

FREE ACTIONS OF SOME COMPACT GROUPS ON MILNOR MANIFOLDS

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Abstract. In this paper, we investigate free actions of some compact groups on cohomology real and complex Milnor manifolds. More precisely, we compute the mod 2 cohomology algebra of the orbit space of an arbitrary free \mathbb{Z}_2 and S^1 -action on a compact Hausdorff space with mod 2 cohomology algebra of a real or a complex Milnor manifold. As applications, we deduce some Borsuk–Ulam type results for equivariant maps between spheres and these spaces. For the complex case, we obtain a lower bound on the Schwarz genus, which further establishes the existence of coincidence points for maps to the Euclidean plane.

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1. Introduction. A basic problem in the theory of transformation groups is to determine groups that can act freely on a given topological space. Once we know that a group acts freely on a given space, the next natural problem is to determine all actions of the group up to conjugation. Determining the homeomorphism or the homotopy type of the orbit space is, in general, a difficult problem. A non-trivial result of Oliver [20] states that the orbit space of any action of a compact Lie group on a Euclidean space is contractible. For spheres, Milnor [13] proved that for any free involution on S^n , the orbit space has the homotopy type of $\mathbb{R}P^n$. Free actions of finite groups on spheres, particularly S^3 , have been well studied in the past, see for example, [23, 24, 26]. But, not many results are known for compact manifolds other than spheres. In [19], Myers investigated orbit spaces of free involutions on three-dimensional lens spaces. In [30], Tao determined orbit spaces of free involutions on $S^1 \times S^2$, and Ritter [25] extended these results to free actions of cyclic groups of order 2^n . Tollefson [31] proved that there are precisely four conjugacy classes of involutions on $S^1 \times S^2$. Fairly recently, Jahren and Kwasik [14] classified, up to conjugation, all free involutions on $S^1 \times S^n$ for $n \geq 3$, by showing that there are exactly four possible homotopy types of orbit spaces.

Various attempts have been made towards the weaker problem of determining possible cohomology algebra of orbit spaces of free actions of finite groups on some specific classes of manifolds, for example, products of spheres, spherical space forms and their products. Dotzel et al. [10] determined the cohomology algebra of orbit spaces of free \mathbb{Z}_p (p prime) and S^1 -actions on cohomology product of two spheres. Orbit spaces of free involutions on cohomology lens spaces were investigated by Singh [28]. The cohomology algebra of orbit spaces of free involutions on product of two projective spaces was computed in [27]. Recently, Pergher et al. [21] and Mattos et al. [9]

considered free \mathbb{Z}_2 and \mathbb{S}^1 -actions on spaces of type (a, b) , which are certain products or wedge sums of spheres and projective spaces. As applications, they also established some bundle theoretic analogues of Borsuk–Ulam theorem for these spaces.

Viewing the product of two projective spaces as a trivial bundle, it is interesting to consider non-trivial projective space bundles over projective spaces. Milnor manifolds are fundamental examples of such spaces. It is well known that the unoriented cobordism algebra of smooth manifolds is generated by the cobordism classes of real projective spaces and real Milnor manifolds [17, Lemma 1]. Therefore, determining various invariants of these manifolds is of interest. Free actions of elementary abelian 2-groups on products of Milnor manifolds were investigated in [29], wherein some bounds on the rank of these groups were obtained.

The aim of this paper is to investigate free \mathbb{Z}_2 and \mathbb{S}^1 -actions on mod 2 cohomology real and complex Milnor manifolds. More precisely, we determine the possible mod 2 cohomology algebra of orbit spaces of free \mathbb{Z}_2 and \mathbb{S}^1 -actions on these spaces. We also find necessary and sufficient conditions for the existence of free actions on these spaces. As applications, we obtain some Borsuk–Ulam kind results for these spaces. We also determine some lower bound on the genus and results on the existence of non-empty coincidence set.

The paper is organized as follows. In Section 2, we recall the definition and cohomology of Milnor manifolds. In Section 3, we construct free \mathbb{Z}_n and \mathbb{S}^1 -actions on these manifolds. Section 4 consists of some preliminaries from the theory of compact transformation groups that will be used in subsequent sections. Induced action on cohomology is investigated in Section 5. In Section 6, we prove our main results as Theorems 6.1–6.3. Finally, in Section 7, we deduce some Borsuk–Ulam type results for equivariant maps between spheres and these spaces. For the complex case, we obtain a lower bound on the Schwarz genus, which establishes the existence of coincidence points for maps to the Euclidean plane.

2. Milnor manifolds. Let r and s be integers such that $0 \leq s \leq r$. A real Milnor manifold, denoted by $\mathbb{R}H_{r,s}$, is the non-singular hypersurface of degree $(1, 1)$ in the product $\mathbb{R}P^r \times \mathbb{R}P^s$ of real projective spaces. Milnor [17] introduced these manifolds in search for generators for the unoriented cobordism algebra. Clearly, $\mathbb{R}H_{r,s}$ is a $(s + r - 1)$ -dimensional closed smooth manifold, and can be described in terms of homogeneous coordinates of real projective spaces as

$$\mathbb{R}H_{r,s} = \left\{ ([x_0, \dots, x_r], [y_0, \dots, y_s]) \in \mathbb{R}P^r \times \mathbb{R}P^s \mid x_0y_0 + \dots + x_sy_s = 0 \right\}.$$

Alternatively, $\mathbb{R}H_{r,s}$ is given as the total space of the fibre bundle

$$\mathbb{R}P^{r-1} \xrightarrow{i} \mathbb{R}H_{r,s} \xrightarrow{p} \mathbb{R}P^s.$$

This is projectivization of the vector bundle

$$\mathbb{R}^r \hookrightarrow E^\perp \longrightarrow \mathbb{R}P^s,$$

where E^\perp is the orthogonal complement in $\mathbb{R}P^s \times \mathbb{R}^{r+1}$ of the canonical line bundle

$$\mathbb{R} \hookrightarrow E \longrightarrow \mathbb{R}P^s.$$

Similarly, a complex Milnor manifold, denoted by $\mathbb{C}H_{r,s}$, is a $2(s+r-1)$ -dimensional closed smooth manifold, given in terms of homogeneous coordinates as

$$\mathbb{C}H_{r,s} = \left\{ ([z_0, \dots, z_r], [w_0, \dots, w_s]) \in \mathbb{C}P^r \times \mathbb{C}P^s \mid z_0\bar{w}_0 + \dots + z_s\bar{w}_s = 0 \right\}.$$

As in the real case, $\mathbb{C}H_{r,s}$ is the total space of the fibre bundle

$$\mathbb{C}P^{r-1} \xrightarrow{i} \mathbb{C}H_{r,s} \xrightarrow{p} \mathbb{C}P^s.$$

It is known due to Conner and Floyd [8, p. 63] that $\mathbb{C}H_{r,s}$ is unoriented cobordant to $\mathbb{R}H_{r,s} \times \mathbb{R}H_{r,s}$.

