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A CHARACTERISATION OF SIMPLE GROUPS $PSL(5, q)$

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Order components of a finite group are introduced in [**4**]. We prove that, for every q, $PSL(5, q)$ can be uniquely determined by its order components. A main consequence of our result is the validity of Thompson's conjecture for the groups under consideration.

1. INTRODUCTION.

If n is an integer, $\pi(n)$ is the set of prime divisors of n and if G is a finite group $\pi(G)$ is defined to be $\pi(|G|)$. The prime graph $\Gamma(G)$ of a group G is a graph whose vertex set is $\pi(G)$, and two distinct primes p and q are linked by an edge if and only if G contains an element of order pq. Let π_i, $i = 1, 2, \ldots, t(G)$ be the connected components of $\Gamma(G)$. For $|G|$ even, π_1 will be the connected component containing 2. Then $|G|$ can be expressed as a product of some positive integers m_i, $i = 1, 2, \ldots, t(G)$ with $\pi(m_i) =$ the vertex set of π_i. The integers m_i's are called the order components of G. The set of order components of G will be denoted by $OC(G)$. If the order of G is even, then m_1 is the even order component and $m_2, \ldots, m_{t(G)}$ will be the odd order components of G. The order components of non-Abelian simple groups having at least three prime graph components are obtained by Chen [**8**, Tables 1,2,3]. The order components of non-Abelian simple groups with two order components are illustrated in Table 1 according to [**13, 18**]. The following groups are uniquely determined by their order components. Suzuki-Ree groups ([**6**]), Sporadic simple groups ([**3**]), $PSL_2(q)$ ([**8**]), $PSL_3(q)$ where q is an odd prime power ([**10**]), $E_8(q)$ ([**7**]), $F_4(q)$ $[q = 2^n]$ ([**12**]), ${}^2G_2(q)$ ([**2**]) and A_p where p and $p-2$ are primes ([**11**]). In this paper, we prove that $PSL(5, q)$ is also uniquely determined by its order components, where q is a prime power.

THE MAIN THEOREM. *Let G be a finite group, $M = PSL(5, q)$ and $OC(G) = OC(M)$. Then $G \cong M$.*

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2. Preliminary Results

In this section we state some preliminary lemmas to be used in the proof of the main theorem.

Definition 2.1: ([**9**]) A finite group G is called a 2-Frobenius group if it has a normal series $G > K > H > 1$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 2.2. ([**18**, Theorem A]) *If G is a finite group and its prime graph has more than one component, then G is one of the following groups:*

- (a) *a simple group;*
- (b) *a Frobenius or 2-Frobenius group;*
- (c) *an extension of a π_1- group by a simple group;*
- (d) *an extension of a simple group by a π_1-solvable group;*
- (e) *an extension of a π_1-group by a simple group by a π_1-group.*

Lemma 2.3. ([**18**, Corollary]) *If G is a solvable group with at least two prime graph components, then G is either a Frobenius group or a 2-Frobenius group and G has exactly two prime graph components one of which consists of the primes dividing the lower Frobenius complement.*

By Lemma 2.3 and the proof of Lemma 2.2 in [**18**] we can state the following lemma.

Lemma 2.4. *If G is a finite group and its prime graph has more than one component, then G is either:*

- (a) *a Frobenius or 2-Frobenius group; or*
- (b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, and $\overline{K} := K/H$ is a non-Abelian simple group with $\pi_i' \subseteq \pi(\overline{K})$ for all $i > 1$. Moreover, $\overline{K} \trianglelefteq G/H \leqslant \mathrm{Aut}(\overline{K})$.*

Lemma 2.5. ([**1**, Theorem 1]) *Let G be a Frobenius group of even order, and let H and K be Frobenius complement and Frobenius kernel of G respectively. Then $t(G) = 2$, the prime graph components of G are $\pi(H)$ and $\pi(K)$, and G has one of the following structures.*

- (a) *$2 \in \pi(K)$, all Sylow subgroups of H are cyclic.*
- (b) *$2 \in \pi(H)$, K is an Abelian group, H is a solvable group, the Sylow subgroups of odd order of H are cyclic and the $2-$Sylow subgroups of H are cyclic or generalised quaternion groups.*
- (c) *$2 \in \pi(H)$, K is an Abelian group, and there exists $H_0 \leqslant H$ such that $|H : H_0| \leqslant 2$, $H_0 = Z \times SL(2,5)$, $\big(|Z|, 2 \cdot 3 \cdot 5\big) = 1$ and the Sylow subgroups of Z are cyclic.*

