

MODULAR FORMS OF DEGREE n AND REPRESENTATION BY QUADRATIC FORMS

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Let $A^{(m)}$, $B^{(n)}$ be positive definite integral matrices and suppose that B is represented by A over each p -adic integers ring Z_p . Using the circle method or theory of modular forms in case of $n = 1$, B , if sufficiently large, is represented by A provided that $m \geq 5$. The approach via the theory of modular forms has been extended by [7] to Siegel modular forms to obtain a partial result in the particular case when $n = 2$, $m \geq 7$. On the other hand Kneser gave an arithmetic approach in case of $n = 1$ in his lectures [4]. Using this idea we proved that B is represented by A over Z provided that $m \geq 2n + 3$ and the minimum of B is sufficiently large [2]. Our aim here is to give an analytic proof in the case when A is an even unimodular positive definite matrix. Under the algebraic preparations of §1 we give the Fourier expansion of Eisenstein series in the sense of Klingen and estimate coefficients from above. In §3 we estimate Fourier coefficients of usual Eisenstein series from above and below and it is applied to our problem in §4.

Notations. Let H_n be the space of $n \times n$ complex symmetric matrices Z with positive real Y . Let Γ_n denote the group of integral $2n \times 2n$ matrices M satisfying

$$MI^tM = I, \quad I = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}.$$

For $M \in \Gamma_n$ we put $M = \begin{pmatrix} A_M & B_M \\ C_M & D_M \end{pmatrix}$ where A_M, \dots, D_M are $n \times n$ matrices. For $1 \leq r \leq n - 1$ we define the subgroup $\Delta_{n,r}$ of Γ_n as the group of all $M \in \Gamma_n$ whose elements in the first $n + r$ columns and last $n - r$ rows vanish. The transposed matrix of a matrix M is denoted by tM . We don't use the usual convention $A[B] = {}^tBAB$. σ stands for the trace of matrices. $e(x)$ means $\exp(2\pi ix)$.

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§1.

Through this section we fix natural numbers r, n which satisfy $1 \leq r \leq n - 1$. For a matrix $M \in M_{p,q}(C)$ ($p \geq n - r$) we denote by \tilde{M} the last $n - r$ rows of M , i.e., $M = \begin{pmatrix} * \\ * \\ \tilde{M} \end{pmatrix}$. For $M \in M_n(C)$ we decompose M as $M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$ where $M_1 \in M_r(C)$, $M_2 \in M_{r,n-r}(C)$, $M_3 \in M_{n-r,r}(C)$, $M_4 \in M_{n-r}(C)$. M_i are used in this sense if we don't refer.

LEMMA 1. For $M, N \in \Gamma_n$, $\Delta_{n,r}M = \Delta_{n,r}N$ is equivalent to $\tilde{M} = g\tilde{N}$ for some $g \in GL(n - r, Z)$.

Proof. Let M, N be elements of Γ_n . Suppose that $M = KN$ where $K = \left(\begin{array}{c|c} * & * \\ * & * \\ \hline 0 & D_4 \end{array} \right) \in \Delta_{n,r}$. Then we have $\tilde{M} = D_4\tilde{N}$. Since K is a unimodular matrix, D_4 is unimodular. Conversely, suppose that $\tilde{M} = g\tilde{N}$ for some $g \in GL(n - r, Z)$. Put $G = \left(\begin{array}{c|c} * & 0 \\ * & 1_r & 0 \\ \hline 0 & 0 & g \end{array} \right) \in \Delta_{n,r}$; then we have $\tilde{GN} = g\tilde{N}$. Hence we may assume $\tilde{M} = \tilde{N}$. Then $\tilde{M}\tilde{N}^{-1} = (0^{(n+r,n-r)}, 1_{n-r})$ holds. Hence $MN^{-1} \in \Delta_{n,r}$. Q.E.D.

LEMMA 2. For an element $N \in \Gamma_n$ with $\text{rank } \tilde{C}_N < n - r$, there is an element M in $\Delta_{n,n-1}$ such that $\Delta_{n,r}N \ni M \begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1} \end{pmatrix}$ for some $U \in GL(n, Z)$.

Proof. By the assumption $\text{rank } \tilde{C}_N < n - r$ there are unimodular matrices $g \in GL(n - r, Z)$, $V \in GL(n, Z)$ such that the last row of $g\tilde{C}_N V$ vanishes. Put $K = \left(\begin{array}{c|c} * & 0 \\ * & 1_r & 0 \\ \hline 0 & 0 & g \end{array} \right) N \begin{pmatrix} V & \\ & {}^tV^{-1} \end{pmatrix}$; then the last row of C_K vanishes and the elements of the last row of D_K are relatively prime. Taking a unimodular matrix $W \in GL(n, Z)$ such that $D_K W = \begin{pmatrix} * & \\ & \dots & 0 & 1 \end{pmatrix}$, we put $M = K \begin{pmatrix} {}^tW^{-1} & \\ & W \end{pmatrix}$; then we have $M \in \Delta_{n,n-1}$ since the last row of M is $(0, \dots, 0, 1)$. We may take V^tW^{-1} as U . Q.E.D.

LEMMA 3. If tAC is symmetric for $A, C \in M_m(Z)$, then there is a symmetric coprime pair $(\mathfrak{C}^{(m)}, \mathfrak{D}^{(m)})$ such that $\mathfrak{C}A + \mathfrak{D}C = 0$.

Proof. If $C = 0$, then we may take $\mathfrak{C} = 0$, $\mathfrak{D} = 1_m$. Suppose $C \neq 0$. First we assume $|C| \neq 0$. Then AC^{-1} is a symmetric rational matrix. Hence there is a symmetric coprime pair $(\mathfrak{C}^{(m)}, \mathfrak{D}^{(m)})$ such that $|\mathfrak{C}| \neq 0$,

$\mathfrak{C}^{-1}\mathfrak{D} = -AC^{-1}$ (p. 166 in [5]). Thus we have $\mathfrak{C}A + \mathfrak{D}C = 0$. Next we assume $|C| = 0$; then there are unimodular matrices $g_1, g_2 \in GL(m, \mathbb{Z})$ such that $g_1 C g_2^{-1} = c = \begin{pmatrix} c_1^{(s)} & 0 \\ 0 & 0 \end{pmatrix}$ ($|c_1| \neq 0, s \geq 1$). Putting $a = {}^t g_1^{-1} A g_2^{-1}$, we have ${}^t a c = {}^t g_2^{-1} {}^t A C g_2^{-1}$ and this is a symmetric matrix. Decompose a as $a = \begin{pmatrix} a_1^{(s)} & a_2 \\ a_3 & a_4 \end{pmatrix}$. ${}^t a c = {}^t c a$ implies that ${}^t a_1 c_1$ is symmetric and ${}^t a_2 c_1 = 0$ and so $a_2 = 0$. Take a symmetric coprime pair $(\mathfrak{C}_1^{(s)}, \mathfrak{D}_1^{(s)})$ such that $\mathfrak{C}_1 a_1 + \mathfrak{D}_1 c_1 = 0$, and put $\mathfrak{C} = \begin{pmatrix} \mathfrak{C}_1 & 0 \\ 0 & 0 \end{pmatrix}$. $\mathfrak{D} = \begin{pmatrix} \mathfrak{D}_1 & \\ & 1_{m-s} \end{pmatrix}$. Then $(\mathfrak{C}, \mathfrak{D})$ is a symmetric coprime pair and

$$\mathfrak{C}a + \mathfrak{D}c = \begin{pmatrix} \mathfrak{C}_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} \mathfrak{D}_1 & \\ & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ 0 \end{pmatrix} = 0.$$

On the other hand, $\mathfrak{C}a + \mathfrak{D}c = (\mathfrak{C} {}^t g_1^{-1} A + \mathfrak{D} g_1 C) g_1^{-1}$ implies $\mathfrak{C} {}^t g_2^{-1} A + \mathfrak{D} g_1 C = 0$. Here it is easy to see that $(\mathfrak{C} {}^t g_1^{-1}, \mathfrak{D} g_1)$ is a symmetric coprime pair. This completes the proof. Q.E.D.

LEMMA 4. For $M \in \Gamma_n$ there are a unimodular matrix $U \in GL(n, \mathbb{Z})$ and $N \in \Delta_{n,r} M \begin{pmatrix} {}^t U & \\ & U^{-1} \end{pmatrix}$ such that the first r columns of C_N vanishes.

Proof. Take $g \in GL(n-r, \mathbb{Z}), U \in GL(n, \mathbb{Z})$ such that $g \tilde{C}_M {}^t U = (0, C_4)$; $C_4 \in M_{n-r}(\mathbb{Z})$. Put $K = \left(\begin{array}{c|c} * & \\ \hline & 1_r \\ \hline & g \end{array} \right) M \begin{pmatrix} {}^t U & \\ & U^{-1} \end{pmatrix}$; then we have $C_K = \begin{pmatrix} C_1 & C_2 \\ 0 & C_4 \end{pmatrix}$. ${}^t AC = {}^t CA$ implies ${}^t A_1 C_1 = {}^t C_1 A_1$ where we put $A = A_K, C = C_K$. By Lemma 3 there is a symmetric coprime pair $(\mathfrak{C}_1^{(r)}, \mathfrak{D}_1^{(r)})$ such that $\mathfrak{C}_1 A_1 + \mathfrak{D}_1 C_1 = 0$. Take an element $G \in \Delta_{n,r}$ such that $C_G = \begin{pmatrix} \mathfrak{C}_1 & 0 \\ 0 & 0 \end{pmatrix}, D_G = \begin{pmatrix} \mathfrak{D}_1 & 0 \\ 0 & 1 \end{pmatrix}$; then $C_{GK} = \begin{pmatrix} \mathfrak{C}_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} + \begin{pmatrix} \mathfrak{D}_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 & C_2 \\ & C_4 \end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$. This completes the proof. Q.E.D.

We put $P_{n,r} = \left\{ A = \begin{pmatrix} A_1^{(r)} & A_2 \\ A_3 & A_4 \end{pmatrix} \in GL(n, \mathbb{Z}) \mid A_3 = 0 \right\}$.

LEMMA 5. Let M, N be elements of Γ_n such that $\begin{pmatrix} C_1 \\ C_3 \end{pmatrix} = \begin{pmatrix} C'_1 \\ C'_3 \end{pmatrix} = 0, |C_4| |C'_4| \neq 0$ where we put $C_M = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, C_N = \begin{pmatrix} C'_1 & C'_2 \\ C'_3 & C'_4 \end{pmatrix}$. If $KM \begin{pmatrix} {}^t V & \\ & V^{-1} \end{pmatrix} = N \begin{pmatrix} {}^t U & \\ & U^{-1} \end{pmatrix}$ for $K \in \Delta_{n,r}, U, V \in GL(n, \mathbb{Z})$, then we have $C_K = 0, {}^t U \in P_{n,r} {}^t V$.

Proof. Put $W = {}^t U {}^t V^{-1}$; then $C_{KM} = C_K A_M + D_K C_M = C_N W$. Putting $A_M = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, C_K = \begin{pmatrix} \mathfrak{C}_1 & 0 \\ 0 & 0 \end{pmatrix}, D_K = \begin{pmatrix} \mathfrak{D}_1 & \mathfrak{D}_2 \\ & \mathfrak{D}_4 \end{pmatrix}, W = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix}$, we have

$C_4'W_3 = 0$ and $\mathfrak{C}_1A_1 = C_2'W_3$. The assumption $|C_4'| \neq 0$ implies $W_3 = 0$. Hence ${}^tU = W^tV \in P_{n,r}{}^tV$ and $\mathfrak{C}_1A_1 = 0$. We have only to prove $|A_1| \neq 0$. Since ${}^tA_M C_M = \begin{pmatrix} * & {}^tA_1C_2 + {}^tA_3C_4 \\ 0 & * \end{pmatrix}$ is symmetric, we have ${}^tA_1C_2 + {}^tA_3C_4 = 0$ and $A_3 = -{}^tC_4^{-1}{}^tC_2A_1$. Since the rank of the first r columns of M is r , we have $r = \text{rank} \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} = \text{rank} \begin{pmatrix} 1_r \\ -{}^tC_4^{-1}{}^tC_2 \end{pmatrix} A_1 \leq \text{rank } A_1 \leq r$. Thus we have $\text{rank } A_1 = r$, i.e., $|A_1| \neq 0$ and so $\mathfrak{C}_1 = 0$. Q.E.D.

The following lemma is a key in this paper.