These manifolds have been well studied in the past. See [11, 15, 29] for some recent results. Their cohomology algebra is also well known [5, 18], and we need it for the proofs of our main results.

THEOREM 2.1. *Let $0 \leq s \leq r$. Then the following holds:*

- (1) $H^*(\mathbb{R}H_{r,s}; \mathbb{Z}_2) \cong \mathbb{Z}_2[a, b]/\langle a^{s+1}, b^r + ab^{r-1} + \dots + a^s b^{r-s} \rangle$, where a and b are homogeneous elements of degree one each.
- (2) $H^*(\mathbb{C}H_{r,s}; \mathbb{Z}_2) \cong \mathbb{Z}_2[g, h]/\langle g^{s+1}, h^r + gh^{r-1} + \dots + g^s h^{r-s} \rangle$, where g and h are homogeneous elements of degree two each.

Note that $\mathbb{R}H_{r,0} = \mathbb{R}P^{r-1}$ and $\mathbb{C}H_{r,0} = \mathbb{C}P^{r-1}$. Since orbit spaces of free involutions on real and complex projective spaces are well known, we, henceforth, assume that $1 \leq s \leq r$.

3. Free actions on Milnor manifolds.

3.1. Circle actions. We give examples of free \mathbb{S}^1 -actions on $\mathbb{R}H_{r,s}$ in the case when both r and s are odd, and later prove that this is indeed a necessary condition for the existence of a free \mathbb{S}^1 -action. We first give a free \mathbb{S}^1 -action on $\mathbb{R}P^s$. Note that only odd-dimensional real projective spaces admit free \mathbb{S}^1 -actions. Let $s = 2m + 1$ and write an element of $\mathbb{R}P^{2m+1}$ as $[w_0, \dots, w_m]$, where w_i are complex numbers.

Define a map $\mathbb{S}^1 \times \mathbb{R}P^{2m+1} \rightarrow \mathbb{R}P^{2m+1}$ by

$$(\xi, [w_0, \dots, w_m]) = [\sqrt{\xi}w_0, \dots, \sqrt{\xi}w_m].$$

It can be checked that the preceding map gives a free \mathbb{S}^1 -action on $\mathbb{R}P^{2m+1}$.

Let $r = 2n + 1$ and write an element of $\mathbb{R}P^r$ as $[z_0, \dots, z_n]$, where z_i are complex numbers. Define an action of \mathbb{S}^1 on $\mathbb{R}P^r$ by

$$(\xi, [z_0, \dots, z_n]) = [\sqrt{\xi}z_0, \dots, \sqrt{\xi}z_n].$$

The diagonal action on $\mathbb{R}P^r \times \mathbb{R}P^s$ is free, and $\mathbb{R}H_{r,s}$ is invariant under this action giving rise to a free \mathbb{S}^1 -action on $\mathbb{R}H_{r,s}$. Restricting the above \mathbb{S}^1 -action gives free \mathbb{Z}_n (in particular \mathbb{Z}_2) action on $\mathbb{R}H_{r,s}$.

If X and Y are two spaces, then $X \cong Y$ means that X and Y have isomorphic mod 2 cohomology algebras, not necessarily induced by a map between X and Y . For the complex case, we have the following result.

PROPOSITION 3.1. *There is no free \mathbb{S}^1 -action on a compact Hausdorff space $X \simeq_2 \mathbb{C}H_{r,s}$.*

Proof. Recall that, we have a fibre bundle

$$\mathbb{C}P^{r-1} \hookrightarrow \mathbb{C}H_{r,s} \longrightarrow \mathbb{C}P^s$$

with $\chi(\mathbb{C}H_{r,s}) = r(s+1)$. Suppose there is a free \mathbb{S}^1 -action on X . Restriction of this action gives free \mathbb{Z}_p -actions for each prime p . By Floyd's Euler characteristic formula [4, Chapter III, Theorem 7.10], we have

$$r(s+1) = \chi(X) = p \chi(X/\mathbb{Z}_p)$$

for each prime p , which is a contradiction. Hence, there is no free \mathbb{S}^1 -action on a space $X \simeq_2 \mathbb{C}H_{r,s}$. \square

3.2. Involutions. If $s = r$, then interchanging the coordinates i.e.,

$$([z_0, \dots, z_s], [w_0, \dots, w_s]) \mapsto ([w_0, \dots, w_s], [z_0, \dots, z_s])$$

gives a free involution on Milnor manifolds. But, if $1 < s < r$ and $r \not\equiv 2 \pmod{4}$, then we show that $\mathbb{R}H_{r,s}$ (respectively $\mathbb{C}H_{r,s}$) admits a free involution if and only if both r and s are odd. We have seen examples of free involutions on real Milnor manifolds before.

For the complex case, it is known that $\mathbb{C}P^n$ admits a free action by a finite group if and only if n is odd, and in that case the only possible group is \mathbb{Z}_2 [12].

If s is odd, then the map

$$[z_0, z_1, \dots, z_{s-1}, z_s] \mapsto [-\bar{z}_1, \bar{z}_0, \dots, -\bar{z}_s, \bar{z}_{s-1}],$$

defines a free involution on $\mathbb{C}P^s$. Similarly, for r odd, the map

$$[z_0, z_1, \dots, z_{r-1}, z_r] \mapsto [-\bar{z}_1, \bar{z}_0, \dots, -\bar{z}_r, \bar{z}_{r-1}],$$

is a free involution on $\mathbb{C}P^r$. Hence, the diagonal action on $\mathbb{C}P^r \times \mathbb{C}P^s$ is free and its restriction gives a free involution on $\mathbb{C}H_{r,s}$.

4. Preliminaries. For the convenience of the reader, we recall some facts that we use without mentioning explicitly. For further details, we refer the reader to [1, 4, 16]. Throughout, we use cohomology with \mathbb{Z}_2 coefficients, and suppress it from the notation.