Lemma 2.6. ([1, Theorem 2]) *Let G be a 2-Frobenius group of even order. Then $t(G)=2$ and there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that, $\pi_1 = \pi(G/K) \cup \pi(H)$, $\pi(K/H) = \pi_2$, G/K and K/H are cyclic, $|G/K| \mid |\mathrm{Aut}(K/H)|$, $(|G/K|, |K/H|) = 1$ and $|G/K| < |K/H|$. Moreover, H is a nilpotent group.*

Lemma 2.7. ([5, Lemma 8]) *Let G be a finite group with $t(G) \geqslant 2$ and N a normal subgroup of G. If N is a π_i-group for some prime graph component of G and $m_1, m_2, \ldots, m_r$ are some of the order components of G but not a π_i-number, then, $m_1 \cdot m_2 \cdot \ldots \cdot m_r$ is a divisor of $|N|-1$.*

Lemma 2.8. *Let G be a finite group with $OC(G) = OC(M)$ where $M = PSL(5,q)$, and suppose $m_1(q)$ and $m_2(q)$ are the even and odd order components of M respectively. Then:*

(a) *If $p \in \pi(G)$, then $|S_p| < q^7$ or is equal to q^{10} where $S_p \in Syl_p(G)$.*

(b) *If q' is a power of a prime number, $q' \mid |G|$ and $q'-1 \equiv 0 \pmod{m_2(q)}$, then $q' = q^5$ or q^{10}.*

(c) *If q' is a power of a prime number, $q' \mid |G|$, then $q'+1 \not\equiv 0 \pmod{m_2(q)}$.*

(d) *$m_2(q) - \varepsilon$ for $\varepsilon = -1, 2, 3$ and $q^\alpha + 1$ for $\alpha = 5$, 10 do not divide $m_1(q)$.*

Proof: Since $|M| = |G| = q^{10}(q^2-1)(q^3-1)(q^4-1)(q^5-1)/k$ where $k = (5, q-1)$, it is easy to show that (a) holds. To prove (b) and (c), let $q' = p^\alpha$ where p is a prime number. Since $|G| = q^{10}(q-1)^4(q+1)^2(q^2+1)(q^2+q+1) \cdot m_2(q)$ where $m_2(q) = (q^5-1)/\bigl(k(q-1)\bigr)$, q' must divide one of the coprime factors q^{10}, $(q-1)^4$, $3(q-1)^4$, $(q+1)^2$, $(q^2+q+1)/3$, q^2+q+1 or q^2+1 if q is an even number or q^{10}, $(q-1)^4/16$, $3(q-1)^4/16$, $8(q-1)^4$, $12(q-1)^4$, $16(q-1)^4$, $48(q-1)^4$, $(q+1)^2/8$, $(q+1)^2/4$, $32(q+1)^2$, $(q^2+1)/2$, q^2+1, $(q^2+q+1)/3$ or q^2+q+1 if q is an odd number. Therefore, it is sufficient to consider the cases where q' divides q^{10}, $48(q-1)^4$, $32(q+1)^2$, q^2+1 or q^2+q+1. Assume that $q' + \varepsilon \equiv 0 \bmod \bigl(m_2(q)\bigr)$ where $\varepsilon = \pm 1$, that is, $q' + \varepsilon = rm_2(q)$ for some positive integer r. If $q' \mid 48(q-1)^4$, then $48(q-1)^4 = sq'$ for some integers s and therefore, $48(q-1)^4 + \varepsilon s = rsm_2(q)$. If $k=5$, we have $A(q) = (rs-240)q^4 + (rs+960)q^3 + (rs-1440)q^2 + (rs+960)q + rs - 5\varepsilon s - 240 = 0$, but for $rs > 240$ it is easy to see $A(q) > 0$ and for $rs \leqslant 240$ we can get a contradiction by calculation. For the case $k=1$, we also get a contradiction by a similar method. If $q' \mid 32(q+1)^2$, q^2+1 or q^2+q+1, then $m_2(q) \leqslant 32(q+1)^2+1$, q^2+2 or q^2+q+2 respectively, and by calculation we get a contradiction. Therefore, $q' \mid q^{10}$. If $q \leqslant 5$, by calculation we can see that (b) and (c) are valid. Thus we may assume that $q > 5$. Since $m_2(q) \leqslant q' + \varepsilon$, we have $q' > q^3$, $q' = q^3 p^n$ and $q' + \varepsilon = rm_2(q)$ for some positive integer n.