LEMMA 6. *Let M be an element of Γ_n with $\text{rank } \tilde{C}_M = n - r$. Then there is an element $N \in \Delta_{n,r}M$ such that $|C_N| \neq 0$, $(A_N C_N^{-1})_1 \equiv 0 \pmod{1}$.*

Proof. By virtue of Lemma 4 there exist $P \in \Gamma_n$, $U \in GL(n, \mathbb{Z})$ such that $\Delta_{n,r}M \ni P \begin{pmatrix} {}^tU \\ U^{-1} \end{pmatrix}$, $C_P = \begin{pmatrix} 0 & C_2 \\ 0 & C_4 \end{pmatrix}$. $\text{rank } \tilde{C}_M = n - r$ yields $|C_4| \neq 0$.

Put $A_P = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$; then $|A_1| \neq 0$ holds as in the proof of Lemma 5.

Take $K = \left(\begin{array}{cc|c} 1_n & 0 & \\ \hline 1_r & 0 & \\ 0 & 0 & 1_n \end{array} \right) \in \Delta_{n,r}$ and put $KP = N'$. Then $A_{N'} = A_P$, $C_{N'} =$

$\begin{pmatrix} A_1 & A_2 + C_2 \\ 0 & C_4 \end{pmatrix}$ imply $|C_{N'}| \neq 0$, $(A_{N'} C_{N'}^{-1})_1 = 1_r$. Taking $N' \begin{pmatrix} {}^tU \\ U^{-1} \end{pmatrix}$ as N , we have $|C_N| \neq 0$, $(A_N C_N^{-1})_1 = 1_r$ and $\Delta_{n,r}M \ni N$. Q.E.D.

LEMMA 7. *Let C_4, D_3, D_4 be elements of $M_{n-r}(\mathbb{Z})$, $M_{n-r,r}(\mathbb{Z})$, $M_{n-r}(\mathbb{Z})$ respectively. Suppose that $|C_4| \neq 0$, $C_4^t D_4$ is symmetric and (C_4, D_3, D_4) is primitive. Then there is a symmetric coprime pair $(C^{(n)}, D^{(n)})$ such that $\tilde{C} = (0, C_4)$, $\tilde{D} = (D_3, D_4)$.*

Proof. Since $\text{rank}(C_4, D_4) = n - r$, there exist matrices $U_4 \in M_{n-r}(\mathbb{Z})$, $V \in GL(2(n - r), \mathbb{Z})$ such that $(C_4, D_4) = U_4(0, 1_{n-r})V$ and $|U_4| \neq 0$. Put $\mathfrak{C}_4 = U_4^{-1}C_4$, $\mathfrak{D}_4 = U_4^{-1}D_4$; then $(\mathfrak{C}_4, \mathfrak{D}_4)$ is primitive and $\mathfrak{C}_4^t \mathfrak{D}_4 = U_4^{-1}C_4^t D_4 U_4^{-1}$ is symmetric. Thus $(\mathfrak{C}_4, \mathfrak{D}_4)$ is a symmetric coprime pair. Since (C_4, D_3, D_4) is primitive, $(D_3, C_4, D_4) = (D_3, U_4(0, 1_{n-r})V)$ is also primitive. Hence (D_3, U_4) is primitive and there is a unimodular matrix $U = \begin{pmatrix} * & * \\ D_3 & U_4 \end{pmatrix} \in GL(n, \mathbb{Z})$. Put $C = U \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{C}_4 \end{pmatrix}$, $D = U \begin{pmatrix} 1_r & \\ & \mathfrak{D}_4 \end{pmatrix}$; then (C, D) is a symmetric coprime pair and $\tilde{C} = (0, U_4 \mathfrak{C}_4) = (0, C_4)$, $\tilde{D} = (D_3, U_4 \mathfrak{D}_4) = (D_3, D_4)$. This completes the proof. Q.E.D.

Let C_4, D_3, D_4 be those as in Lemma 7 and let M be an element of Γ_n such that $\tilde{M} = (0, C_4, D_3, D_4)$. By Lemma 6 there is $K \in \Delta_{n,r}$ such that

$|C_N| \neq 0$, $(A_N C_N^{-1})_1 \equiv 0 \pmod{1}$ for $N = KM$. By Lemma 1 $g\tilde{N} = \tilde{M}$ holds for some $g \in GL(n - r, Z)$. Put $P = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & 1_r \\ & g \end{array} \right) N$; then $\tilde{P} = g\tilde{N} = \tilde{M} = (0, C_4, D_3, D_4)$, $|C_P| \neq 0$, and $(A_P C_P^{-1})_1 \equiv 0 \pmod{1}$. We denote one of such P 's by $M[C_4, D_3, D_4]$. Then $\Delta_{n,r} M[C_4, D_3, D_4]$ is uniquely determined by (C_4, D_3, D_4) by Lemma 1.

Put

$$\mathfrak{S}_{n,r} = \left\{ (C_4, D_3, D_4) \mid \begin{array}{l} C_4, D_4 \in M_{n-r}(Z), D_3 \in M_{n-r,r}(Z), |C_4| \neq 0, \\ C_4 {}^t D_4 = D_4 {}^t C_4 \text{ and } (C_4, D_3, D_4) \text{ is primitive} \end{array} \right\}.$$

For $S, S' \in \mathfrak{S}_{n,r}$ we define $S \sim S'$ by $S' = gS$ for some $g \in GL(n - r, Z)$.

LEMMA 8. $\bigcup_{\substack{M \in \Gamma_n \\ \text{rank } \tilde{C}_M = n-r}} \Delta_{n,r} M = \bigcup \Delta_{n,r} M[C_4, D_3, D_4] \begin{pmatrix} {}^t U & \\ & U^{-1} \end{pmatrix}$, where the right hand is a disjoint union, (C_4, D_3, D_4) (resp. ${}^t U$) runs over representatives of $\mathfrak{S}_{n,r}/\sim$ (resp. $P_{n,r} \setminus GL(n, Z)$).

Proof. Take an element M of Γ_n such that $\text{rank } \tilde{C}_M = n - r$. Then there exist $g \in GL(n - r, Z)$, $U \in GL(n, Z)$ such that $g^{-1} \tilde{C}_M {}^t U^{-1} = (0, C_4^{(n-r)})$, $|C_4| \neq 0$. Hence $M \in \Delta_{n,r} M[C_4, D_3, D_4] \begin{pmatrix} {}^t U & \\ & U^{-1} \end{pmatrix}$. For $W = \begin{pmatrix} W_1 & W_2 \\ & W_4 \end{pmatrix} \in P_{n,r}$

and $h \in GL(n - r, Z)$, we put $N = \left(\begin{array}{c|c} * & \\ \hline 1_r & h \end{array} \right) M[C_4, D_3, D_4] \begin{pmatrix} W & \\ & {}^t W^{-1} \end{pmatrix}$. Then

$C_N = \begin{pmatrix} * & * \\ 0 & h C_4 W_4 \end{pmatrix}$ holds. Thus M is contained in some coset $\Delta_{n,r} M[C_4, D_3, D_4] \begin{pmatrix} {}^t U & \\ & U^{-1} \end{pmatrix}$ for any specified representatives $(C_4, D_3, D_4), U$. We must

prove the disjointness of the right hand. Suppose $KM[C_4, D_3, D_4] \begin{pmatrix} {}^t U & \\ & U^{-1} \end{pmatrix} = M[C'_4, D'_3, D'_4] \begin{pmatrix} {}^t U' & \\ & U'^{-1} \end{pmatrix}$ for $(C_4, D_3, D_4), (C'_4, D'_3, D'_4) \in \mathfrak{S}_{n,r}/\sim$, ${}^t U, {}^t U' \in P_{n,r} \setminus GL(n, Z)$, $K \in \Delta_{n,r}$. There are some $G, G' \in \Delta_{n,r}$ such that the first r -columns of the C -parts of $GM[C_4, D_3, D_4]$, $G'M[C'_4, D'_3, D'_4]$ vanish as in the proof of Lemma 4. Hence Lemma 5 implies ${}^t U \in P_{n,r} {}^t U'$. Thus we have $U = U'$ and then $KM[C_4, D_3, D_4] = M[C'_4, D'_3, D'_4]$ implies $g'(C_4, D_3, D_4) = (C'_4, D'_3, D'_4)$ where g' is a unimodular matrix defined by the right lower $(n - r) \times (n - r)$ submatrix of K . Hence $(C_4, D_3, D_4) = (C'_4, D'_3, D'_4)$.

Q.E.D.

We introduce another equivalence relation \approx in $\mathfrak{S}_{n,r}$. For (C_4, D_3, D_4) ,

$(C'_4, D'_3, D'_4) \in \mathfrak{S}_{n,r}$ we define $(C_4, D_3, D_4) \approx (C'_4, D'_3, D'_4)$ by $g(C'_4, D'_3, D'_4) = (C_4, D_3 + C_4 S_3, D_4 + C_4 S_4)$ for some $g \in GL(n-r, \mathbb{Z})$, $S_3 \in M_{n-r,r}(\mathbb{Z})$, $S_4 = {}^t S_4 \in M_{n-r}(\mathbb{Z})$. It is easy to see that $(C'_4, D'_3, D'_4) \approx (C_4, D_3, D_4)$ if and only if $\Delta_{n,r} M[C'_4, D'_3, D'_4] = \Delta_{n,r} M[C_4, D_3, D_4] \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}$ for some $S = {}^t S = \begin{pmatrix} 0 & S_2 \\ S_3 & S_4 \end{pmatrix} \in M_n(\mathbb{Z})$.

THEOREM.

$$\bigcup_{\substack{M \in \Gamma_n \\ \text{rank } C_M = n-r}} \Delta_{n,r} M = \bigcup \Delta_{n,r} M[C_4, D_3, D_4] \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^t U & \\ & U^{-1} \end{pmatrix},$$

where the right hand is a disjoint union, (C_4, D_3, D_4) (resp. ${}^t U$) runs over representatives of $\mathfrak{S}_{n,r}/\approx$ (resp. $P_{n,r} \backslash GL(n, \mathbb{Z})$) and S runs over $\{S = {}^t S \in M_n(\mathbb{Z}) \mid S_1 = 0\}$.

Proof. This is an immediate corollary of Lemma 8. Q.E.D.

We remark the following two propositions although they are not used for our aim.

PROPOSITION 1. Take $(C_4, D_3, D_4) \in \mathfrak{S}_{n,r}$ and $M, N \in \Gamma_n$ such that $\tilde{M} = \tilde{N} = (0, C_4, D_3, D_4)$, $|C_M| |C_N| \neq 0$ $(A_M C_M^{-1})_1 \equiv (A_N C_N^{-1})_1 \equiv 0 \pmod{1}$. Then we have $GL(r, \mathbb{Z})(C_M)_1 = GL(r, \mathbb{Z})(C_N)_1$.

Proof. Put $A_M = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, $C_M = \begin{pmatrix} C_1 & C_2 \\ & C_4 \end{pmatrix}$; ${}^t A_M C_M = {}^t C_M A_M$ implies ${}^t A_1 C_1 = {}^t C_1 A_1$ and $A_1 C_1^{-1} = {}^t(A_1 C_1^{-1})$ is an integral matrix. Put $K = \left(\begin{array}{cc|cc} 0 & 0 & -1_r & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ \hline 1_r & 0 & -A_1 C_1^{-1} & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{array} \right)$; then $K \in \Delta_{n,r}$ and $C_{KM} = \begin{pmatrix} 0 & * \\ 0 & C_4 \end{pmatrix}$. We define similarly $K' \in \Delta_{n,r}$ for N . Then $\Delta_{n,r} K M = \Delta_{n,r} K' N$ and Lemma 5 imply

$$K' N = \begin{pmatrix} U & US \\ & {}^t U^{-1} \end{pmatrix} K M,$$

where $\begin{pmatrix} U & US \\ & {}^t U^{-1} \end{pmatrix} \in \Delta_{n,r}$, $U = \begin{pmatrix} U_1^{(r)} & \\ U_3 & U_4 \end{pmatrix} \in GL(n, \mathbb{Z})$. Then $(A_{K'N})_1 = (U A_{KM} + U S C_{KM})_1$ implies $-(C_N)_1 = -U_1(C_M)_1$. Thus we have $GL(r, \mathbb{Z})(C_M)_1 = GL(r, \mathbb{Z})(C_N)_1$. Q.E.D.

