\sin|\theta-\phi|}{2r(1-\cos(\theta-\phi))} + \frac{2r\cos\theta\sin\phi(1-\cos t)}{2r\sin\theta\sin\phi(1-\cos t)} \le \frac{C}{|\theta-\phi|}, \\ \Big|\frac{\partial_\theta\partial_r D_r}{D_r}\Big| &= \Big|\frac{1/r\partial_\theta D_r}{D_r}\Big| \le \frac{C}{|\theta-\phi|}, \end{aligned} \tag{4.12}$$

we easily get that

$$|\partial_\theta\tau_r^1(\theta,\phi)| \le \frac{C}{\sigma^\lambda|\theta-\phi|}\sqrt{r\log\frac{1}{r}}\left(r^{\lambda-1}1_{(0,1/2)}(r) + I_{\lambda+1}^{2\lambda-1}1_{(1/2,1)}(r)\right),$$

and therefore, for θ and ϕ satisfying $2\theta/3 \le \phi \le 3\theta/2$, we have

$$\|\partial_\theta \tau_r^1(\theta,\phi)\|_{L^2(dr)} \le \frac{C}{|\theta-\phi|^2}\frac{1}{(\sin(\theta\vee\phi))^{2\lambda}}.$$

The derivative in ϕ is treated similarly, by using the parallel estimates to (4.9) and (4.12). Similar arguments also hold for τ_r^2. ■

Lemma 4.3 *For every $\theta,\phi \in [0,\pi/2]$ and ϕ outside the region $2\theta/3 \le \phi \le 3\theta/2$, we also have* (4.6) *and* (4.7).

Proof We use that from (4.10) one can achieve the following inequality

$$|\tau_r^1(\theta,\phi)| \le C\sqrt{r\log\frac{1}{r}}\left(r^{\lambda-1}1_{(0,1/2)}(r) + I_{\lambda+1}^{2\lambda-1}1_{(1/2,1)}(r)\right),$$

and from here, using that for $r\in(1/2,1)$, $r\log\frac{1}{r}\sim 1-r$, we get that

$$\begin{aligned}\|\tau_r^1(\theta,\phi)\|^2_{L^2(dr)} &\le C + C\int_{1/2}^1 \frac{r\log\frac{1}{r}}{\Delta_r^{2(\lambda+1)}}\,dr, \\ &\le C + C\int_{1/2}^1 \frac{1-r}{((1-r)^2+\Delta)^{2(\lambda+1)}}\,dr, \le \frac{C}{\Delta^{2\lambda+1}},\end{aligned} \tag{4.13}$$

where the last estimate can easily be obtained by the change of variable $u=\frac{1-r}{\sqrt{\Delta}}$. For $\phi < 2\theta/3$, $\sin(\theta-\phi)\sim\sin\theta$ and $1-\cos(\theta-\phi)\sim|\theta-\phi|^2\sim\sin(\theta-\phi)^2$. Thus, by these properties, (4.13) implies

$$\|\tau_r^1(\theta,\phi)\|_{L^2(dr)} \le \frac{C}{(1-\cos(\theta-\phi))^{\lambda+1/2}} \le \frac{C}{|\theta-\phi|}\frac{C}{(\sin(\theta\vee\phi))^{2\lambda}}.$$

Now, for $\phi > 3\theta/2$, $1-\cos(\theta-\phi)\sim|\theta-\phi|^2\sim(\sin(\theta-\phi))^2$ and also $|\sin(\theta-\phi)|\sim \sin\phi$. Then from (4.13), we get as before

$$\|\tau_r^1(\theta,\phi)\|_{L^2(dr)} \le \frac{C}{|\theta-\phi|}\frac{C}{(\sin(\theta\vee\phi))^{2\lambda}}.$$