Now let $\varepsilon = -1$ and $q' - 1 = rm_2(q)$. If $q' \leqslant q^5$, then $rm_2(q) = q' - 1 \leqslant q^5 - 1$, thus $r \leqslant k(q-1)$. On the other hand, $r(q^4+q^3+q^2+q) + r + k = kq' = kq^3p^n$, so $q \mid r+k$, thus, $q \leqslant r+k \leqslant kq$. If $k=1$, then $r+1 = q$ and thus $q' = q^5$. If $k=5$, then $r+k = tq$ where $t = 1, 2, \ldots, 5$. For $t \neq 5$ we have $r(q^3+q^2+q) + r + t = 5q^2p^n$, so $q \mid r+t$

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but $q \mid r+k = r+5$, thus $q \mid 5-t$, that is, $q < 5$ which is a contradiction. If $t = 5$, then $r + 5 = 5q$, and thus $q' = q^5$. Now let $q' > q^5$, then $q' = q^5 p^m$ for some positive integer m. By (a), $q' \leqslant q^{10}$, so $p^m \leqslant q^5$. Since $q' - 1 = kp^m(q-1)m_2(q) + p^m - 1$ and $m_2(q) \mid q' - 1$, then $p^m - 1 \equiv 0 \pmod{m_2(q)}$ with $p^m \leqslant q^5$ and by a similar method as the last case we must have $p^m = q^5$, thus $q' = q^{10}$ and therefore, (b) is proved. Suppose that $\varepsilon = 1$ and $q' + 1 = rm_2(q)$. If $q' \leqslant q^5$, as above, $r < k(q-1)$ and $q \mid r - k$, and thus, $q \leqslant r - k < k(q-2)$. If $k = 1$, then $q \leqslant r - 1 \leqslant q - 1$ which is impossible. If $k = 5$, then $r - 5 = tq$ where $t = 1, 2, 3, 4$. Thus $r(q^3 + q^2 + q) + r + t = 5q^2p^n$, so $q \mid r + t$. But $q \mid r - 5$, thus $q \mid t + 5$, that is, $q = 2, 3, 4, 7, 8$ or 9 and it contradicts $k = 5$. Therefore, $q' > q^5$, that is, $q' = q^5p^m$ for some positive integer m. By (a), $q' \leqslant q^{10}$, so $p^m \leqslant q^5$. Since $q' + 1 = kp^m(q-1)m_2(q) + p^m + 1$ and $m_2(q) \mid q' + 1$, then $p^m + 1 \equiv 0 \pmod{m_2(q)}$ with $p^m \leqslant q^5$ and as above we get a contradiction. Now the proof of (c) is completed. Since the proof of (d) is similar for each cases, we present one of them. Let $\varepsilon = -1$. Since $m_1(q) = f(q)(m_2(q) + 1) + r(q)$ where $r(q) = -20q^3 + 40q^2 - 84q + 40$ for $5 \nmid q - 1$, and $r(q) = -4980q^3 + 7800q^2 - 8820q - 3240$ for $5 \mid q - 1$, if $m_2(q) + 1 \mid m_1(q)$, then $r(q) = 0$. This has no solution which is contradiction. ∎

Lemma 2.9. *Let G be a finite group and $OC(G) = OC(M)$ where $M = PSL(5,q)$. Then G is neither a Frobenius group nor a 2−Frobenius group.*

Proof: If G is a Frobenius group, then by Lemma 2.5, $OC(G) = \{|H|, |K|\}$ where H and K are the Frobenius complement and the Frobenius kernel of G, respectively. Suppose that $2 \mid |K|$, then $|K| = m_1(q)$ and $|H| = m_2(q)$. Let p be a prime number which divides $|K|$ and $p \nmid q$. By nilpotency of K, S_p must be a unique normal subgroup of G where S_p is p−Sylow subgroup of K. Thus $m_2(q) \mid |S_p| - 1$ by Lemma 2.7. Therefore, $|S_p| = q^5$ or q^{10} by Lemma 2.8(b), which is a contradiction. If $2 \mid |H|$, then $|H| = m_1(q)$ and $|K| = m_2(q)$. Since $|H|$ divides $|K| - 1$, $m_1(q) \mid m_2(q) - 1$, which is a contradiction.