Let (C_4, D_3, D_4) be an element of $\mathfrak{S}_{n,r}$ and define matrices $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$, $(\mathfrak{C}_4, \mathfrak{D}_4)$ and (C, D) as in the proof of Lemma 7; then $C_4 = U_4 \mathfrak{C}_4$, $D_4 = U_4 \mathfrak{D}_4$

and $D_3 = U_3$. Taking, $\mathfrak{A}_4, \mathfrak{B}_4$ such that $\begin{pmatrix} \mathfrak{A}_4 & \mathfrak{B}_4 \\ \mathfrak{C}_4 & \mathfrak{D}_4 \end{pmatrix} \in \Gamma_{n-r}$ we put

$$M = \left(\begin{array}{c|c} 1_n & 0 \\ \hline 1_r & 0 \\ 0 & 0 \end{array} \middle| \begin{array}{c} 0 \\ \hline 1_n \end{array} \right) \left(\begin{array}{c|c} {}^tU^{-1} & 0 \\ \hline 0 & U \end{array} \right) \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \hline 0 & \mathfrak{A}_4 & 0 & \mathfrak{B}_4 \\ \hline 0 & 0 & 1 & 0 \\ 0 & \mathfrak{C}_4 & 0 & \mathfrak{D}_4 \end{array} \right).$$

Then $C_4 = U_4\mathfrak{C}_4$ and $|C_4| \neq 0$ imply $|U_4| \neq 0$ and $|U_1 - U_2U_4^{-1}U_3| \neq 0$ follows from

$$\begin{pmatrix} 1 & -U_2U_4^{-1} \\ 0 & 1 \end{pmatrix} U = \begin{pmatrix} U_1 - U_2U_4^{-1}U_3 & 0 \\ * & * \end{pmatrix}.$$

Hence we have

$$U^{-1} = \begin{pmatrix} (U_1 - U_2U_4^{-1}U_3)^{-1} & -(U_1 - U_2U_4^{-1}U_3)^{-1}U_2U_4^{-1} \\ -U_4^{-1}U_3(U_1 - U_2U_4^{-1}U_3)^{-1} & U_4^{-1} + U_4^{-1}U_3(U_1 - U_2U_4^{-1}U_3)^{-1}U_2U_4^{-1} \end{pmatrix}.$$

Putting ${}^tU^{-1} = V$, we have

$$\begin{aligned} A_M &= \begin{pmatrix} V_1 & V_2\mathfrak{A}_4 \\ V_3 & V_4\mathfrak{A}_4 \end{pmatrix}, & B_M &= \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}, \\ C_M &= \begin{pmatrix} V_1 & V_2\mathfrak{A}_4 + U_2\mathfrak{C}_4 \\ 0 & C_4 \end{pmatrix}, & D_M &= \begin{pmatrix} U_1 & V_2\mathfrak{B}_4 + U_2\mathfrak{D}_4 \\ D_3 & D_4 \end{pmatrix}, \end{aligned}$$

For this special extension $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ of $(0, C_4, D_3, D_4)$ we have

LEMMA 8. $X = (B_1 - A_1C_1^{-1}D_1 - A_2C_4^{-1}D_3 + A_1C_1^{-1}C_2C_4^{-1}D_3){}^tC_1$ is integral.

Proof. $X = (-U_1 - V_2\mathfrak{A}_4C_4^{-1}D_3 + (V_2\mathfrak{A}_4 + U_2\mathfrak{C}_4)C_4^{-1}D_3){}^tV_1 = (-U_1 + U_2\mathfrak{C}_4C_4^{-1}D_3){}^tV_1 = -1_r$. Q.E.D.

PROPOSITION 2. Take $N = M[C_4, D_3, D_4]$ for $(C_4, D_3, D_4) \in \mathfrak{S}_{n,r}$ and a half-integral symmetric matrix $P^{(n)}$ such that ${}^t(C_N)_1^{-1}P_1(C_N)_1^{-1}$ is half-integral. Then $\sigma(PC_N^{-1}D_N) \bmod Z$ is uniquely determined by (C_4, D_3, D_4) and P .

Proof. Take $M \in \Gamma_n$ such that $\tilde{M} = (0, C_4, D_3, D_4) \mid C_M| \neq 0, (A_M C_M^{-1})_1 \equiv 0 \pmod 1$ and put $C_M = C, D_M = D$; then we have

$$C^{-1}D = \begin{pmatrix} C_1^{-1}(D_1 - C_2C_4^{-1}D_3) & {}^t(C_4^{-1}D_3) \\ C_4^{-1}D_3 & C_4^{-1}D_4 \end{pmatrix}$$

and $(C^{-1}D)_2, (C^{-1}D)_3, (C^{-1}D)_4$ are only dependent of (C_4, D_3, D_4) . Take any

extension $N = M[C_4, D_3, D_4]$ and define K, K', U, S as in the proof of Proposition 1; then $\tilde{C}_N = \tilde{C}_M$ implies $U_4 = 1_{n-r}$ and we have

$$C_N = \begin{pmatrix} -U_1 S_1 & 0 \\ 0 & 0 \end{pmatrix} A_M + \begin{pmatrix} U_1 + U_1 S_1 A_1 C_1^{-1} & -U_1 S_2 \\ 0 & 1 \end{pmatrix} C_M,$$

$$D_N = \begin{pmatrix} -U_1 S_1 & 0 \\ 0 & 0 \end{pmatrix} B_M + \begin{pmatrix} U_1 + U_1 S_1 A_1 C_1^{-1} & -U_1 S_2 \\ 0 & 1 \end{pmatrix} D_M.$$

Hence $(C_M^{-1} D_M - C_N^{-1} D_N)_1 = C_1^{-1} S_1 (B_1 - A_1 C_1^{-1} D_1 - A_2 C_4^{-1} D_3 + A_1 C_1^{-1} C_2 C_4^{-1} D_3)$. Now we suppose that M is a special extension in Lemma 8; then $C_1(C_M^{-1} D_M - C_N^{-1} D_N)^t C_1 \equiv 0 \pmod{1}$. Therefore $\sigma(PC_M^{-1} D_M) - \sigma(PC_N^{-1} D_N) = \sigma(P_1(C_M^{-1} D_M - C_N^{-1} D_N)_1) = \sigma({}^t C_1^{-1} P_1 C_1^{-1} (C_1(C_M^{-1} D_M - C_N^{-1} D_N)^t C_1)) \equiv 0 \pmod{1}$. Q.E.D.

§ 2.

Through this section we fix natural numbers r, n, k such that $1 \leq r \leq n - 1, k \geq n + r + 2, k \equiv 0 \pmod{2}$. Denote by f a cusp form of degree r and weight k which is also fixed.

For $M \in \Gamma_n$ we put

$$(f|M)(Z) = f(M\langle Z \rangle)_1 |C_M Z + D_M|^{-k},$$

where $M\langle Z \rangle = (A_M Z + B_M)(C_M Z + D_M)^{-1}, (Z \in H_n)$, and $(M\langle Z \rangle)_1$ is the upper left $r \times r$ submatrix of $M\langle Z \rangle$ as in § 1, and $(f|M)(Z) = f(M\langle Z \rangle^*) M\{Z\}^{-k}$ in the notation of Klingen [3]. It is easy to see that $(f|M)(Z) = (f|NM)(Z)$ for any $N \in A_{n,r}$.

Put $E_1(Z) = \sum_M (f|M)(Z)$ (resp. $E_2(Z) = \sum_M (f|M)(Z)$) where M runs over representatives of $A_{n,r} \backslash \Gamma_n$ such that $\text{rank } \tilde{C}_M = n - r$ (resp. $\text{rank } \tilde{C}_M < n - r$).

LEMMA 1. *Let N be an element of Γ_n such that $\text{rank } \tilde{C}_N < n - r$ and put $\frac{\partial}{\partial Y} = \left(\frac{1}{2}(1 + \delta_{ij}) \frac{\partial}{\partial y_{ij}}\right)$. Then we have $\left|\frac{\partial}{\partial Y}\right| (f|N)(Z) = 0$ where $Y = \text{Im } Z$.*

Proof. By Lemma 2 in § 1 there exist $M \in A_{n,n-1}, U \in GL(n, \mathbf{Z})$ such that $A_{n,r} N \ni M \begin{pmatrix} U \\ {}^t U^{-1} \end{pmatrix}$. Put $A_M = \begin{pmatrix} A_1 & 0 \\ A_3 & A_4 \end{pmatrix}, B_M = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, C_M = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}, D_M = \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix}$ and $Z = \begin{pmatrix} Z_1 & Z_2 \\ {}^t Z_2 & Z_4 \end{pmatrix}$ where A_1, \dots, D_1 and Z_1 are $(n - 1) \times (n - 1)$ matrices. Then we have $C_M Z + D_M = \begin{pmatrix} C_1 Z_1 + D_1 & * \\ 0 & D_4 \end{pmatrix}, M\langle Z \rangle = \begin{pmatrix} (A_1 Z_1 + B_1)(C_1 Z_1 + D_1)^{-1} & * \\ * & * \end{pmatrix}$. Hence $(f|M)(Z)$ does not depend on Z_2, Z_4

and so $|\partial/\partial Y|(f|M)(Z) = 0$. Since $(f|N)(Z) = (f|M)(UZ^tU)$ and $|Y||\partial/\partial Y|$ is invariant under the transformation $Y \rightarrow UY^tU$ on $\{Y^{(n)} | Y > 0\}$, we have

$$\begin{aligned} |Y| \left| \frac{\partial}{\partial Y} \right| (f|N)(Z) &= |Y| \left| \frac{\partial}{\partial Y} \right| (f|M)(UZ^tU) \\ &= |Y| \left| \frac{\partial}{\partial Y} \right| (f|M)(UX^tU + iY) \Big|_{Y \rightarrow UY^tU} = 0, \end{aligned}$$

where $Z = X + iY$. Q.E.D.

PROPOSITION 1. $E_2(Z)$ has Fourier expansion $\sum_T a_2(T)e(\sigma(TZ))$ such that $a_2(T) = 0$ for $T > 0$.

Proof. Put $N = M \begin{pmatrix} 1 & S \\ & 1 \end{pmatrix}$ where $N, M \in \Gamma_n$, ${}^tS = S \in M_n(\mathbb{Z})$. Then $\text{rank } \tilde{C}_N = \text{rank } \tilde{C}_M$. Hence $E_2(Z + S) = E_2(Z)$ for any $S = {}^tS \in M_n(\mathbb{Z})$. Thus $E_2(Z)$ has Fourier expansion $\sum_T a_2(T)e(\sigma(TZ))$. From Lemma 1 follows

$$\left| \frac{\partial}{\partial Y} \right| E_2(Z) = \sum_T a_2(T) | -2\pi T | e(\sigma(TZ)) = 0.$$

Hence $a_2(T)$ vanishes if T is positive definite. Q.E.D.

For a natural number m we put

$$\begin{aligned} A_m &= \{S \in M_m(\mathbb{Z}) | S = {}^tS\}, \\ A_m^* &= \{S \in M_m(\mathbb{Q}) | S = {}^tS: \text{half-integral}\}. \end{aligned}$$

A_m^* is the dual lattice of A_m via $\sigma(SS')$.

The following is well known ([1], [5], [8]).

LEMMA 2. For a positive definite matrix $Y^{(m)}$ and $\rho > m + 1$,

$$\sum_{F \in A_m} |Y + 2\pi i F|^{-\rho}$$

is absolutely convergent and

$$\Gamma_m(\rho) \sum_{F \in A_m} |Y + 2\pi i F|^{-\rho} = 2^{-m(m-1)/2} \sum_{\substack{T > 0 \\ T \in A_m^*}} |T|^{\rho - (m+1)/2} e^{-\sigma(TY)},$$

where $\Gamma_m(\rho) = \pi^{m(m-1)/4} \prod_{\nu=0}^{m-1} \Gamma(\rho - \nu/2)$.

LEMMA 3. For a positive number a ,

$$\begin{aligned} \sum_{F \in A_m} |(2\pi i)^{-1}(Z + aF)|^{-\rho} \\ = 2^{-m(m-1)/2} (2\pi)^{2m\rho} \Gamma_m(\rho)^{-1} a^{-m\rho} \sum_{\substack{T > 0 \\ T \in A_m^*}} |T|^{\rho - (m+1)/2} e(a^{-1}\sigma(TZ)), \end{aligned}$$

where $Z \in H_m$ and $\rho > m + 1$.

Proof. This is an immediate corollary of Lemma 2. Q.E.D.

LEMMA 4. If a, b are complex numbers such that $\operatorname{Re} a > 0$, then we have

$$\int_{\mathbb{R}} \exp(-ax^2 + 2bx)dx = \sqrt{\pi/a} \exp(b^2/a),$$

where $\sqrt{\pi/a}$ is real positive if a is real.

Proof. This is also well known. Q.E.D.

The following is an easy generalization.