Let G be a group and X a G -space. Let

$$G \hookrightarrow E_G \longrightarrow B_G$$

be the universal principal G -bundle and

$$X \xrightarrow{i} X_G \xrightarrow{\pi} B_G$$

the associated Borel fibration [3, Chapter IV]. Our main computational tool is the Leray–Serre spectral sequence associated to the Borel fibration [16, Theorem 5.2]. The

E_2 -term of this spectral sequence is given by

$$E_2^{k,l} = H^k(B_G; \mathcal{H}^l(X)),$$

where $\mathcal{H}^l(X)$ is a locally constant sheaf with stalk $H^l(X)$ and group G . Further, the spectral sequence converges to $H^*(X_G)$ as an algebra. If $\pi_1(B_G)$ acts trivially on $H^*(X)$, then the system of local coefficient is simple and we get

$$E_2^{k,l} \cong H^k(B_G) \otimes H^l(X).$$

Further, if the system of local coefficient is simple, then the edge homomorphisms

$$H^k(B_G) = E_2^{k,0} \longrightarrow E_3^{k,0} \longrightarrow \dots \longrightarrow E_k^{k,0} \longrightarrow E_{k+1}^{k,0} = E_\infty^{k,0} \subset H^k(X_G)$$

and

$$H^l(X_G) \longrightarrow E_\infty^{0,l} = E_{l+1}^{0,l} \subset E_l^{0,l} \subset \dots \subset E_2^{0,l} = H^l(X)$$

are the homomorphisms

$$\pi^* : H^k(B_G) \rightarrow H^k(X_G) \quad \text{and} \quad i^* : H^l(X_G) \rightarrow H^l(X),$$

respectively [16, Theorem 5.9].

On passing to quotients, the G -equivariant projection $X \times E_G \rightarrow X$ yields the fibre bundle

$$E_G \hookrightarrow X_G \xrightarrow{h} X/G$$

with contractible fibre E_G . By [1, p. 20], h is a homotopy equivalence, and consequently

$$h^* : H^*(X/G) \xrightarrow{\cong} H^*(X_G).$$

Next, we recall some results regarding free \mathbb{Z}_2 and \mathbb{S}^1 -actions on compact Hausdorff spaces. For free actions, vanishing of $H^*(X)$ implies vanishing of $H^*(X/G)$ in higher range [4, p. 374, Theorem 1.5].

PROPOSITION 4.1. *Let $G = \mathbb{Z}_2$ act freely on a compact Hausdorff space X . Suppose that $H^j(X) = 0$ for all $j > n$, then $H^j(X/G) = 0$ for all $j > n$.*

For $G = \mathbb{S}^1$, one can derive an analogue of the preceding result by using the Gysin sequence for the principle bundle $X \rightarrow X/G$.

PROPOSITION 4.2. *Let $G = \mathbb{S}^1$ act freely on a compact Hausdorff space X . Suppose that $H^j(X) = 0$ for all $j > n$, then $H^j(X/G) = 0$ for all $j \geq n$.*

We use the well known facts that $H^*(B_{\mathbb{Z}_2}; \mathbb{Z}_2) = \mathbb{Z}_2[t]$ and $H^*(B_{\mathbb{S}^1}; \mathbb{Z}_2) = \mathbb{Z}_2[u]$, where $\deg(t) = 1$ and $\deg(u) = 2$, respectively.

5. Induced action on cohomology. When a group acts on a topological space, in general, it is difficult to determine the induced action on cohomology. In our context, we have the following.

PROPOSITION 5.1. *Let $G = \mathbb{Z}_2$ act freely on a compact Hausdorff space $X \simeq_2 \mathbb{R}H_{r,s}$, where $1 < s < r$ and $r \not\equiv 2 \pmod{4}$. Then, the induced action on $H^*(X)$ is trivial.*

Proof. Let $G = \langle g \rangle$ and $a, b \in H^1(X)$ be generators of the cohomology algebra $H^*(X)$. By the naturality of cup product, we have

$$g^*(a^i b^j) = (g^*(a))^i (g^*(b))^j$$

for all $i, j \geq 0$. Therefore, it is enough to consider

$$g^* : H^1(X) \rightarrow H^1(X).$$

Suppose that g^* is non-trivial. Then, it cannot preserve both a and b . Assuming that $g^*(b) \neq b$, we have $g^*(b) = a$ or $a + b$. If $g^*(b) = a$, then $g^*(b^{s+1}) = a^{s+1} = 0$, which implies $b^{s+1} = 0$. Hence, $b^r = 0$, contradicting the fact that top-dimensional cohomology must be non-zero with \mathbb{Z}_2 coefficients. So we must have $g^*(b) = a + b$ and $g^*(a) = a$. Suppose that r is odd. Then,

$$g^*(a^{s-1} b^r) = a^{s-1} (a + b)^r = r a^s b^{r-1} + a^{s-1} b^r = a^s b^{r-1} + a^{s-1} b^r = 0.$$

This gives $a^{s-1} b^r = 0$, which is a contradiction. Hence, for r odd, the induced action on $H^*(X)$ must be trivial.

Suppose that $r \equiv 0 \pmod{4}$. Notice that, $b^{r+1} = 0$ implies $g^*(b^{r+1}) = (a + b)^{r+1} = 0$. But, from the binomial expansion,

$$(a + b)^{r+1} = a^{r+1} + \dots + \binom{r+1}{2} a^2 b^{r-1} + (r+1) a b^r,$$

we see that the last term is non-zero and the second last term is zero modulo 2. This gives $(a + b)^{r+1} \neq 0$, a contradiction. Hence, the induced action must be trivial in this case as well. \square

REMARK 5.2. If $s = 1$ and $r > 1$ is an odd integer, then orders of b and $a + b$ are $r + 1$ and r , respectively. Hence, in this case also, g^* is identity. For $s = 1$ or $r \equiv 2 \pmod{4}$, the induced action on $H^*(X)$ may be non-trivial.

Similarly, for the complex case, we obtain the following.

PROPOSITION 5.3. *Let $G = \mathbb{Z}_2$ act freely on a compact Hausdorff space $X \simeq_2 \mathbb{C}H_{r,s}$, where $1 < s < r$ and $r \not\equiv 2 \pmod{4}$. Then, the induced action on $H^*(X)$ is trivial.*

REMARK 5.4. If $r = s$, then the free involution on $\mathbb{R}H_{s,s}$ given by

$$([x_0, \dots, x_s], [y_0, \dots, y_s]) \mapsto ([y_0, \dots, y_s], [x_0, \dots, x_s]),$$

and the similar free involution on $\mathbb{C}H_{s,s}$ given by

$$([z_0, \dots, z_s], [w_0, \dots, w_s]) \mapsto ([w_0, \dots, w_s], [z_0, \dots, z_s])$$

are cohomologically non-trivial [29, Propositions 5.1 and 5.2].