Let G be a 2−Frobenius group. By Lemma 2.6, there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $|K/H| = m_2(q)$ and $|G/K| < |K/H|$. Since $|K/H| = m_2(q) = (q^5-1)/\big((q-1)(5, q-1)\big) < (q^2+1)(q^2+q+1)$, there exists a prime number p such that $p \mid (q^2 + 1)(q^2 + q + 1)$ and p does not divide $|G/K|$, that is, $p \mid |H|$ since $\pi_1 = \pi(G/K) \cup \pi(H)$. But S_p, the p−Sylow subgroup of H, must be a normal subgroup of K because H is nilpotent. Therefore, $m_2(q) \mid |S_p| - 1$ by Lemma 2.7 which contradicts Lemma 2.8(b). ∎

Lemma 2.10. *Let G be a finite group. If the order components of G are the same as those of $M = PSL(5,q)$, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ satisfying the following two conditions:*

(a) *H and G/K are π_1−groups, K/H is a non-Abelian simple group and H is a nilpotent group.*

(b) *The odd order component of M is equal to the odd order component of some K/H. Especially, $t(K/H) \geqslant 2$.*

PROOF: Part (a) of the Lemma follows from Lemma 2.4 and 2.9 because the prime graph of M has two prime graph components.

To prove (b) note if p and q are prime numbers, then if K/H has an element of order pq, so G has an element of order pq. Hence by the definition of prime graph components, an odd order component of G must be an odd order component of K/H. If q is odd, then by Table I, the number of order components of M is equal to two. Therefore, $t(K/H) \geqslant 2$. ∎

In the next section we prove the Main Theorem.

3. PROOF OF THE MAIN THEOREM.

By Lemma 2.10, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-Abelian simple group where $t(K/H) \geqslant 2$, and the odd order component of M is the odd order component of some K/H, that is, one of the odd order component of K/H is equal to $m_2(q) = (q^5 - 1)/\big((q-1)(5, q-1)\big)$. We summarise the relevant information in Tables I-III below.

Table I

The order components of simple groups[1] with $t(G) = 2$

Group	Orcmp1	Orcmp2
A_p, $p \neq 5, 6$ p and $p-2$ not both prime	$3 \cdot 4 \cdots (p-3)(p-2)(p-1)$	p
A_{p+1}, $p \neq 4, 5$ $p-1$ and $p+1$ not both prime	$3 \cdot 4 \cdots (p-2)(p-1)(p+1)$	p
A_{p+2}, $p \neq 3, 4$ p and $p+2$ not both prime	$3 \cdot 4 \cdots (p-1)(p+1)(p+2)$	p
$A_{p-1}(q)$, $(p, q) \neq (3, 2), (3, 4)$	$q^{p(p-1)/2}\prod_{i=1}^{p-1}(q^i - 1)$	$\dfrac{q^p - 1}{(q-1)(p, q-1)}$
$A_p(q)$, $q - 1 \mid p + 1$	$q^{p(p+1)/2}(q^{p+1} - 1)\prod_{i=2}^{p-1}(q^i - 1)$	$\dfrac{q^p - 1}{q - 1}$
${}^2A_{p-1}(q)$	$q^{p(p-1)/2}\prod_{i=1}^{p-1}(q^i - (-1)^i)$	$\dfrac{q^p + 1}{(q+1)(p, q+1)}$
${}^2A_p(q)$, $q + 1 \mid p + 1$ $(p, q) \neq (3, 3), (5, 2)$	$q^{p(p+1)/2}(q^{p+1} - 1)\prod_{i=2}^{p-1}(q^i - (-1)^i)$	$\dfrac{q^p + 1}{q + 1}$

[1] p is an odd prime number.

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Table I (continued)