LEMMA 5. If A is a symmetric matrix of $M_m(\mathbb{C})$ such that $\operatorname{Re} A > 0$ and b is an element of \mathbb{C}^m , then we have

$$\int_{\mathbb{R}^m} \exp(-{}^t x A x + 2{}^t b x)dx = \sqrt{\det(\pi A^{-1})} \exp({}^t b A^{-1} b),$$

where $\sqrt{\det(\pi A^{-1})}$ is real positive if A is real.

We need the following generalization.

LEMMA 6. If A is a symmetric matrix of $M_{n-r}(\mathbb{C})$ such that $\operatorname{Re} A > 0$ and $W_1^{(r)} > 0$ and Q is an element of $M_{n-r,r}(\mathbb{C})$, then we have

$$\begin{aligned} & \int_{X \in M_{r,n-r}(\mathbb{R})} \exp(-2\pi\sigma(W_1 X A {}^t X) + 2\pi\sigma(XQ))dX \\ &= |W_1|^{((r-n)/2)2^{r(r-n)/2}} \sqrt{(\det A^{-1})^r} \exp\left(\frac{\pi}{2}\sigma({}^t Q A^{-1} Q W_1^{-1})\right), \end{aligned}$$

where $\sqrt{(\det A^{-1})^r}$ is real positive if A is real.

Proof. Put ${}^t X = ({}^t x_1, \dots, {}^t x_r)$ and ${}^t x = (x_1, \dots, x_r) \in M_{1,(n-r)r}(\mathbb{R})$. Then we have $\sigma(XA{}^t X) = {}^t x \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix} x$ where rA 's are on the diagonal. Denoting QW^{-1} by (y_1, \dots, y_r) where $W = \sqrt{W_1} > 0$, we have $\sigma(XQW^{-1}) = ({}^t y_1, \dots, {}^t y_r)x$. Thus the integral of the left side is

$$\begin{aligned} & |W|^{r-n} \int_{X \in M_{r,n-r}(\mathbb{R})} \exp(-2\pi\sigma(XA{}^t X) + 2\pi\sigma(XQW^{-1}))dX \\ &= |W_1|^{(r-n)/2} \int_{\mathbb{R}^{(n-r)r}} \exp\left(-2\pi{}^t x \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix} x + 2\pi({}^t y_1, \dots, {}^t y_r)x\right)dx \end{aligned}$$

$$= |W_1|^{(r-n)/2} 2^{r(\tau-n)/2} \sqrt{(\det A^{-1})^r} \exp\left(\frac{\pi}{2} \sigma({}^t W^{-1} {}^t Q A^{-1} Q W^{-1})\right), \quad \text{Q.E.D.}$$

LEMMA 7. *If N is an element of Γ_n such that $|C_N| \neq 0$, $(A_N C_N^{-1})_1 \equiv 0 \pmod{1}$, then we have*

$$(f|N)(Z) = |C_N|^k |W_4|^{-k} f(W_1 - W_2 W_4^{-1} {}^t W_2),$$

where

$$W = \begin{pmatrix} W_1^{(r)} & W_2 \\ {}^t W_2 & W_4 \end{pmatrix} = C_N Z {}^t C_N + D_N {}^t C_N.$$

Proof. Put $N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Since $B - AC^{-1}D = B - A({}^t C^{-1}D) = (B {}^t C - A {}^t D) {}^t C^{-1} = -{}^t C^{-1}$, we have $N\langle Z \rangle = (AZ + B)(CZ + D)^{-1} = AC^{-1} - {}^t C^{-1}(CZ + D)^{-1} = AC^{-1} - W^{-1}$. From the identity

$$W = \begin{pmatrix} 1 & W_2 W_4^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} W_1 - W_2 W_4^{-1} {}^t W_2 & 0 \\ 0 & W_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ W_4^{-1} {}^t W_2 & 1 \end{pmatrix}$$

follows

$$W^{-1} = \begin{pmatrix} (W_1 - W_2 W_4^{-1} {}^t W_2)^{-1} & * \\ * & * \end{pmatrix}.$$

Hence we have

$$\begin{aligned} (f|N)(Z) &= f(-(W_1 - W_2 W_4^{-1} {}^t W_2)^{-1}) |W {}^t C^{-1}|^{-k} \\ &= f(W_1 - W_2 W_4^{-1} {}^t W_2) |W_1 - W_2 W_4^{-1} {}^t W_2|^k |W|^{-k} |C|^k \\ &= |C|^k |W_4|^{-k} f(W_1 - W_2 W_4^{-1} {}^t W_2). \end{aligned} \quad \text{Q.E.D.}$$

Let $N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an element of Γ_n such that $|C| \neq 0$, $(AC^{-1})_1 \equiv 0 \pmod{1}$, $C_3 = 0$ where we decompose $M \in M_n(\mathbb{C})$ as $M = \begin{pmatrix} M_1^{(r)} & M_2 \\ M_3 & M_4 \end{pmatrix}$ as in §1 and take a natural number p such that pC^{-1} is an integral matrix. We fix N, p till Lemma 13. Now we calculate

$$\sum_{S = \begin{pmatrix} 0 & S_2 \\ {}^t S_2 & S_4 \end{pmatrix} \in p^2 A_n} \left(f|N \left(\frac{1}{0} \middle| \frac{C^{-1} S {}^t C^{-1}}{1} \right) \right) (Z).$$

Put $W = CZ {}^t C + D {}^t C$. For $S = \begin{pmatrix} 0 & S_2 \\ {}^t S_2 & S_4 \end{pmatrix} \in A_n$ we have

$$N \left(\frac{1}{0} \middle| \frac{C^{-1} S {}^t C^{-1}}{1} \right) = \left(\frac{*}{C} \middle| \frac{*}{S {}^t C^{-1} + D} \right)$$

and $CZ^tC + (S^tC^{-1} + D)^tC = W + S$. If, moreover, $S \in p^2\Lambda_n$, then

$$\begin{aligned} & \left(f \mid N \left(\frac{1}{0} \middle| \frac{C^{-1}S^tC^{-1}}{1} \right) \right) (Z) \\ &= |C|^k |W_4 + S_4|^{-k} f(W_1 - (W_1 + S_2)(W_4 + S_4)^{-1}{}^t(W_2 + S_2)). \end{aligned}$$

First we calculate, for $t > 0$, $t \in \Lambda_r^*$,

$$\sum_{S_2 \in p^2M_{r,n-r}(Z)} e(-\sigma(t(W_2 + S_2)W_4^{-1}{}^t(W_2 + S_2))).$$

It equals

$$\begin{aligned} & \sum_{S_2 \in M_{r,n-r}(Z)} e(-\sigma(t(W_2 + p^2S_2)W_4^{-1}{}^t(W_2 + p^2S_2))) \\ &= \sum_{S_2 \in M_{r,n-r}(Z)} \int_{X \in M_{r,n-r}(R)} e(-\sigma(t(W_2 + p^2X)W_4^{-1}{}^t(W_2 + p^2X))) e(\sigma(S_2^tX)) dX \\ &= e(-\sigma(tW_2W_4^{-1}{}^tW_2)) \sum_{S_2 \in M_{r,n-r}(Z)} \int_{X \in M_{r,n-r}(R)} \exp(-2\pi\sigma(p^4tX(iW_4^{-1})^tX)) \\ & \quad \times \exp(2\pi\sigma(X(-2ip^2W_4^{-1}{}^tW_2t + i^tS_2))) dX \\ &= e(-\sigma(tW_2W_4^{-1}{}^tW_2)) \sum_{S_2 \in M_{r,n-r}(Z)} |p^4t|^{(r-n)/2} 2^{r(r-n)/2} \sqrt{\det(i^{-1}W_4)^r} \\ & \quad \times \exp((\pi/2)\sigma^t(Q(i^{-1}W_4)Qp^{-4}t^{-1})), \end{aligned}$$

where we put $Q = -2ip^2W_4^{-1}{}^tW_2t + i^tS_2$.

Since

$$\begin{aligned} & \sigma^t(Q(i^{-1}W_4)Qp^{-4}t^{-1}) \\ &= p^{-4}i\sigma(4p^4W_2W_4^{-1}{}^tW_2t - 4p^2W_2^tS_2 + S_2W_4^tS_2t^{-1}), \end{aligned}$$

we have

$$\begin{aligned} & \sum_{S_2 \in p^2M_{r,n-r}(Z)} e(-\sigma(t(W_2 + S_2)W_4^{-1}{}^t(W_2 + S_2))) \\ &= 2^{r(r-n)/2} \cdot p^{2r(r-n)} |t|^{(r-n)/2} \sqrt{\det(i^{-1}W_4)^r} \\ & \quad \times \sum_{S_2 \in M_{r,n-r}(Z)} e((4p^4)^{-1}\sigma(-4p^2W_2^tS_2 + S_2W_4^tS_2t^{-1})). \end{aligned}$$

Put $f(z^{(r)}) = \sum_{t \in \Lambda_r^*} b(t)e(\sigma(tz))$; then $b(t) = O(|t|^{k/2})$ is known [7].

LEMMA 8.

$$\begin{aligned} & \sum_{S = \begin{pmatrix} 0 & S_2 \\ tS_2 & 0 \end{pmatrix} \in p^2\Lambda_n} \left(f \mid N \left(\frac{1}{0} \middle| \frac{C^{-1}S^tC^{-1}}{1} \right) \right) (Z) \\ &= |C|^k 2^{r(r-n)/2} p^{2r(r-n)} |W_4|^{-k} \sqrt{\det(i^{-1}W_4)^r} \\ & \quad \times \sum_{\substack{t > 0 \\ t \in \Lambda_r^* \\ S_2 \in M_{r,n-r}(Z)}} |t|^{(r-n)/2} b(t) e(\sigma(tW_1) + (4p^4)^{-1}\sigma(-4p^2W_2^tS_2 + S_2W_4^tS_2t^{-1})) \end{aligned}$$

where the right hand is absolutely convergent.

Proof. Define a matrix $P \in M_r(\mathbf{R})$ by

$$P = \text{Im} (W_1 - (W_2 + X)W_4^{-1} {}^t(W_2 + X)) - (X + Q) \text{Im} (-W_4^{-1}) {}^t(X + Q),$$

where $Q = \text{Re } W_2 + \text{Im } W_2 \cdot \text{Re} (-W_4^{-1})(\text{Im} (-W_4^{-1}))^{-1}$, $X \in M_{r, n-r}(\mathbf{R})$. Then P is independent of X . Since $(W_1 - (W_2 + X)W_4^{-1} {}^t(W_2 + X))^{-1}$ is the upper left $r \times r$ matrix of

$$\left(W + \begin{pmatrix} 0 & X \\ {}^tX & 0 \end{pmatrix} \right)^{-1}$$

we have $W_1 - (W_2 + X)W_4^{-1} {}^t(W_2 + X) \in H_r$ and its imaginary part is positive definite. Hence, putting $X = -Q$, we see that P is positive definite. Now we have

$$\begin{aligned} & |e(\sigma(t(W_1 - (W_2 + S_2)W_4^{-1} {}^t(W_2 + S_2))))| \\ &= \exp(-2\pi\sigma(tP + t(S_2 + Q) \text{Im} (-W_4^{-1}) {}^t(S_2 + Q))) \\ &< \exp(-2\pi\varepsilon\sigma(t + (S_2 + Q) {}^t(S_2 + Q))), \end{aligned}$$

where $\varepsilon > 0$ is defined by

$$P > \varepsilon 1_r, \quad \text{Im} (-W_4^{-1}) > \sqrt{\varepsilon} 1_{n-r}, \quad t > \sqrt{\varepsilon} 1_r \quad \text{for } t \in A_r^*, t > 0.$$

Then it is easy to see that

$$\sum_{\substack{t > 0 \\ t \in A_r^* \\ S_2 \in p^2 M_{r, n-r}(\mathbf{Z})}} b(t)e(\sigma(t(W_1 - (W_2 + S_2)W_4^{-1} {}^t(W_2 + S_2))))$$

is absolutely convergent.