Next, we determine conditions on r and s for which a compact Hausdorff space $X \simeq_2 \mathbb{R}H_{r,s}$ or $\mathbb{C}H_{r,s}$ admits a free involution.

PROPOSITION 5.5. *Let $G = \mathbb{Z}_2$ act freely on $X \simeq_2 \mathbb{R}H_{r,s}$, where $1 < s < r$ and $r \not\equiv 2 \pmod{4}$. Then, both r and s are odd.*

Proof. Suppose \mathbb{Z}_2 acts freely on $X \simeq_2 \mathbb{R}H_{r,s}$. Let $a, b \in H^1(X)$ be generators of the cohomology algebra $H^*(X)$. By Proposition 5.1, $\pi_1(B_G) = \mathbb{Z}_2$ acts trivially on $H^*(X)$, so that the fibration $X \hookrightarrow X_G \rightarrow B_G$ has a simple system of local coefficients. Hence, the spectral sequence has the form

$$E_2^{p,q} \cong H^p(B_G) \otimes H^q(X).$$

If $d_2 : E_2^{0,1} \rightarrow E_2^{2,0}$ is trivial, then the spectral sequence degenerates at E_2 -term and we get $H^i(X/G) \neq 0$ for infinitely many values of i . This contradicts Proposition 4.1. Thus, d_2 must be non-trivial. Hence, we have following three possibilities:

- (i) $d_2(1 \otimes a) = t^2 \otimes 1$ and $d_2(1 \otimes b) = 0$.
- (ii) $d_2(1 \otimes a) = 0$ and $d_2(1 \otimes b) = t^2 \otimes 1$.
- (iii) $d_2(1 \otimes a) = t^2 \otimes 1$ and $d_2(1 \otimes b) = t^2 \otimes 1$.

We first prove that the cases (i) and (ii) are not possible.

Assuming that s is odd, first we show that case (i) is not possible. The even case follows similarly. Suppose $d_2(1 \otimes a) = t^2 \otimes 1$ and $d_2(1 \otimes b) = 0$. By the derivation property of the differential, we have

$$d_2(t^k \otimes a^m b^n) = \begin{cases} t^{k+2} \otimes a^{m-1} b^n & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

Then, the relation $b^r + ab^{r-1} + \dots + a^s b^{r-s} = 0$ gives

$$\begin{aligned} 0 &= d_2(1 \otimes (b^r + ab^{r-1} + \dots + a^s b^{r-s})) \\ &= d_2(1 \otimes b^r) + d_2(1 \otimes ab^{r-1}) + \dots + d_2(1 \otimes a^s b^{r-s}) \\ &= 0 + t^2 \otimes b^{r-1} + \dots + t^2 \otimes a^{s-1} b^{r-s} \\ &= t^2 \otimes (b^{r-1} + \dots + a^{s-1} b^{r-s}), \end{aligned}$$

a contradiction. Hence, case (i) is not possible. The same argument works for case (ii) as well.

Hence, we must have $d_2(1 \otimes a) = t^2 \otimes 1$ and $d_2(1 \otimes b) = t^2 \otimes 1$. If s is even, then $a^{s+1} = 0$ gives

$$0 = d_2(1 \otimes a^{s+1}) = t^2 \otimes a^s,$$

a contradiction. Therefore, s must be odd, and a similar argument shows that r is also odd. □

As a consequence of Proposition 5.5 and the previously defined \mathbb{S}^1 -action on $\mathbb{R}H_{r,s}$, we obtain the following.

COROLLARY 5.6. *Let $1 < s < r$ and $r \not\equiv 2 \pmod{4}$. Then, $\mathbb{R}H_{r,s}$ admits a free involution if and only if both r and s are odd.*

We have similar observations for the complex case.

PROPOSITION 5.7. *Let \mathbb{Z}_2 act freely on $X \simeq_2 \mathbb{C}H_{r,s}$ with $1 < s < r$ and $r \not\equiv 2 \pmod{4}$. Then, both r and s are odd.*

COROLLARY 5.8. *Let $1 < s < r$ and $r \not\equiv 2 \pmod{4}$. Then, $\mathbb{C}H_{r,s}$ admits a free involution if and only if both r and s are odd.*

For \mathbb{S}^1 -actions, we have the following.

PROPOSITION 5.9. *Let $1 \leq s \leq r$. Then, \mathbb{S}^1 acts freely on $\mathbb{R}H_{r,s}$ if and only if both r and s are odd.*

Proof. For $G = \mathbb{S}^1$, since $\pi_1(B_G) = 1$, the system of local coefficients is simple. Recall that, $H^*(B_{\mathbb{S}^1}; \mathbb{Z}_2) = \mathbb{Z}_2[u]$, where $\deg(u) = 2$. Hence, the spectral sequence has the form

$$E_2^{p,q} \cong H^p(B_G) \otimes H^q(X).$$

Clearly, for p odd, $E_2^{p,q} = 0$. As in the case of \mathbb{Z}_2 -action, it can be seen that the differential d_2 must be non-zero and the only possibility for d_2 is $d_2(1 \otimes a) = u \otimes 1$ and $d_2(1 \otimes b) = u \otimes 1$. Consequently, both r and s must be odd. \square

6. Main results. We are now in a position to present our main results.

THEOREM 6.1. *Let $G = \mathbb{Z}_2$ act freely on a compact Hausdorff space $X \simeq_2 \mathbb{R}H_{r,s}$ such that induced action on mod 2 cohomology is trivial. Then,*

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y, z, w]/I,$$

where

$$I = \left\langle z^2, w^2 - \gamma_1 zw - \gamma_2 x - \gamma_3 y, x^{\frac{s+1}{2}} + \alpha_0 zw x^{\frac{s-1}{2}} + \alpha_1 zw x^{\frac{s-3}{2}} y + \dots + \alpha_{\frac{s-1}{2}} zw y^{\frac{s-1}{2}}, \right. \\ \left. (w + \beta_0 z)y^{\frac{r-1}{2}} + (w + \beta_1 z)xy^{\frac{r-3}{2}} + \dots + (w + \beta_{\frac{s-1}{2}} z)x^{\frac{s-1}{2}} y^{\frac{r-1}{2}} \right\rangle,$$

with $\deg(x) = 2, \deg(y) = 2, \deg(z) = 1, \deg(w) = 1$ and $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}_2$.