Group	Orcmp1	Orcmp2
${}^2A_3(2)$	$2^6 \cdot 3^4$	5
$B_n(q)$, $n = 2^m \geqslant 4, q$ odd	$q^{n^2}(q^n - 1)\Pi_{i=1}^{n-1}(q^{2i} - 1)$	$\frac{q^n + 1}{2}$
$B_p(3)$	$3^{p^2}(3^p + 1)\Pi_{i=1}^{p-1}(3^{2i} - 1)$	$\frac{3^p - 1}{2}$
$C_n(q)$, $n = 2^m \geqslant 2$	$q^{n^2}(q^n - 1)\Pi_{i=1}^{n-1}(q^{2i} - 1)$	$\frac{q^n + 1}{(2, q-1)}$
$C_p(q)$, $q = 2, 3$	$q^{p^2}(q^p + 1)\Pi_{i=1}^{p-1}(q^{2i} - 1)$	$\frac{q^p - 1}{(2, q-1)}$
$D_p(q)$, $p \geqslant 5$, $q = 2, 3, 5$	$q^{p(p-1)}\Pi_{i=1}^{p-1}(q^{2i} - 1)$	$\frac{q^p - 1}{q - 1}$
$D_{p+1}(q)$, $q = 2, 3$	$\frac{1}{(2, q-1)}q^{p(p+1)}(q^p + 1)(q^{p+1} - 1)\Pi_{i=1}^{p-1}(q^{2i} - 1)$	$\frac{q^p - 1}{(2, q-1)}$
${}^2D_n(q)$, $n = 2^m \geqslant 4$	$q^{n(n-1)}\Pi_{i=1}^{n-1}(q^{2i} - 1)$	$\frac{q^n + 1}{(2, q+1)}$
${}^2D_n(2)$, $n = 2^m + 1 \geqslant 5$	$2^{n(n-1)}(2^n + 1)(2^{n-1} - 1)\Pi_{i=1}^{n-2}(2^{2i} - 1)$	$2^{n-1} + 1$
${}^2D_p(3)$, $p \neq 2^m + 1$, $p \geqslant 5$	$3^{p(p-1)}\Pi_{i=1}^{p-1}(3^{2i} - 1)$	$\frac{3^p + 1}{4}$
${}^2D_n(3)$, $n = 2^m + 1 \neq p, m \geqslant 2$	$\frac{1}{2}3^{n(n-1)}(3^n + 1)(3^{n-1} - 1)\Pi_{i=1}^{n-2}(3^{2i} - 1)$	$3^{n-1} + \frac{1}{2}$
$G_2(q)$, $q \equiv \varepsilon(\mathrm{mod}\, 3), \varepsilon = \pm 1$, $q > 2$	$q^6(q^3 - \varepsilon)(q^2 - 1)(q + \varepsilon)$	$q^2 - \varepsilon q + 1$
${}^3D_4(q)$	$q^{12}(q^6 - 1)(q^2 - 1)(q^4 + q^2 + 1)$	$q^4 - q^2 + 1$
$F_4(q)$, q odd	$q^{24}(q^8 - 1)(q^6 - 1)^2(q^4 - 1)$	$q^4 - q^2 + 1$
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2$	13
$E_6(q)$	$q^{36}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^5 - 1)(q^3 - 1)(q^2 - 1)$	$\frac{q^6 + q^3 + 1}{(3, q-1)}$
${}^2E_6(q)$, $q > 2$	$q^{36}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^5 + 1)(q^3 + 1)(q^2 - 1)$	$\frac{q^6 - q^3 + 1}{(3, q+1)}$
M_{12}	$2^6 \cdot 3^3 \cdot 5$	11
J_2	$2^7 \cdot 3^3 \cdot 5^2$	7
Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$	29
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3$	17
Mcl	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7$	11
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13$	23
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11$	23
Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11$	13
$F_5 = HN$	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11$	19

Table II

The order components of simple groups[1] with $t(G) \geqslant 3$

Group	Orcmp 1	Orcmp 2	Orcmp 3	Orcmp 4
A_p, p and $p-2$ are primes	$3 \cdot 4 \cdots (p-3)(p-1)$	$p-2$	p	
$A_1(q)$, $4 \mid q+1$	$q+1$	q	$(q-1)/2$	
$A_1(q)$, $4 \mid q-1$	$q-1$	q	$(q+1)/2$	
$A_1(q)$, $2 \mid q$	q	$q+1$	$q-1$	
$A_2(2)$	8	3	7	
$A_2(4)$	2^6	5	7	9
${}^2A_5(2)$	$2^{15} \cdot 3^6 \cdot 5$	7	11	
${}^2B_2(q)$ $q = 2^{2n+1} > 2$	q^2	$q - \sqrt{2q} + 1$	$q + \sqrt{2q} + 1$	$q-1$
${}^2D_p(3)$ $p = 2^n + 1, n \geqslant 2$	$2 \cdot 3^{p(p-1)}(3^{p-1} - 1)$ $\times \Pi_{i=1}^{p-2}(3^{2i} - 1)$	$(3^{p-1}+1)/2$	$(3^p + 1)/4$	
${}^2D_{p+1}(2)$ $p = 2^n - 1, n \geqslant 2$	$2^{p(p+1)}(2^p - 1)$ $\times \Pi_{i=1}^{p-1}(2^{2i} - 1)$	$2^p + 1$	$2^{p+1} + 1$	
$E_7(2)$	$2^{63} \cdot 3^{11} \cdot 5^2 \cdot 7^3$ $\cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43$	73	127	
$F_4(q)$ $2 \mid q, q > 2$	$q^{24}(q^6-1)^2(q^4-1)^2$	$q^4 + 1$	$q^4 - q^2 + 1$	
${}^2F_4(q)$ $q = 2^{2n+1} > 2$	$q^{12}(q^4-1)(q^3+1)$ $\times (q^2+1)(q-1)$	$q^2 - \sqrt{2q^3}$ $+q - \sqrt{2q} + 1$	$q^2 + \sqrt{2q^3}$ $+q + \sqrt{2q} + 1$	
$G_2(q)$, $3 \mid q$	$q^6(q^2-1)^2$	$q^2 + q + 1$	$q^2 - q + 1$	
${}^2G_2(q)$, $q = 3^{2n+1}$	$q^3(q^2-1)$	$q - \sqrt{3q} + 1$	$q + \sqrt{3q} + 1$	