To prove that the right hand is absolutely convergent, it is enough to show

$$\begin{aligned} & \sum_{S_2 \in M_{r, n-r}(\mathbf{Z})} |e(\sigma(tW_1) + (4p^4)^{-1}\sigma(-4p^2W_2 {}^tS_2 + S_2W_4 {}^tS_2t^{-1}))| \\ &= O(|t|^{n-r} \exp(-2\pi\varepsilon\sigma(t))) \quad \text{for some } \varepsilon > 0. \end{aligned}$$

$\text{Im} (\sigma(tW_1) + (4p^4)^{-1}\sigma(-4p^2W_2 {}^tS_2 + S_2W_4 {}^tS_2t^{-1}))$ is equal to

$$\begin{aligned} & \sigma\left(\text{Im } W\right) \begin{pmatrix} t & -(2p^2)^{-1}S_2 \\ -(2p^2)^{-1} {}^tS_2 & (4p^4)^{-1} {}^tS_2t^{-1}S_2 \end{pmatrix} \\ &= \sigma\left(\begin{pmatrix} 1 & -(2p^2)^{-1}t^{-1}S_2 \\ 0 & 1 \end{pmatrix} \text{Im } W\right) \begin{pmatrix} 1 & 0 \\ -(2p^2)^{-1} {}^tS_2t^{-1} & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and, taking a positive number ε_1 such that $\text{Im } W > \varepsilon_1 1_n$, we have

$$\begin{aligned} & |e(\sigma(tW_1) + (4p^4)^{-1}\sigma(-4p^2W_2^tS_2 + S_2W_4^tS_2t^{-1}))| \\ & \leq \exp(-2\pi\varepsilon_1\sigma(t + (4p^4)^{-1}S_2^tS_2t^{-1})). \end{aligned}$$

Hence we have only to prove that, for $\varepsilon' = 2\pi\varepsilon_1(4p^4)^{-1}$,

$$\sum_{S_2 \in M_{r,n-r}(\mathbb{Z})} \exp(-\varepsilon'\sigma(S_2^tS_2t^{-1})) = O(|t|^{n-r}) \quad \text{for } t > 0, t \in A_r^*.$$

Without loss of generality we may assume that t^{-1} is in some Siegel domain. Then there are positive constants $\varepsilon_2, \varepsilon_3$ such that

$$\varepsilon_2\delta < t^{-1} < \varepsilon_3\delta,$$

where δ is a diagonal matrix defined by

$$\delta = \begin{pmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_r \end{pmatrix} = {}^tT^{-1}t^{-1}T^{-1}, \quad T = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

Then $t < \varepsilon_2^{-1}\delta^{-1}$ and $t \in A_r^*$ imply that $\delta_i < \varepsilon_2^{-1}$. Therefore we have

$$\begin{aligned} & \sum_{S_2 \in M_{r,n-r}(\mathbb{Z})} \exp(-\varepsilon'\sigma(S_2^tS_2t^{-1})) < \sum_{S_2 \in M_{r,n-r}(\mathbb{Z})} \exp(-\varepsilon'\varepsilon_2\sigma(S_2^tS_2\delta)) \\ & = \sum \exp\left(-\varepsilon_4 \sum_{i,j} \delta_i s_{ij}^2\right) \quad (\varepsilon_4 = \varepsilon'\varepsilon_2) \\ & = \prod_i \left(\sum_{s \in \mathbb{Z}} \exp(-\varepsilon_4\delta_i s^2)\right)^{n-r} \leq \prod_i \left(1 + 2 \sum_{s \geq 1} \exp(-\varepsilon_4\delta_i s)\right)^{n-r} \\ & = \prod_i \left(1 + 2 \frac{\exp(-\varepsilon_4\delta_i)}{1 - \exp(-\varepsilon_4\delta_i)}\right)^{n-r} \\ & = \prod_i \left(1 + \frac{2}{\exp(\varepsilon_4\delta_i) - 1}\right)^{n-r} < \prod_i \left(1 + \frac{2}{\varepsilon_4\delta_i}\right)^{n-r} \\ & = \prod_i \left(\frac{\varepsilon_4\delta_i + 2}{\varepsilon_4\delta_i}\right)^{n-r} < \left(\prod_i \varepsilon_4^{-1}(\varepsilon_4\varepsilon_2^{-1} + 2)\right)^{n-r} |t|^{n-r}. \end{aligned}$$

Now the calculation before Lemma 8 implies the identity in Lemma 8. Q.E.D.

LEMMA 9. For $t > 0, t \in A_r^*$ we have

$$\begin{aligned} & \sum_{S_4 \in p^{2k}A_{n-r}} |W_4 + S_4|^{-k} \sqrt{\det(i^{-1}(W_4 + S_4))} e((4p^4)^{-1}\sigma((W_4 + S_4)^t S_2 t^{-1} S_2)) \\ &= i^{(n-r)k} (2\pi)^{(n-r)(k-r/2)} 2^{(r-n)(n-r-1)/2} p^{(r-n)(n-r+1)} (4p^4 b_i)^{(r-n)(2k-n-1)/2} \\ & \quad \times \Gamma_{n-r}(k-r/2)^{-1} e((4p^4)^{-1}\sigma(W_4^t S_2 t^{-1} S_2)) \\ & \quad \times \sum_T |T|^{k-(n+1)/2} e((4p^4 b_i)^{-1}\sigma(TW_4)) , \end{aligned}$$

where b_i is any fixed natural number such that $b_i t^{-1}$ is integral, $S_2 \in M_{r, n-r}(\mathbb{Z})$, and T runs over

$$\{T \mid T > 0, T \in A_{n-r}^*, (4p^2)^{-1}({}^t S_2 t^{-1} S_2 + b_i^{-1} T) \in A_{n-r}^*\} .$$

Proof. The left side equals

$$\begin{aligned} & i^{k(n-r)} e((4p^4)^{-1}\sigma(W_4^t S_2 t^{-1} S_2)) \sum_{S_4 \in A_{n-r}} |-i(W_4 + p^2 S_4)|^{-(k-r/2)} e((4p^2)^{-1}\sigma(S_4^t S_2 t^{-1} S_2)) \\ &= i^{k(n-r)} e((4p^4)^{-1}\sigma(W_4^t S_2 t^{-1} S_2)) \sum_{S_4 \in A_{n-r} \bmod 4p^2 b_i} e((4p^2)^{-1}\sigma(S_4^t S_2 t^{-1} S_2)) \\ & \quad \times \sum_{S_4 \in A_{n-r}} |-i(W_4 + p^2 S_4 + 4p^4 b_i S_4)|^{-(k-r/2)} \\ &= i^{k(n-r)} e((4p^4)^{-1}\sigma(W_4^t S_2 t^{-1} S_2)) \sum_{S_4 \in A_{n-r} \bmod 4p^2 b_i} e((4p^2)^{-1}\sigma(S_4^t S_2 t^{-1} S_2)) \\ & \quad \times (2\pi)^{(r-n)(k-r/2)} 2^{(r-n)(n-r-1)/2} (2\pi)^{2(n-r)(k-r/2)} \Gamma_{n-r}(k-r/2)^{-1} \\ & \quad \times (4p^4 b_i)^{-(n-r)(k-r/2)} \sum_{\substack{T > 0 \\ T \in A_{n-r}^*}} |T|^{k-(n+1)/2} e((4p^4 b_i)^{-1}\sigma(T(W_4 + p^2 S_4))) . \end{aligned}$$

From

$$\begin{aligned} & \sum_{S_4 \in A_{n-r} \bmod 4p^2 b_i} e((4p^2)^{-1}\sigma(S_4^t S_2 t^{-1} S_2) + (4p^4 b_i)^{-1}\sigma(p^2 T S_4)) \\ &= \begin{cases} (4p^2 b_i)^{(n-r)(n-r+1)/2} & \text{if } (4p^2)^{-1}({}^t S_2 t^{-1} S_2 + b_i^{-1} T) \in A_{n-r}^* , \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

follows the identity in Lemma 9.

Q.E.D.

Now we have

$$\begin{aligned} & \sum_{S = \begin{pmatrix} 0 & S_2 \\ {}^t S_2 & S_4 \end{pmatrix} \in p^{2k}A_n} \left(f \mid N \left(\frac{1}{0} \mid \frac{C^{-1} S^t C^{-1}}{1} \right) \right) (\mathbb{Z}) \\ &= \sum_{S_4 \in p^{2k}A_{n-r}} |C|^k 2^{r(r-n)/2} p^{2r(r-n)} \sum_{\substack{t > 0 \\ t \in A_n^* \\ S_2 \in M_{r, n-r}(\mathbb{Z})}} |t|^{(r-n)/2} b(t) \\ & \quad \times e(\sigma(tW_4) - p^{-2}\sigma(W_2^t S_2)) |W_4 + S_4|^{-k} \sqrt{\det(i^{-1}(W_4 + S_4))} \\ & \quad \times e((4p^4)^{-1}\sigma((W_4 + S_4)^t S_2 t^{-1} S_2)) \\ &= |C|^k 2^{(r-n)(n-1)/2} p^{(r-n)(n+r+1)} i^{(n-r)k} (2\pi)^{(n-r)(k-r/2)} \\ & \quad \times \Gamma_{n-r}(k-r/2)^{-1} \sum_{t, S_2, T} |t|^{(r-n)/2} (4p^4 b_i)^{(r-n)(2k-n-1)/2} \end{aligned}$$

$$\begin{aligned} &\times |T|^{k-(n+1)/2} b(t) e(\sigma(tW_1) - p^{-2}\sigma(W_2^t S_2) + (4p^4)^{-1}\sigma(W_4^t S_2 t^{-1} S_2) \\ &\qquad\qquad\qquad + (4p^4 b_t)^{-1}\sigma(TW_4)), \end{aligned}$$

where

$$\begin{aligned} &t > 0, \quad t \in A_r^*, \quad S_2 \in M_{r,n-r}(\mathbb{Z}), \quad T > 0, \\ &T \in A_{n-r}^*, \quad (4p^2)^{-1}({}^t S_2 t^{-1} S_2 + b_t^{-1} T) \in A_{n-r}^*. \end{aligned}$$

Put

$$P = \begin{pmatrix} t & -(2p^2)^{-1} S_2 \\ -(2p^2)^{-1} {}^t S_2 & (4p^4)^{-1} (b_t^{-1} T + {}^t S_2 t^{-1} S_2) \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ {}^t P_2 & P_4 \end{pmatrix};$$

then

$$P = \begin{pmatrix} 1 & 0 \\ -(2p^2)^{-1} {}^t S_2 t^{-1} & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & (4p^4 b_t)^{-1} T \end{pmatrix} \begin{pmatrix} 1 & -(2p^2)^{-1} t^{-1} S_2 \\ 0 & 1 \end{pmatrix}$$

implies that P is positive definite and $|P| = (4p^4 b_t)^{r-n} |t| |T|$. Our assumptions on t, S_2, T mean that $P_1 \in A_r^*, 2p^2 P_2 \in M_{r,n-r}(\mathbb{Z}), p^2 P_4 \in A_{n-r}^*$, and $b_t 4p^4 P_4 - b_t {}^t S_2 t^{-1} S_2 \in A_{n-r}^*$. $\{t, S_2, T\}$ and P correspond bijectively and

$$\sigma(WP) = \sigma(W_1 t) - p^{-2}\sigma(W_2^t S_2) + (4p^4)^{-1}\sigma(W_4(b_t^{-1} T + {}^t S_2 t^{-1} S_2)).$$

Thus we have

LEMMA 10.

$$\begin{aligned} &\sum_{S = \begin{pmatrix} 0 & S_2 \\ {}^t S_2 & S_4 \end{pmatrix} \in p^2 A_n} \left(f \mid N \left(\frac{1}{0} \mid \frac{C^{-1} S^t C^{-1}}{1} \right) \right) (\mathbb{Z}) \\ &= |C|^{k 2^{(r-n)(n-1)/2} p^{(r-n)(n+r+1)} i^{(n-r)k} (2\pi)^{(n-r)(k-r/2)}} \\ &\quad \times \Gamma_{n-r}(k-r/2)^{-1} \sum_{P > 0} b(P) |P_1|^{(r+1)/2-k} |P|^{k-(n+1)/2} e(\sigma(PW)), \end{aligned}$$

where

$$P_1 \in A_r^*, \quad 2p^2 P_2 \in M_{r,n-r}(\mathbb{Z}), \quad p^2 P_4 \in A_{n-r}^*.$$

LEMMA 11. Put $G = \{({}^t S_2 {}^t C_1^{-1} - S_4 {}^t (C_1^{-1} C_2 C_4^{-1}), S_4 {}^t C_4^{-1}) \mid S_2 \in p^2 M_{r,n-r}(\mathbb{Z}), S_4 \in p^2 A_{n-r}\}$, $G' = \{(C_4 {}^t S_2, C_4 S_4) \mid S_2 \in M_{r,n-r}(\mathbb{Z}), S_4 \in A_{n-r}\}$. Then we have $[G': G] = p^{(n-r)(n+r+1)} \text{abs}(|C_1|^{r-n} |C_4|^{-n-1})$.