Proof. Let $a, b \in H^1(X)$ be generators of the cohomology algebra $H^*(X)$. By similar argument as in Proposition 5.5, we see that both r and s must be odd and

$$d_2(1 \otimes a) = t^2 \otimes 1 \quad \text{and} \quad d_2(1 \otimes b) = t^2 \otimes 1.$$

By the derivation property of the differential, we have

$$d_2(1 \otimes a^m b^n) = \begin{cases} t^2 \otimes a^{m-1} b^n + t^2 \otimes a^m b^{n-1} & \text{if } m \text{ and } n \text{ are odd} \\ t^2 \otimes a^{m-1} b^n & \text{if } m \text{ is odd and } n \text{ is even} \\ t^2 \otimes a^m b^{n-1} & \text{if } m \text{ is even and } n \text{ is odd} \\ 0 & \text{if } m \text{ and } n \text{ are even.} \end{cases}$$

It suffices to look at

$$d_2 : E_2^{0,q} \rightarrow E_2^{2,q-1}.$$

- For $q \leq s$, a basis of $E_2^{0,q} \cong H^q(X)$ consists of

$$\{a^q, a^{q-1}b, \dots, ab^{q-1}, b^q\}.$$

If q is even, then $\text{rk}(\text{Ker } d_2) = \frac{q}{2} + 1$ and $\text{rk}(\text{Im } d_2) = \frac{q}{2}$. If q is odd, then $\text{rk}(\text{Ker } d_2) = \frac{q+1}{2} = \text{rk}(\text{Im } d_2)$.

- For $s < q \leq r - 1$, a basis consists of

$$\{a^s b^{q-s}, a^{s-1} b^{q-s+1}, \dots, ab^{q-1}, b^q\}.$$

In this case, $\text{rk}(\text{Ker } d_2) = \frac{s+1}{2} = \text{rk}(\text{Im } d_2)$.

- For $r \leq q \leq s + r - 1$, a basis consists of

$$\{a^s b^{q-s}, a^{s-1} b^{q-s+1}, \dots, a^{q-r+1} b^{r-1}\}.$$

If q is odd, then $\text{rk}(\text{Ker } d_2) = \frac{s+r-1-q}{2}$ and $\text{rk}(\text{Im } d_2) = \frac{s+r+1-q}{2}$. And, if q is even, then $\text{rk}(\text{Ker } d_2) = \frac{r+s-q}{2} = \text{rk}(\text{Im } d_2)$.

From the above observation, we get that for all $k \geq 2$ and for all l , $E_3^{k,l} = 0$ as $\text{rk}(E_3^{k,l}) = 0$. This gives

$$E_3^{k,l} = \begin{cases} \text{Ker}\{d_2 : E_2^{k,l} \rightarrow E_2^{k+2,l-1}\} & k = 0, 1 \text{ and for all } l. \\ 0 & k \geq 2 \text{ and for all } l. \end{cases}$$

Note that $d_r : E_r^{k,l} \rightarrow E_r^{k+r,l-r+1}$ is trivial for all $r \geq 3$ and for all k, l , and hence $E_\infty^{*,*} \cong E_3^{*,*}$. Since $H^*(X_G) \cong \text{Tot} E_\infty^{*,*}$, the total complex of $E_\infty^{*,*}$, we have

$$H^n(X_G) \cong \bigoplus_{i+j=n} E_\infty^{i,j} = E_\infty^{0,n} \oplus E_\infty^{1,n-1}$$

for all $0 \leq n \leq r + s - 1$.

Note that $t \otimes 1$ is a permanent cocycle and let $z = \pi^*(t) \in E_\infty^{1,0} \subseteq H^1(X_G)$ be determined by $t \otimes 1 \in E_2^{1,0}$. As $E_\infty^{2,0} = 0$, we have $z^2 = 0$. Also, $1 \otimes (a + b) \in E_2^{0,1}$ is a permanent cocycle. Let $w \in H^1(X_G)$ such that $i^*(w) = a + b$. Notice that, $1 \otimes a^2 \in E_2^{0,2}$ and $1 \otimes b^2 \in E_2^{0,2}$ are permanent cocycles, and hence they determine elements in $E_\infty^{0,2}$. Let $x, y \in H^2(X_G)$ such that $i^*(x) = a^2$ and $i^*(y) = b^2$. As $a^{s+1} = 0$, we get the following relation:

$$x^{\frac{s+1}{2}} + \alpha_0 z w x^{\frac{s-1}{2}} + \alpha_1 z w x^{\frac{s-3}{2}} y + \dots + \alpha_{\frac{s-1}{2}} z w y^{\frac{s-1}{2}} = 0,$$

where $\alpha_i \in \mathbb{Z}_2$. Notice that,

$$i^*(w y^{\frac{r-1}{2}} + w x y^{\frac{r-3}{2}} + \dots + w x^{\frac{s-1}{2}} y^{\frac{r-s}{2}}) = 0.$$

Hence, it satisfies

$$w y^{\frac{r-1}{2}} + w x y^{\frac{r-3}{2}} + \dots + w x^{\frac{s-1}{2}} y^{\frac{r-s}{2}} = \beta_0 z y^{\frac{r-1}{2}} + \beta_1 z x y^{\frac{r-3}{2}} + \beta_{\frac{s-1}{2}} z x^{\frac{s-1}{2}} y^{\frac{r-s}{2}},$$

where $\beta_i \in \mathbb{Z}_2$. Note that, we can write w^2 as

$$w^2 = \gamma_1 z w + \gamma_2 x + \gamma_3 y,$$

where $\gamma_i \in \mathbb{Z}_2$. Therefore,

$$H^*(X/G) \cong H^*(X_G) \cong \mathbb{Z}_2[x, y, z, w]/I,$$

where

$$I = \left\langle z^2, w^2 - \gamma_1 zw - \gamma_2 x - \gamma_3 y, x^{\frac{s+1}{2}} + \alpha_0 zw x^{\frac{s-1}{2}} + \alpha_1 zw x^{\frac{s-3}{2}} y + \dots + \alpha_{\frac{s-1}{2}} zw y^{\frac{s-1}{2}}, \right. \\ \left. (w + \beta_0 z)y^{\frac{r-1}{2}} + (w + \beta_1 z)xy^{\frac{r-3}{2}} + \dots + (w + \beta_{\frac{s-1}{2}} z)x^{\frac{s-1}{2}} y^{\frac{r-s}{2}} \right\rangle,$$

with $\deg(x) = 2, \deg(y) = 2, \deg(z) = 1, \deg(w) = 1$ and $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}_2$. □

For the complex case, we prove the following.