The estimate for τ_r^2 follows in an analogous way. We will obtain (4.7) for the derivative in θ of τ_r^1. The proof for the other derivative is completely analogous, and also for τ_r^2. Let us observe that for $r\in(0,1/2)$, $D_r \ge C$ and therefore $|\partial_\theta\tau_r^i(\theta,\phi)| \le C\sqrt{r\log(1/r)}(r^{\lambda-1}+1)$. If $r\in(1/2,1)$, it is easy to see that

$$\begin{aligned} &|\partial_\theta D_r|, |\partial_\theta\partial_r D_r| \le C(\sin|\theta-\phi|+\sin\phi) \qquad |\partial_\theta^2 D_r| \le C, \\ &|\partial_\theta D_r|^2 \le C((\sin|\theta-\phi|)^2+(\sin\phi)^2).\end{aligned} \tag{4.14}$$

Thus, from (4.11), (4.9) and (4.14), it is not difficult to obtain that

$$|\partial_\theta \tau_r^1(\theta,\phi)| \le C\sqrt{r\log\frac{1}{r}}(r^{\lambda-1}+1)\chi_{(0,1/2)}(r) + C\sqrt{r\log\frac{1}{r}}\frac{\sin|\theta-\phi|+\sin\phi}{\Delta_r^{\lambda+2}}.$$

From here, $\|\partial_\theta \tau^1(\theta,\phi)\|_{L^2(dr)} \le C + C(\sin|\theta-\phi|+\sin\phi)A$, where

$$A^2 = \int_{1/2}^1 \frac{r\log\frac{1}{r}}{((1-r)^2+\Delta)^{2\lambda+4}}\,dr \le \frac{C}{\Delta^{2\lambda+3}}.$$

In the former inequality we have used for $r \in (1/2,1)$ that $r\log\frac{1}{r} \sim 1-r$, and we have performed the change of variables $u = (1-r)/\sqrt{\Delta}$. Thus, we obtain

$$\|\partial_\theta \tau^1(\theta,\phi)\|_{L^2(dr)} \le C + C\frac{\sin|\theta-\phi|+\sin\phi}{\Delta^{\lambda+3/2}}.$$

In the region $\phi < 2\theta/3$, we have $|\theta-\phi| \sim \sin|\theta-\phi| \sim \sin\theta$, and $\Delta \sim (\sin|\theta-\phi|)^2$. Thus,

$$\|\partial_\theta \tau^1(\theta,\phi)\|_{L^2(dr)} \le C + C\frac{|\theta-\phi|}{\Delta^{3/2}}\frac{1}{\Delta^\lambda} \le \frac{C}{|\theta-\phi|^2}\frac{1}{(\sin(\theta\vee\phi))^{2\lambda}}.$$

On the other hand, for $\phi > 3\theta/2$, $|\theta-\phi| \sim \sin|\theta-\phi| \sim \sin\phi$, and $\Delta \sim (\sin|\theta-\phi|)^2$. Thus, we obtain the desired estimate in the same way. ■

4.3 The Muckhenhoupt–Stein g-Function

The g-function defined by Muckhenhoupt and Stein [5] associated to the ultraspherical polynomials is

$$gf(\theta) = \Big(\int_0^1 (1-r)|\partial_r f(r,\theta)|^2\,dr\Big)^{1/2},$$

where $f(r,\theta)$ is described in Subsection 3.2. They show that this operator is bounded in $L^p(dm_\lambda)$ for every $p \in (1,\infty)$. A natural question is whether this operator can be handled with the technique developed in this paper. In other words, we would like to see if the operator can be described as a Calderón–Zygmund operator. It turns out that our computations hold for the operator

$$g_1 f(\theta) = \Big(\int_{1/2}^1 (1-r)|\partial_r f(r,\theta)|^2\,dr\Big)^{1/2}.$$

The remaining part

$$g_0 f(\theta) = \Big(\int_0^{1/2} (1-r)|\partial_r f(r,\theta)|^2\,dr\Big)^{1/2}$$

is easier to handle, and one easily gets that g_0 maps $L^p(dm_\lambda)$ into $L^p(dm_\lambda)$ for every $p \in [1,\infty]$.

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