Proof. By definition $C = \begin{pmatrix} C_1 & C_2 \\ & C_4 \end{pmatrix}$ and $pC^{-1} = p \begin{pmatrix} C_1^{-1} & -C_1^{-1} C_2 C_4^{-1} \\ & C_4^{-1} \end{pmatrix}$ are integral. It implies $G' \supset G$. Put $G_0 = M_{n-r,r}(\mathbb{Z}) \times \{C_4 S_4 \mid S_4 \in A_{n-r}\}$; then $[G_0: G'] = \text{abs} |C_4|^r$. As representatives of G_0/G we can take representatives of $\{C_4 S_4 \mid S_4 \in A_{n-r}\} / \{S_4 {}^t C_4^{-1} \mid S_4 \in p^2 A_{n-r}\}$ and then representatives of

$$M_{n-r,r}(Z) / \{ {}^t S_2 {}^t C_1^{-1} | S_2 \in p^2 M_{r,r-n}(Z) \} .$$

Hence we have

$$\begin{aligned} [G_0 : G] &= [\{ C_4 S_4 | S_4 \in A_{n-r} \} : \{ S_4 {}^t C_4^{-1} | S_4 \in p^2 A_{n-r} \}] \\ &\quad \times [M_{n-r,r}(Z) : \{ {}^t S_2 {}^t C_1^{-1} | S_2 \in p^2 M_{n-r,r}(Z) \}] \\ &= [A_{n-r} : p^2 C_4^{-1} A_{n-r} {}^t C_4^{-1}] abs | p^2 {}^t C_1^{-1} |^{n-r} \\ &= abs | p C_4^{-1} |^{n-r+1} | p^2 {}^t C_1^{-1} |^{n-r} . \end{aligned}$$

Thus we have $[G' : G] = p^{(n-r)(n+r+1)} abs (| C_1 |^{r-n} | C_4 |^{-n-1})$. Q.E.D.

It is obvious that

$$\begin{aligned} G &= \left\{ \widetilde{S^t C^{-1}} | S = \begin{pmatrix} 0 & S_2 \\ {}^t S_2 & S_4 \end{pmatrix} \in p^2 A_n \right\} , \\ G' &= \left\{ \widetilde{CS} | S = \begin{pmatrix} 0 & S_2 \\ {}^t S_2 & S_4 \end{pmatrix} \in A_n \right\} \end{aligned}$$

and

$$N \left(\frac{1}{0} \middle| \frac{C^{-1} S^t C^{-1}}{1} \right) = (\widetilde{C}, \widetilde{D} + \widetilde{S^t C^{-1}}) , \quad N \left(\frac{1}{0} \middle| \frac{S}{1} \right) = (\widetilde{C}, \widetilde{D} + \widetilde{CS}) .$$

We take

$$S'_i = \begin{pmatrix} 0 & S_{i,2} \\ {}^t S_{i,2} & S_{i,4} \end{pmatrix} \in A_n$$

such that $\widetilde{CS'_i}$ is representatives of G'/G . For $M \in \Gamma_n$, $(f|M)(Z)$ is uniquely determined by \widetilde{M} . Hence we may write $(f|\widetilde{M})(Z)$ for $(f|M)(Z)$. Then we have

$$\begin{aligned} \sum_{s = \begin{pmatrix} 0 & S_2 \\ {}^t S_2 & S_4 \end{pmatrix} \in A_n} \left(f | N \left(\frac{1}{0} \middle| \frac{S}{1} \right) \right) (Z) &= \sum_{g \in G'} (f | (\widetilde{C}, \widetilde{D} + g)) (Z) \\ &= \sum_i \sum_{g \in G'} (f | (\widetilde{C}, \widetilde{D} + \widetilde{CS'_i} + g)) (Z) \\ &= \sum_i \sum_{s = \begin{pmatrix} 0 & S_2 \\ {}^t S_2 & S_4 \end{pmatrix} \in p^2 A_n} \left(f | N \left(\frac{1}{0} \middle| \frac{S'_i}{1} \right) \left(\frac{1}{0} \middle| \frac{C^{-1} S^t C^{-1}}{1} \right) \right) (Z) . \end{aligned}$$

For

$$M = N \left(\frac{1}{0} \middle| \frac{S'_i}{1} \right)$$

we have $C_M = C_N = C$, $D_M = CS'_i + D$, $(A_M C_M^{-1})_1 \equiv 0 \pmod{1}$ and $C_M Z^t C_M$

+ $D_M {}^t C_M = W + CS_i {}^t C$. Applying Lemma 10, we have

$$\begin{aligned} & \sum_{S = \begin{pmatrix} 0 & S_2 \\ {}^t S_2 & S_4 \end{pmatrix} \in A_n} \left(f \left| N \left(\frac{1}{0} \middle| \frac{S}{1} \right) \right. \right) (Z) \\ &= \alpha |C|^k p^{(r-n)(n+r+1)} \sum_{P > 0} b(P_i) |P_1|^{(r+1)/2-k} |P|^{k-(n+1)/2} \\ & \quad \times e(\sigma(P(W + CS_i {}^t C))), \end{aligned}$$

where $\alpha = i^{(n-r)k} 2^{(r-n)(n-1)/2} (2\pi)^{(n-r)(k-r/2)} \Gamma_{n-r}(k-r/2)^{-1}$, and P runs over

$$\{P > 0 \mid P_1 \in A_r^*, 2p^2 P_2 \in M_{r,n-r}(Z), p^2 P_4 \in A_{n-r}^*\}.$$

We fix P such that $P_1 \in A_r^*, 2p^2 P_2 \in M_{r,n-r}(Z), p^2 P_4 \in A_{n-r}^*$ and we put $\chi(g) = e(2\sigma(g_1 {}^t C_1 P_2) + 2\sigma(g_2 {}^t C_2 P_2) + \sigma(g_2 {}^t C_4 P_4))$ for $g = (g_1^{(n-r,r)}, g_2^{(n-r,n-r)}) \in G'$. It is easy to see that $\chi(g) = 1$ for $g \in G$ and $e(\sigma(PCS {}^t C)) = \chi((C_4 {}^t S_2, C_4 S_4)) = \chi(\widetilde{CS})$ for $S = \begin{pmatrix} 0 & S_2 \\ {}^t S_2 & S_4 \end{pmatrix} \in A_n$. Therefore the definition of S_i implies

$$\sum_i e(\sigma(PCS_i {}^t C)) = \sum_{g \in G'/G} \chi(g).$$

χ is trivial if and only if $2 {}^t C_1 P_2 C_4 \in M_{r,n-r}(Z)$ and ${}^t C_2 P_2 C_4 + {}^t C_4 {}^t P_2 C_2 + {}^t C_4 P_4 C_4 \in A_{n-r}^*$. Put $T = {}^t CPC$; then we have

LEMMA 12. $P_1 \in A_r^*, 2p^2 P_2 \in M_{r,n-r}(Z), p^2 P_4 \in A_{n-r}^*, 2 {}^t C_1 P_2 C_4 \in M_{r,n-r}(Z)$ and ${}^t C_2 P_2 C_4 + {}^t C_4 {}^t P_2 C_2 + {}^t C_4 P_4 C_4 \in A_{n-r}^*$ if and only if $T \in A_n^*$ and ${}^t C_1^{-1} T_1 C_1^{-1} = ({}^t C^{-1} T C^{-1})_1 \in A_r^*$.

Proof. $T = {}^t CPC$ implies $T_1 = {}^t C_1 P_1 C_1, T_2 = {}^t C_1 P_1 C_2 + {}^t C_1 P_2 C_4, T_4 = {}^t C_2 P_1 C_2 + {}^t C_4 {}^t P_2 C_2 + {}^t C_2 P_2 C_4 + {}^t C_4 P_4 C_4$. The assumptions on P imply ${}^t C_1^{-1} T_1 C_1^{-1} = P_1 \in A_r^*$ and so $T_1 \in A_r^*$. $2T_2 = 2 {}^t C_1 P_1 C_2 + 2 {}^t C_1 P_2 C_4 \in M_{r,n-r}(Z)$ holds since $P_1 \in A_r^*, 2 {}^t C_1 P_2 C_4 \in M_{r,n-r}(Z)$. $T_4 = {}^t C_2 P_1 C_2 + ({}^t C_4 {}^t P_2 C_2 + {}^t C_2 P_2 C_4 + {}^t C_4 P_4 C_4) \in A_{n-r}^*$. Thus “only if” part has been proved. Conversely, assume $T \in A_n^*, {}^t C_1^{-1} T_1 C_1^{-1} \in A_r^*$. Then $P_1 = {}^t C_1^{-1} T_1 C_1^{-1} \in A_r^*, {}^t C_2 P_2 C_4 + {}^t C_4 {}^t P_2 C_2 + {}^t C_4 P_4 C_4 = T_4 - {}^t C_2 P_1 C_2 \in A_{n-r}^*, 2 {}^t C_1 P_2 C_4 = 2T_2 - 2 {}^t C_1 P_1 C_2 \in M_{r,n-r}(Z), 2p^2 P_2 = 2(p {}^t C_1^{-1}) ({}^t C_1 P_2 C_4) (p C_4^{-1}) \in M_{r,n-r}(Z)$ follow easily. From $P_4 - {}^t C_4^{-1} T_4 C_4^{-1} + {}^t C_4^{-1} {}^t C_2 P_1 C_2 C_4^{-1} = -{}^t P_2 C_2 C_4^{-1} - {}^t C_4^{-1} {}^t C_2 P_2 = -{}^t C_4^{-1} ({}^t T_2 - {}^t C_2 P_1 C_1) C_1^{-1} C_2 C_4^{-1} - {}^t (C_1^{-1} C_2 C_4^{-1}) (T_2 - {}^t C_1 P_1 C_2) C_4^{-1}$ and $p C^{-1} = p \begin{pmatrix} C_1^{-1} & -C_1^{-1} C_2 C_4^{-1} \\ 0 & C_4^{-1} \end{pmatrix} \in M_n(Z)$ follows $p^2 P_4 \in A_{n-r}^*$. Q.E.D.

Summarizing we have

LEMMA 13.

$$\begin{aligned} & \sum_{S = \begin{pmatrix} 0 & S_2 \\ {}^t S_2 & S_1 \end{pmatrix} \in A_n} \left(f | N \left(\frac{1}{0} \middle| \frac{S}{1} \right) \right) (Z) \\ &= \alpha |C_1|^k |C_4|^{-k} \sum_{\substack{T > 0 \\ T \in A_n^*}} b(({}^t C^{-1} T C^{-1})_1) |T_1|^{(r+1)/2-k} \\ & \quad \times |T|^{k-(n+1)/2} e(\sigma(TC^{-1}D)) e(\sigma(TZ)), \end{aligned}$$

where we suppose $b(({}^t C^{-1} T C^{-1})_1) = 0$ if $({}^t C^{-1} T C^{-1})_1 \notin A_n^*$.

This lemma, Proposition 1 and Theorem in § 1 imply the former part of the following

THEOREM. *Let r, n, k , be natural numbers such that $1 \leq r \leq n - 1$, $k \geq n + r + 2$, $k \equiv 0 \pmod 2$, and let $f(z^{(r)}) = \sum_{t>0} b(t)e(\sigma(tz))$ be a cusp form of degree r and weight k . If we put*

$$E_{n,r}^k(Z, f) = \sum_{M \in J_{n,r} \backslash \Gamma_n} (f | M)(Z) = \sum_{\substack{T > 0 \\ T \in A_n^*}} a(T, f) e(\sigma(TZ)),$$

then we have

$$\begin{aligned} a(T, f) &= \alpha \sum_{\substack{(C_4, D_3, D_4) \in \mathfrak{S}_{n,r}/\approx \\ {}^t U \in P_{n,r} \backslash GL(n, \mathbb{Z})}} |C_1|^k |C_4|^{-k} b(({}^t C^{-1} U^{-1} T^t U^{-1} C^{-1})_1) \\ & \quad \times |(U^{-1} T^t U^{-1})_1|^{(r+1)/2-k} |T|^{k-(n+1)/2} e(\sigma(U^{-1} T^t U^{-1} C^{-1} D)) \end{aligned}$$

for $T > 0$, where M_1 stands for the upper left $r \times r$ matrix of M , (C, D) stands for any fixed symmetric coprime pair such that

$$(\tilde{C}, \tilde{D}) = (0, C_4, D_3, D_4), \quad C = \begin{pmatrix} C_1 & C_2 \\ 0 & C_4 \end{pmatrix}, \quad |C| \neq 0$$

and $(AC^{-1})_1 \equiv 0 \pmod 1$ for some $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$, and

$$\begin{aligned} \alpha &= i^{(n-r)k} 2^{(r-n)(n-1)/2} (2\pi)^{(n-r)(k-r/2)} \\ & \quad \times \left(\pi^{(n-r)(n-r-1)/4} \prod_{\nu=0}^{n-r-1} \Gamma(k - (r + \nu)/2) \right)^{-1}. \end{aligned}$$

Moreover we have $a(T, f) = O(|T_1|^{-(k-r-1)/2} |T|^{k-(n+1)/2})$ if $T > 0$ runs over any fixed Siegel domain.