THEOREM 6.2. *Let $G = \mathbb{Z}_2$ act freely on a compact Hausdorff space $X \simeq_2 \mathbb{C}H_{r,s}$, such that induced action on mod 2 cohomology is trivial. Then,*

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y, z, w]/J,$$

where

$$J = \left\langle z^3, w^2 - \gamma_1 z^2 w - \gamma_2 x - \gamma_3 y, x^{\frac{s+1}{2}} + \alpha_0 z^2 w x^{\frac{s-1}{2}} + \alpha_1 z^2 w x^{\frac{s-3}{2}} y + \dots + \alpha_{\frac{s-1}{2}} z^2 w y^{\frac{s-1}{2}}, \right. \\ \left. (w + \beta_0 z^2)y^{\frac{r-1}{2}} + (w + \beta_1 z^2)xy^{\frac{r-3}{2}} + \dots + (w + \beta_{\frac{s-1}{2}} z^2)x^{\frac{s-1}{2}} y^{\frac{r-s}{2}} \right\rangle,$$

with $\deg(x) = 4, \deg(y) = 4, \deg(z) = 1, \deg(w) = 2$ and $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}_2$.

Proof. Let $G = \mathbb{Z}_2$ act freely on $X \simeq_2 \mathbb{C}H_{r,s}$. Note that $E_2^{k,l} = 0$ for l odd. This gives

$$d_2 : E_2^{k,l} \rightarrow E_2^{k+2,l-1}$$

is zero, and hence $E_3^{k,l} = E_2^{k,l}$ for all k, l . Let $a, b \in H^2(X)$ be generators of the cohomology algebra $H^*(X)$. As in the proof of the Theorem 6.1, the only possibility for d_3 is

$$d_3(1 \otimes a) = t^3 \otimes 1 \quad \text{and} \quad d_3(1 \otimes b) = t^3 \otimes 1.$$

Note that r and s must be odd. For various values of l , we consider the differentials

$$d_3 : E_3^{0,2l} \rightarrow E_3^{3,2l-2}.$$

If we compute the ranks of $\text{Ker } d_3$ and $\text{Im } d_3$, we get that $\text{rk}(E_4^{k,2l}) = 0$ for all $k \geq 3$. This implies that $E_4^{k,2l} = 0$ for all $k \geq 3$ and $E_4^{k,2l} = \text{Ker}\{d_3 : E_3^{k,2l} \rightarrow E_3^{k+3,2l-2}\}$ for $k = 0, 1, 2$. Also,

$$d_r : E_r^{k,l} \rightarrow E_r^{k+r,l-r+1}$$

is zero for all $r \geq 4$. Hence, $E_\infty^{*,*} \cong E_4^{*,*}$. Since $H^*(X_G) \cong \text{Tot} E_\infty^{*,*}$, we get

$$H^n(X_G) \cong \bigoplus_{i+j=n} E_\infty^{i,j} = E_\infty^{0,n} \oplus E_\infty^{1,n-1} \oplus E_\infty^{2,n-2}$$

for all $0 \leq p \leq 2(s + r - 1)$.

Note that $t \otimes 1$ is a permanent cocycle and let $z = \pi^*(t) \in E_\infty^{1,0} \subseteq H^1(X_G)$ be determined by $t \otimes 1 \in E_2^{1,0}$. As $E_\infty^{3,0} = 0$, we have $z^3 = 0$. Also, $1 \otimes (a + b) \in E_2^{0,2}$ is a permanent cocycle. Let $w \in H^2(X_G)$ such that $i^*(w) = a + b$. Also, $1 \otimes a^2 \in E_2^{0,4}$ and $1 \otimes b^2 \in E_2^{0,4}$ are permanent cocycles, and hence they determine elements in $E_\infty^{0,4}$. Let $x, y \in H^4(X_G)$ such that $i^*(x) = a^2$ and $i^*(y) = b^2$. As $a^{s+1} = 0$, we get the following relation:

$$x^{\frac{s+1}{2}} + \alpha_0 z^2 w x^{\frac{s-1}{2}} + \alpha_1 z^2 w x^{\frac{s-3}{2}} y + \dots + \alpha_{\frac{s-1}{2}} z^2 w y^{\frac{s-1}{2}} = 0,$$

where $\alpha_i \in \mathbb{Z}_2$. Notice that,

$$i^*(w y^{\frac{r-1}{2}} + w x y^{\frac{r-3}{2}} + \dots + w x^{\frac{s-1}{2}} y^{\frac{r-s}{2}}) = 0.$$

Hence, it satisfies

$$w y^{\frac{r-1}{2}} + w x y^{\frac{r-3}{2}} + \dots + w x^{\frac{s-1}{2}} y^{\frac{r-s}{2}} = \beta_0 z^2 y^{\frac{r-1}{2}} + \beta_1 z^2 x y^{\frac{r-3}{2}} + \beta_{\frac{s-1}{2}} z^2 x^{\frac{s-1}{2}} y^{\frac{r-s}{2}},$$

where $\beta_i \in \mathbb{Z}_2$. Note that, w^2 satisfies the following relation:

$$w^2 = \gamma_1 z^2 w + \gamma_2 x + \gamma_3 y,$$

where $\gamma_i \in \mathbb{Z}_2$. Therefore,

$$H^*(X/G) \cong H^*(X_G) \cong \mathbb{Z}_2[x, y, z, w]/J,$$

where

$$J = \left\langle z^3, w^2 - \gamma_1 z^2 w - \gamma_2 x - \gamma_3 y, x^{\frac{s+1}{2}} + \alpha_0 z^2 w x^{\frac{s-1}{2}} + \dots + \alpha_{\frac{s-1}{2}} z^2 w y^{\frac{s-1}{2}}, \right. \\ \left. (w + \beta_0 z^2) y^{\frac{r-1}{2}} + (w + \beta_1 z^2) x y^{\frac{r-3}{2}} + \dots + (w + \beta_{\frac{s-1}{2}} z^2) x^{\frac{s-1}{2}} y^{\frac{r-s}{2}} \right\rangle,$$

with $\deg(x) = 4, \deg(y) = 4, \deg(z) = 1, \deg(w) = 2$ and $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}_2$. This completes the proof. □

For \mathbb{S}^1 -actions, we obtain the following.