To prove the latter part we prepare the following

LEMMA 14. *The number of (D_3, D_4) such that (C_4, D_3, D_4) is representatives of $\mathfrak{S}_{n,r}/\approx$ for fixed C_4 ($|C_4| \neq 0$) is at most $\text{abs} |C_4|^r \delta_1^{n-r} \cdots \delta_{n-r}$ where $\delta_1 | \cdots | \delta_{n-r}$ are elementary divisors of C_4 .*

Proof. By the definition of the relation \approx the number of inequivalent (C_4, D_3, D_4) for fixed C_4 is at most $[M_{n-r,r}(\mathbf{Z}):C_4M_{n-r,r}(\mathbf{Z})] \times [\{D_4 \in M_{n-r}(\mathbf{Z}) \mid C_4^{-1}D_4 = {}^t(C_4^{-1}D_4)\}:C_4A_{n-r}]$. $[M_{n-r,r}(\mathbf{Z}):C_4M_{n-r,r}(\mathbf{Z})] = abs|C_4|^r$ is obvious. The latter part equals $[\{S = {}^tS \in M_{n-r}(\mathbf{Q}) \mid C_4S \in M_{n-r}(\mathbf{Z})\}:A_{n-r}]$. Put $C_4 = g_1\delta g_2$ where $g_i \in GL(n-r, \mathbf{Z})$,

$$\delta = \begin{pmatrix} \delta_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \delta_{n-r} \end{pmatrix}, \quad \delta_1 \mid \cdots \mid \delta_{n-r}.$$

Then $\{S = {}^tS \in M_{n-r}(\mathbf{Q}) \mid C_4S \in M_{n-r}(\mathbf{Z})\} = \{S = {}^tS \in M_{n-r}(\mathbf{Q}) \mid \delta S \in M_{n-r}(\mathbf{Z})\}$. Hence the latter part equals $\delta_1^{n-r} \cdots \delta_{n-r}$. Q.E.D.

We can take

$$\left\{ \begin{pmatrix} c_1 & & & c_{ij} \\ & \ddots & & \\ 0 & & \ddots & \\ & & & c_{n-r} \end{pmatrix} \mid c_1, \dots, c_{n-r} > 0, 0 \leq c_{ij} < c_2 \right\}$$

as representatives of $GL(n-r, \mathbf{Z}) \setminus \{C_4 \in M_{n-r}(\mathbf{Z}) \mid |C_4| \neq 0\}$. If

$$C_4 = \begin{pmatrix} c_1 & & & c_{ij} \\ & \ddots & & \\ & & \ddots & \\ & & & c_{n-r} \end{pmatrix},$$

then $\delta_1^{n-r} \cdots \delta_{n-r} = \delta_1(\delta_1\delta_2) \cdots (\delta_1 \cdots \delta_{n-r}) \leq c_1(c_1c_2) \cdots (c_1 \cdots c_{n-r})$ where $\delta_1 \mid \cdots \mid \delta_{n-r}$ are elementary divisors of C_4 . Hence we have

$$\begin{aligned} a(T, f) &= O\left(\sum_{\substack{(C_4, D_3, D_4) \in \mathcal{S}_{n,r} / \approx \\ {}^tU \in P_{n,r} \setminus GL(n, \mathbf{Z})}} |C_4|^{-k} \times |(U^{-1}T^tU^{-1})_1|^{(\tau+1-k)/2} |T|^{k-(n+1)/2} \right) \\ &= \sum_{c_i=1}^{\infty} \left(\prod_{i=1}^{n-r} c_i \right)^{-k+\tau} (c_1^{n-r} \cdots c_{n-r})(c_2 \cdots c_{n-r}^{n-r-1}) \\ &\quad \times O\left(\sum_{{}^tU \in P_{n,r} \setminus GL(n, \mathbf{Z})} |(U^{-1}T^tU^{-1})_1|^{(\tau+1-k)/2} |T|^{k-(n+1)/2} \right), \end{aligned}$$

where the sum of c_i is equal to $\zeta(k-n)^{n-r}$, and the last sum is a so-called Selberg's zeta function and the order of the magnitude is $|T_1|^{(\tau+1-k)/2} |T|^{k-(n+1)/2}$ if T runs over any fixed Siegel domain (p. 143 and Theorem in p. 144 in [5]). This completes the proof of Theorem.

§3.

Let k, n be natural numbers such that $k \geq n + 2, k \equiv 0 \pmod{2}$ and put

$$E_n^k(\mathbf{Z}) = \sum |C\mathbf{Z} + D|^{-k} \quad (\mathbf{Z} \in H_n),$$

where (C, D) runs over representatives of $GL(n, \mathbf{Z}) \setminus \{\text{symmetric coprime pairs}\}$. Our aim is to prove

THEOREM. *Put*

$$E_n^k(\mathbf{Z}) = \sum_{\substack{T \geq 0 \\ T \in A_n^*}} a(T) e(\sigma(T\mathbf{Z})).$$

Then there are positive numbers c_1, c_2 such that

$$c_1 |T|^{k-(n+1)/2} < |a(T)| < c_2 |T|^{k-(n+1)/2} \quad \text{for } T > 0.$$

Put $S_T = \sum_R e(-\sigma(TR)) \nu(R)^{-k}$ for $T \in A_n^*$ where R runs over all $n \times n$ rational symmetric matrices modulo 1 and $\nu(R)$ is the product of denominators of elementary divisors of R . Then it is known that

$$a(T) = \text{const.} \times |T|^{k-(n+1)/2} S_T \quad \text{for } T > 0 \text{ ([9])}.$$

On the other hand $S_T = \sum |C|^{-k} e(-\sigma(TC^{-1}D))$ where (C, D) runs over the set $\{(C, D) | \text{symmetric coprime pair, } |C| \neq 0\} / \sim$. Here by definition $(C', D') \sim (C, D)$ if and only if there are $U \in GL(n, \mathbf{Z}), S \in A_n$ such that $(C', D') = U(C, D + CS)$. We take as representatives of C matrices of the form

$$\begin{pmatrix} c_1 & & & \\ & \ddots & & \\ & & c_{ij} & \\ & & & c_n \end{pmatrix}, \quad 0 \leq c_{ij} < c_j;$$

then the number of the choice of D for C is at most $c_1^n \cdots c_n$ as in the proof of Theorem in § 2. Thus we have

$$|S_T| \leq \sum_{c_i=1}^{\infty} (\prod c_i)^{-k} (\prod c_i^{i-1}) \prod c_i^{n+1-i} = \zeta(k-n)^n.$$

To complete the proof we must show $|S_T| > \epsilon$ where ϵ is a positive number independent of T .

Put $S_p(T) = \sum_R e(-\sigma(TR)) \nu(R)^{-k}$ where R runs over all $n \times n$ rational symmetric matrices modulo 1 such that $\nu(R)$ is a power of p . Then $S_T = \prod_p S_p(T)$ for $T > 0, T \in A_n^*$. Put

$$J = \frac{1}{2} \begin{pmatrix} 0 & 1_k \\ 1_k & 0 \end{pmatrix} \quad \text{and} \quad A_q(T) = \# \{C \in M_{2k,n}(\mathbf{Z}) \bmod q \mid q^{-1} {}^t C J C - T \in A_n^*\}.$$

Then we have

$$S_p(T) = (p^a)^{n(n+1)/2-2kn} A_{p^a}(T)$$

for sufficiently large a .

LEMMA 1. For $W \in \Lambda_n$ we put

$$G(W; p^a) = \sum_{\substack{C \in M_{2k,n}(\mathbb{Z}) \\ C \pmod{p^a} \\ \text{primitive mod } p}} e(p^{-a}\sigma({}^tCJCW)).$$

Then we have

$$G(W; p^a) = \begin{cases} p^{2kn(a-1)} G(p^{-(a-1)}W; p) & \text{if } W \equiv 0 \pmod{p^{a-1}}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For a primitive element C_1 of $M_{2k,n}(\mathbb{Z})$ we take a unimodular matrix $U = (C_1 \ *)$. From

$$\begin{pmatrix} W \\ 0 \end{pmatrix} = \begin{pmatrix} 1_n \\ 0 \end{pmatrix} W = U^{-1} C_1 W$$

follows that $W \equiv 0 \pmod{p}$ if and only if $C_1 W \equiv 0 \pmod{p}$. Lemma is obvious for $a = 1$. We assume $a \geq 2$. Decompose $C \in M_{2k,n}(\mathbb{Z})$ as $C = C_1 + p^{a-1}C_2$. Then C is primitive mod p if and only if C_1 is primitive mod p .

Hence we have

$$G(W; p^a) = \sum_{\substack{C_1 \pmod{p^{a-1}} \\ C_1: \text{primitive mod } p}} e(p^{-a}\sigma({}^tC_1JC_1W)) \sum_{C_2 \pmod{p}} e(2p^{-1}\sigma({}^tC_1JC_2W)).$$

$2\sigma({}^tC_1JC_2W) \equiv 0 \pmod{p}$ for any C_2 if and only if $2W{}^tC_1J \equiv 0 \pmod{p}$, and it is equivalent to $W \equiv 0 \pmod{p}$. Thus we have

$$G(W; p^a) = \begin{cases} p^{2kn} G(p^{-1}W; p^{a-1}) & \text{if } W \equiv 0 \pmod{p}, \\ 0 & \text{otherwise.} \end{cases}$$

Now our lemma is inductively proved.

Q.E.D.

Put

$$A'_{p^a}(T) = \# \left\{ C \in M_{2k,n}(\mathbb{Z}) \pmod{p^a} \mid \begin{array}{l} C; \text{ primitive mod } p, \\ p^{-a}({}^tCJC - T) \in \Lambda_n^* \end{array} \right\}.$$

LEMMA 2.

$$(p^a)^{n(n+1)/2-2kn} A'_{p^a}(T) = p^{n(n+1)/2-2kn} A'_p(T) \quad \text{for } a \geq 1, T \in \Lambda_n^*.$$

Proof.

$$\begin{aligned}
 (p^a)^{n(n+1)/2} A'_{p^a}(T) &= \sum_{W \in \Lambda_n \pmod{p^a}} \sum_{\substack{C \pmod{p^a} \\ \text{primitive} \pmod{p}}} e(p^{-a}\sigma((CJC - TW))) \\
 &= \sum_{W \in \Lambda_n \pmod{p^a}} G(W; p^a)e(-p^{-a}\sigma(TW)) \\
 &= \sum_{W \in \Lambda_n \pmod{p}} p^{2kn(a-1)} G(W; p)e(-p^{-1}\sigma(TW)) \\
 &= p^{2kn(a-1)} p^{n(n+1)/2} A'_p(T). \qquad \text{Q.E.D.}
 \end{aligned}$$

LEMMA 3.

$$S_p(T) \geq p^{n(n+1)/2 - 2kn} A'_p(T) \quad \text{for } T \in \Lambda_n^* .$$

Proof. This follows from $A_{p^a}(T) \geq A'_p(T)$. Q.E.D.

LEMMA 4. *There is a positive number ε such that $S_2(T) > \varepsilon$ for $T \in \Lambda_n^*$.*

Proof. $A'_2(T)$ is uniquely determined by $T \pmod 2$. Hence the values of $A'_2(T)$ is a finite set. Hence we have only to prove $A'_2(T) \neq 0$ for $T \in \Lambda_n^*$. By the theory of quadratic forms $T \in \Lambda_n^*$ is equivalent over \mathbb{Z}_2 to a direct sum of

$$2^{a-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad 2^{a-1} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad 2^a u \quad (a \geq 0, u \in \mathbb{Z}_2^\times).$$

Since $A'_2(T + 2T') = A'_2(T)$ for $T, T' \in \Lambda_n^*$, we may assume that T is a direct sum of

$$\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad u \in \mathbb{Z}_2^\times$$

and 0. Hence we may suppose

$$T = \begin{bmatrix} A_1 & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & A_r & & \\ & & & & u_1 & \\ & & & & & \ddots \\ & & & & & & u_{n-2r} \end{bmatrix},$$

where

$$A_1 = \dots = A_{r-1} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_r = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$u_3 = \dots = u_{n-2r} = 0$ and $u_1, u_2 = 0$ or $\in \mathbb{Z}_2^\times$ (93: 18 in [6]). Denote by t the number of u_i such that $u_i = 0$. It is easy to see that $\perp_{r+t-1} \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents primitively $\perp_{i=1}^{r-1} A_i \perp (0)$. To prove $A'_2(T) \neq 0$ we have only to show that $\perp_{k-(r+t-1)} \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents $A_r, A_r \perp (u_1), A_r \perp (u_1) \perp (u_2), (u_i \in \mathbb{Z}_2^\times)$, primitively according as $t = n - 2r, n - 2r - 1, n - 2r - 2$.

(i) The case of $t = n - 2r, n - 2r - 1$; then $2(k - r - t + 1) - 3 \geq 3$ and $\perp_{k-r-t+1} \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is \mathbb{Z}_2 -maximal. Hence $\perp_{k-r-t+1} \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents $A_r \perp (u_1), (u_1 \in \mathbb{Z}_2^\times)$. $2^{-1} \det(2A_r \perp (2u_1)) \in \mathbb{Z}_2^\times$ implies that this representation is primitive.

(ii) The case of $t = n - 2r - 2$; then $k - r + 1 - t \geq 5$.

$$\frac{1}{2}(u, 1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ 1 \end{pmatrix} = u, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

yield that $\perp_{\frac{k}{4}} \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents $A_r \perp (u_1) \perp (u_2)$ primitively. Thus we have proved $A'_2(T) \neq 0$ for $T \in A_n^*$. Q.E.D.

If $S_p(T) \geq (1 - p^{-2})^{4n}$ for odd prime p , then

$$S_T = \prod S_p(T) \geq S_2(T) \prod_{p \neq 2} (1 - p^{-2})^{4n} > \epsilon' > 0$$

holds, and it completes the proof. From now we show

$$S_p(T) \geq (1 - p^{-2})^{4n} \quad \text{for odd prime } p, T \in A_n^* .$$

We fix an odd prime p . Let L be a hyperbolic space of $\dim 2k$ over $F_p = \mathbb{Z}/(p)$; then $A'_p(T)$ is equal to the number of isometries from the quadratic space over F_p corresponding to T to L where isometries are supposed to be injective. For quadratic spaces M, N over F_p we denote by $A(M, N)$ the number of isometries from M to N . Our aim is to prove that

$$A(T, L) \geq p^{2kn - n(n+1)/2} (1 - p^{-2})^{4n} \quad \text{for any quadratic space}$$

T of dimension n .

Let T be a quadratic space of $\dim n$ over F_p and $T = T_0 \perp R$ where R is the radical of T . Define a quadratic space L_1 by $L \cong T_0 \perp L_1$; then $A(T, L) = A(T_0, L)A(R, L_1)$ and

$$A(T_0, L) = p^{-t(t+1)/2+2kt} \begin{cases} (1 \pm p^{-k})(1 \pm p^{-k+t/2}) \prod_{s=1}^{t/2-1} (1 - p^{2s-2k}) & \text{if } t \equiv 0 \pmod{2}, \\ (1 \pm p^{-k}) \prod_{s=1}^{(t-1)/2} (1 - p^{2s-2k}) & \text{if } t \equiv 1 \pmod{2}, \end{cases}$$

where $t = \dim T_0$ ([8]).

From $k \geq n + 2$ and $n \geq t$ follows

$$A(T_0, L) \geq p^{-t(t+1)/2+2kt}(1 - p^{-2})^{2n}.$$

Since, then, $A(T, L) \geq p^{2kn-n(n+1)/2}(1 - p^{-2})^{4n}$ follows from $A(R, L_i) \geq p^{-(n-t)(n-t+1)/2+(2k-t)(n-t)}(1 - p^{-2})^{2(n-t)}$, we have only to prove

LEMMA 5. *Let M be a regular quadratic space over F_p of $\dim M = m$ and let N be a totally isotropic quadratic space over F_p of $\dim N = s$. Then we have*

$$A(N, M) \geq p^{-s(s+1)/2+ms}(1 - p^{-2})^{2s} \quad \text{if } m \geq 2s + 3.$$

Proof. Put $N = F_p[v_1, \dots, v_s]$, $M = H \perp M_1$ where H is a hyperbolic plane. For quadratic spaces we denote by Q, B associated quadratic forms and bilinear forms ($Q(x) = B(x, x)$). Take a basis $\{e, f\}$ of H such that $Q(e) = Q(f) = 0, B(e, f) = 1$. Let σ be an isometry from N to M such that $\sigma(v_i) = e$, and put $\sigma(v_i) = a_i e + b_i f + u_i$ ($a_i, b_i \in F_p, u_i \in M_1$). Then $B(v_i, v_i) = B(\sigma(v_i), \sigma(v_i)) = b_i = 0$. Since σ is injective, u_2, \dots, u_s are linearly independent and $B(u_i, u_j) = 0$ for i, j . If, conversely, $w_2, \dots, w_s \in M_1$ are linearly independent and $B(w_i, w_j) = 0$ for i, j , then $\mu(v_i) = e, \mu(v_i) = a_i e + w_i$ ($a_i \in F_p, i \geq 2$) define an isometry from N to M . Thus we have

$$A(N_s, M) = p^{s-1} a(M) A(N_{s-1}, M_1),$$

where N_i denotes a totally isotropic quadratic space of $\dim i$ and $a(M)$ is the number of (non-zero) isotropic vectors of M .

Put $M = \perp_{s-1} H \perp M_0$; then we have

$$\begin{aligned} A(N_s, M) &= p^{s-1} a\left(\perp_{s-1} H \perp M_0\right) A\left(N_{s-1}, \perp_{s-2} H \perp M_0\right) \\ &= \dots \\ &= p^{s(s-1)/2} \prod_{i=0}^{s-1} a\left(\perp_i H \perp M_0\right). \end{aligned}$$

If $\dim M_0 \equiv 1 \pmod{2}$, then we have

$$\begin{aligned} a\left(\frac{\perp}{i} H \perp M_0\right) &= p^{2i+m-2(s-1)-1} - 1 \\ &\geq p^{2i+m-2(s-1)-1}(1-p^{-2}) \\ &\geq p^{2i+m-2(s-1)-1}(1-p^{-2})^2, \end{aligned}$$

since $2i + m - 2(s - 1) - 1 \geq 2$.

If $\dim M_0 \equiv 0 \pmod{2}$, then we have

$$a\left(\frac{\perp}{i} H \perp M_0\right) = \begin{cases} p^{2r-1} + p^r - p^{r-1} - 1 & \text{if } M_0 \text{ is hyperbolic,} \\ p^{2r-1} - p^r + p^{r-1} - 1 & \text{otherwise,} \end{cases}$$

where

$$2r = 2i + m - 2(s - 1) = \dim \frac{\perp}{i} H \perp M_0.$$

Hence

$$a\left(\frac{\perp}{i} H \perp M_0\right) = (p^r \mp 1)(p^{r-1} \pm 1) \geq p^{2r-1}(1-p^{-2})^2$$

holds.

Thus we have

$$\begin{aligned} A(N_s, M) &\geq p^{s(s-1)/2} \prod_{i=0}^{s-1} p^{2i+m-2(s-1)-1}(1-p^{-2})^2 \\ &= p^{m \cdot s - s(s+1)/2} (1-p^{-2})^{2s}. \end{aligned} \quad \text{Q.E.D.}$$

COROLLARY. *If n, k are natural numbers such that $k \geq 2n + 2$, $k \equiv 0 \pmod{2}$ and $f(Z) = \sum a(T)e(\sigma(TZ))(Z \in H_n)$ is a modular form of degree n and weight k , then we have*

$$a(T) = O(|T|^{k-(n+1)/2}) \quad \text{for } T > 0.$$

Proof. It is known that there exist cusp forms f_r of degree r and weight k such that $f(Z) = \sum_{r=1}^{n-1} E_{n,r}^k(Z, f_r) + aE_n^k(Z) + f_n(Z)$, ($a \in \mathbb{C}$) ([3]). Since $a({}^tUTU) = a(T)$ for $U \in GL(n, \mathbb{Z})$, we may assume that T is in some fixed Siegel domain. Theorem in § 2 and our theorem imply the corollary. Q.E.D.

§ 4.

Let A be an even integral unimodular positive definite symmetric matrix of rank m ; then $m \equiv 0 \pmod{8}$. Put

$$\theta_n(Z, A) = \sum_{C \in \mathcal{M}_{m,n}(Z)} e\left(\frac{1}{2} \sigma({}^tCACZ)\right), \quad Z \in H_n.$$

Then $\theta_n(Z, A)$ is a modular form of degree n and weight $m/2$. Then there exist cusp forms f_r of degree r and weight $m/2$ such that

$$\theta_n(Z, A) = E_n^{m/2}(Z) + \sum_{r=1}^{n-1} E_{n,r}^{m/2}(Z, f_r) + f_n(Z) \quad \text{for } n \leq m/4 - 1.$$

If

$$E_n^{m/2}(Z) = \sum_{\substack{T \geq 0 \\ T \in A_n^*}} a_{m/2,n}(T) e(\sigma(TZ)),$$

then

$$a_{m/2,n}(T) = 2^{n(m-n+1)/2} \prod_{t=0}^{n-1} \frac{\pi^{(m-t)/2}}{\Gamma((m-t)/2)} \cdot |T|^{(m-n-1)/2} \prod_p S_p(T) \quad \text{for } T > 0,$$

where $S_p(T) = \sum_{R \in \mathbb{Z}} e(-\sigma(TR)) \nu(R)^{-m/2}$ where R runs over all $n \times n$ rational symmetric matrices modulo 1 such that the product $\nu(R)$ of denominators of elementary divisors of R is a power of p .

Put

$$\theta_n(Z, A) = \sum_{\substack{T \geq 0 \\ T \in A_n^*}} N_n(T, A) e(\sigma(TZ)),$$

and

$$E_{n,r}^{m/2}(Z, f_r) = \sum_{\substack{T \geq 0 \\ T \in A_n^*}} a(T, f_r) e(\sigma(TZ)).$$

Then we have, summarizing,

THEOREM. *If $n \leq m/4 - 1$, $T \in A_n^*$, $T > 0$, then $N_n(T, A) = a_{m/2,n}(T) + \sum_{r=1}^{n-1} a(T, f_r) + O(|T|^{m/4})$. If, moreover, T runs in any fixed Siegel domain, then*

$$a_{m/2,n}(T) \sim |T|^{(m-n-1)/2}, \quad a(T, f_r) = O(|T_r|^{-(m/2-r-1)/2} |T|^{(m-n-1)/2}),$$

where T_r stands for the upper left $r \times r$ submatrix of T .

For $n \times n$ positive definite matrix S we denote by $m(S)$ the minimal value of ${}^t x S x$ ($x \in \mathbb{Z}^n - \{0\}$). It is well known that there is a constant μ_n such that $m(S) \leq \mu_n \sqrt[n]{|S|}$ for any $n \times n$ positive definite matrix S .

COROLLARY. *If $n \leq m/4 - 1$, then*

$$N_n(T, A) = a_{m/2, n}(T) + O(m(T)^{1-m/4} |T|^{(m-n-1)/2}) \quad \text{for } T > 0, T \in A_n^* .$$

Especially $N_n(T, A) > 0$ if $m(T)$ is sufficiently large.

Proof.

$$\begin{aligned} |T_r|^{-(m/2-r-1)/2} &= O(m(T_r)^{-r(m/2-r-1)/2}) \\ &= O(m(T)^{-r(m/2-r-1)/2}) . \end{aligned}$$

On the other hand

$$\begin{aligned} |T|^{m/4} (m(T)^{1-m/4} |T|^{(m-n-1)/2})^{-1} &= m(T)^{m/4-1} |T|^{(2n+2-m)/4} \\ &\leq \mu_n^{m/4-1} |T|^{(m/4-1)/n + (2n+2-m)/4} . \end{aligned}$$

Since $r(m/2 - r - 1)/2 \geq m/4 - 1$ for $1 \leq r \leq n \leq m/4 - 1$, we have $a(T, f_r) = O(m(T)^{1-m/4} |T|^{(m-n-1)/2})$, $|T|^{m/4} = O(m(T)^{1-m/4} |T|^{(m-n-1)/2})$, if $|T| \geq 1$.

There are only finitely many equivalence classes of $T \in A_n^*$ such that $T > 0, |T| < 1$. This completes the proof. Q.E.D.

Remark. Let $f(Z) = \sum a(T)e(\sigma(TZ))$ be a modular form of degree n , weight $k (\in \frac{1}{2}\mathbf{Z})$ with level such that the constant term of $f(Z)$ at any cusps vanishes. Results in [2] and here seem to suggest that $a(T) = O(m(T)^{1-k/2} |T|^{k-(n+1)/2})$ for $T > 0$ if, at least, $2k \geq 2n + 3$.

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