THEOREM 6.3. *Let $G = \mathbb{S}^1$ act freely on a compact Hausdorff space $X \simeq_2 \mathbb{R}H_{r,s}$. Then,*

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y, w]/\langle x^{\frac{s+1}{2}}, w y^{\frac{r-1}{2}} + x w y^{\frac{r-3}{2}} + \dots + w x^{\frac{s-1}{2}} y^{\frac{r-s}{2}}, w^2 - \alpha x - \beta y \rangle,$$

where $\deg(x) = 2, \deg(y) = 2, \deg(w) = 1$ and $\alpha, \beta \in \mathbb{Z}_2$.

Proof. By Proposition 5.9, the only possibility for the differential d_2 is that $d_2(1 \otimes a) = u \otimes 1, d_2(1 \otimes b) = u \otimes 1$ and both r and s are odd. As in the proof of Theorem 6.1, if we compute the ranks of $\text{Ker } d_2$ and $\text{Im } d_2$, we get $\text{rk}(E_3^{k,l}) = 0$, and hence $E_3^{k,l} = 0$ for all $k \geq 1$ and for all l . Also, $E_3^{0,l} = \text{Ker}\{d_2 : E_2^{0,l} \rightarrow E_2^{2,l-1}\}$ for all l .

Note that, $d_r : E_r^{k,l} \rightarrow E_r^{k+r,l-r+1}$ is trivial for all $r \geq 3$ and for all k, l . Hence, $E_\infty^{*,*} = E_3^{*,*}$. Since $H^*(X_G) \cong \text{Tot}E_\infty^{*,*}$, we have

$$H^n(X_G) \cong \bigoplus_{i+j=n} E_\infty^{i,j} = E_\infty^{0,n}$$

for all $0 \leq n \leq r + s - 1$.

We see that $1 \otimes (a + b) \in E_2^{0,1}$ is a permanent cocycle. Let $w \in H^1(X_G)$ such that $i^*(w) = a + b$. Also, $1 \otimes a^2 \in E_2^{0,2}$ and $1 \otimes b^2 \in E_2^{0,2}$ are permanent cocycles. Hence, they determine elements in $E_\infty^{0,2}$. Let $x, y \in H^2(X_G)$ such that $i^*(x) = a^2$ and $i^*(y) = b^2$. As $a^{s+1} = 0$, we get $x^{\frac{s+1}{2}} = 0$. Note that

$$i^*(wy^{\frac{r-1}{2}} + xwy^{\frac{r-3}{2}} + \dots + wx^{\frac{s-1}{2}}y^{\frac{r-s}{2}}) = 0.$$

Hence, we get the following relation:

$$wy^{\frac{r-1}{2}} + xwy^{\frac{r-3}{2}} + \dots + wx^{\frac{s-1}{2}}y^{\frac{r-s}{2}} = 0.$$

Note that, we can write w^2 as

$$w^2 = \alpha x + \beta y,$$

for some $\alpha, \beta \in \mathbb{Z}_2$. Therefore,

$$H^*(X/G) \cong H^*(X_G) \cong \mathbb{Z}_2[x, y, w]/\langle x^{\frac{s+1}{2}}, wy^{\frac{r-1}{2}} + xwy^{\frac{r-3}{2}} + \dots + wx^{\frac{s-1}{2}}y^{\frac{r-s}{2}}, w^2 - \alpha x - \beta y \rangle,$$

where $\deg(x) = 2, \deg(y) = 2, \deg(w) = 1$ and $\alpha, \beta \in \mathbb{Z}_2$. □

EXAMPLE 6.4. Take $r = 3$ and $s = 1$. Recall that, $\mathbb{R}H_{3,1}$ is a three-dimensional closed smooth manifold. A free \mathbb{S}^1 -action on $\mathbb{R}H_{3,1}$ gives a principal \mathbb{S}^1 -bundle $\mathbb{R}H_{3,1} \rightarrow \mathbb{R}H_{3,1}/\mathbb{S}^1$ with compact two-dimensional base. Now, using the Leray–Serre spectral sequence associated to the Borel fibration, one can see that $H^1(\mathbb{R}H_{3,1}/\mathbb{S}^1; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Hence, the orbit space must be $\mathbb{R}P^2$ and its cohomology algebra matches with our result for $\beta = 1$.

COROLLARY 6.5. *Let $G = \mathbb{S}^1$ act freely on a compact Hausdorff space $X \simeq_2 \mathbb{R}H_{r,s}$. Then, the Euler class of the principal G -bundle $X \xrightarrow{q} X/G$ is zero.*

Proof. From Theorem 6.3, we get $H^i(X/G) = \mathbb{Z}_2$ for $i = 0, 1$ and $H^2(X/G) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The Gysin sequence of the G -bundle $X \xrightarrow{q} X/G$ is

$$0 \rightarrow H^1(X/G) \xrightarrow{q^*} H^1(X) \rightarrow H^0(X/G) \xrightarrow{\smile e} H^2(X/G) \xrightarrow{q^*} \dots,$$

where $e \in H^2(X/G)$ is the Euler class. The conclusion now follows from the sequence. □

7. Applications to equivariant maps. Let X be a compact Hausdorff space with a free involution and \mathbb{S}^n the unit n -sphere equipped with the antipodal involution. Conner and Floyd [7] asked; for which integer n , there exists a \mathbb{Z}_2 -equivariant map from \mathbb{S}^n to X , but no such map from \mathbb{S}^{n+1} to X .

For $X = \mathbb{S}^n$, by the Borsuk–Ulam theorem, the answer to the preceding question is n . In the same paper, Conner and Floyd defined the index of the involution on X as

$$\text{ind}(X) = \max \{ n \mid \text{there exists a } \mathbb{Z}_2\text{-equivariant map } \mathbb{S}^n \rightarrow X \}.$$

The characteristic classes with \mathbb{Z}_2 coefficients can be used to derive a cohomological criteria to study the above question. Let $w \in H^1(X/G; \mathbb{Z}_2)$ be the Stiefel–Whitney class of the principal G -bundle $X \rightarrow X/G$. Conner and Floyd also defined

$$\text{co-ind}_{\mathbb{Z}_2}(X) = \max \{ n \mid w^n \neq 0 \}.$$

Since $\text{co-ind}_{\mathbb{Z}_2}(\mathbb{S}^n) = n$, by [7, (4.5)], we obtain

$$\text{ind}(X) \leq \text{co-ind}_{\mathbb{Z}_2}(X).$$

Using these indices, we obtain the following results.

PROPOSITION 7.1. *Let $X \simeq_2 \mathbb{R}H_{r,s}$ be a compact Hausdorff space, where $1 \leq s < r$. Then, there is no \mathbb{Z}_2 -equivariant map $\mathbb{S}^k \rightarrow X$ for $k \geq 2$.*

Proof. Take a classifying map

$$f : X/G \rightarrow B_G$$

for the principal G -bundle $X \rightarrow X/G$. Let $\eta : X/G \rightarrow X_G$ is a homotopy inverse of the homotopy equivalence $h : X_G \rightarrow X/G$. Then, $\pi\eta : X/G \rightarrow B_G$ also classifies the principal G -bundle $X \rightarrow X/G$, and hence it is homotopic to f . Therefore, it suffices to consider the map

$$\pi^* : H^1(B_G) \rightarrow H^1(X_G).$$

The image of the Stiefel–Whitney class of the universal principal G -bundle $G \hookrightarrow E_G \rightarrow B_G$ is the Stiefel–Whitney class of $X \rightarrow X/G$. For $X \simeq_2 \mathbb{R}H_{r,s}$, using the proof of Theorem 6.1, we see that $x \in H^1(X/G)$ is the Stiefel–Whitney class with $x \neq 0$ and $x^2 = 0$. This gives $\text{co-ind}_{\mathbb{Z}_2}(X) = 1$ and $\text{ind}(X) \leq 1$. Hence, there is no \mathbb{Z}_2 -equivariant map $\mathbb{S}^k \rightarrow X$ for $k \geq 2$. □

PROPOSITION 7.2. *Let $X \simeq_2 \mathbb{C}H_{r,s}$ be a compact Hausdorff space, where $1 \leq s < r$. Then, there is no \mathbb{Z}_2 -equivariant map $\mathbb{S}^k \rightarrow X$ for $k \geq 3$.*

Proof. From the proof of Theorem 6.2, $x \in H^1(X/G)$ is the Stiefel–Whitney class with $x^2 \neq 0$ and $x^3 = 0$. This gives $\text{co-ind}_{\mathbb{Z}_2}(X) = 2$ and $\text{ind}(X) \leq 2$. Hence, there is no \mathbb{Z}_2 -equivariant map $\mathbb{S}^k \rightarrow X$ for $k \geq 3$. □

Given a G -space X , Volovikov [33] defined another numerical index $i(X)$ as the smallest r such that for some k , the differential

$$d_r : E_r^{k-r,r-1} \rightarrow E_r^{k,0}$$

in the Leray–Serre spectral sequence of the fibration $X \xrightarrow{i} X_G \xrightarrow{\pi} B_G$ is non-trivial. It is clear that $i(X) = r$ if $E_2^{k,0} = E_3^{k,0} = \dots = E_r^{k,0}$ for all k and $E_r^{k,0} \neq E_{r+1}^{k,0}$ for some k . If $E_2^{*,0} = E_\infty^{*,0}$, then $i(X) = \infty$. Thus, $i(X)$ is either an integer greater than 1 or ∞ . Using this index, Coelho, Mattos and Santos proved the following [6, Theorem 1.1] result.

PROPOSITION 7.3. *Let G be a compact Lie group and X, Y be path-connected compact Hausdorff spaces with free G -actions. Suppose that $i(X) \geq m + 1$ for some natural $m \geq 1$. If $H^{k+1}(Y/G; \mathbb{Z}_2) = 0$ for some $1 \leq k < m$ and $0 < \text{rk}(H^{k+1}(B_G))$, then there is no G -equivariant map $f : X \rightarrow Y$.*

The preceding result together yields the following.

PROPOSITION 7.4. *Suppose \mathbb{Z}_2 acts freely on $X \simeq_2 \mathbb{C}H_{r,s}$ and a path-connected compact Hausdorff space Y such that $H^2(Y/G) = 0$. Then, there is no \mathbb{Z}_2 -equivariant map $X \rightarrow Y$.*

Proof. Note that, we obtained $i(X) = 3$ in the proof of Theorem 6.2. Now, the result is a consequence of Proposition 7.3. □

Let G be a finite group considered as a 0-dimensional simplicial complex and X a paracompact space with a free G -action. The Schwarz genus $g_{\text{free}}(X, G)$ of the free G -space X is the smallest number n such that there exists a G -equivariant map

$$X \rightarrow G * \cdots * G,$$

the n -fold join of G equipped with the diagonal G -action. Note that for $G = \mathbb{Z}_2$, the free genus is the least integer n for which there exists a \mathbb{Z}_2 -equivariant map $f : X \rightarrow S^{n-1}$. See [32, Chapter V] for the original source and [2, 33] for more details and applications. In the literature, the free genus for $G = \mathbb{Z}_2$ is known under different names, for example, B -index [34], co-index [7] and level [22].

PROPOSITION 7.5. *Let $X \simeq_2 \mathbb{C}H_{r,s}$ be a compact Hausdorff space with a free \mathbb{Z}_2 -action. Then, $g_{\text{free}}(X, \mathbb{Z}_2) \geq 3$. In particular, there does not exist any \mathbb{Z}_2 -equivariant map $X \rightarrow \mathbb{S}^1$.*

Proof. It follows from [7] that

$$g_{\text{free}}(X, \mathbb{Z}_2) \geq \text{co-ind}_{\mathbb{Z}_2}(X) + 1,$$

and hence $g_{\text{free}}(X, \mathbb{Z}_2) \geq 3$. □

Let G be a finite group and X a G -space. Given a continuous map $f : X \rightarrow Y$, the coincidence set $A(f, k)$ is defined as

$$A(f, k) = \{x \in X \mid \exists \text{ distinct } g_1, \dots, g_k \in G \text{ such that } f(g_1x) = \cdots = f(g_kx)\}.$$

The following result of Schwarz [32] relates the free genus and the coincidence set.

THEOREM 7.6. *Let X be a paracompact connected space with a free \mathbb{Z}_p -action. Suppose that $g_{\text{free}}(X, \mathbb{Z}_p) > m(p - 1)$. Then, for any continuous map $f : X \rightarrow \mathbb{R}^m$*

$$g_{\text{free}}(A(f, p), \mathbb{Z}_p) \geq g_{\text{free}}(X, \mathbb{Z}_p) - m(p - 1).$$

In particular, the set $A(f, p)$ is non-empty.

As a consequence of the preceding theorem and Proposition 7.5, we obtain the following.

PROPOSITION 7.7. *Let $X \simeq_2 \mathbb{C}H_{r,s}$ be a compact Hausdorff space with a free \mathbb{Z}_2 -action. Then, any continuous map $X \rightarrow \mathbb{R}^2$ has a non-empty coincidence set.*

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