[1] p is an odd prime number.

Table II (continued)

Group	Orcmp 1	Orcmp 2	Orcmp 3	Orcmp 4	Orcmp 5	Orcmp 6
$E_7(3)$	$2^{23}\cdot 3^{63}\cdot 5^2\cdot 7^3$ $\cdot 11^2\cdot 13^3\cdot 19\cdot 37\cdot 41$ $\cdot 61\cdot 73\cdot 547$	757	1093			
${}^2E_6(2)$	$2^{36}\cdot 3^9\cdot 5^2\cdot 7^2\cdot 11$	13	17	19		
M_{11}	$2^4\cdot 3^2$	5	11			
M_{22}	$2^7\cdot 3^2$	5	7	11		
M_{23}	$2^7\cdot 3^2\cdot 5\cdot 7$	11	23			
M_{24}	$2^{10}\cdot 3^3\cdot 5\cdot 7$	11	23			
J_1	$2^3\cdot 3\cdot 5$	7	11	19		
J_3	$2^7\cdot 3^5\cdot 5$	17	19			
J_4	$2^{21}\cdot 3^3\cdot 5\cdot 7\cdot 11^3$	23	29	31	37	43
HS	$2^9\cdot 3^2\cdot 5^3$	7	11			
Sz	$2^{13}\cdot 3^7\cdot 5^2\cdot 7$	11	13			
ON	$2^9\cdot 3^4\cdot 5\cdot 7^3$	11	19	31		
Ly	$2^8\cdot 3^7\cdot 5^6\cdot 7\cdot 11$	31	37	67		
Co_2	$2^{18}\cdot 3^6\cdot 5^3\cdot 7$	11	23			
F_{23}	$2^{18}\cdot 3^{13}\cdot 5^2\cdot 7\cdot 11\cdot 13$	17	23			
F'_{24}	$2^{21}\cdot 3^{16}\cdot 5^2\cdot 7^3\cdot 11\cdot 13$	17	23	29		
$F_1 = M$	$2^{46}\cdot 3^{20}\cdot 5^9\cdot 7^6\cdot 11^2\cdot 13^3$ $\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 47$	41	59	71		
$F_2 = B$	$2^{41}\cdot 3^{13}\cdot 5^6\cdot 7^2\cdot 11\cdot 13$ $\cdot 17\cdot 19\cdot 23$	31	47			
$F_3 = Th$	$2^{15}\cdot 3^{10}\cdot 5^3\cdot 7^2\cdot 13$	19	31			

Table III
The order components of $E_8(q)$

Group	$E_8(q)$, $q \equiv 0,1,4 (\bmod 5)$
Orcmp 1	$q^{120}(q^{18}-1)(q^{14}-1)(q^{12}-1)^2(q^{10}-1)^2(q^8-1)^2(q^4+q^2+1)$
Orcmp 2	$q^8+q^7-q^5-q^4-q^3+q+1$
Orcmp 3	$q^8-q^7+q^5-q^4+q^3-q+1$
Orcmp 4	$q^8-q^6+q^4-q^2+1$
Orcmp 5	q^8-q^4+1

Group	$E_8(q)$, $q \equiv 2,3 (\bmod 5)$
Orcmp 1	$q^{120}(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^8-1)(q^4+1)$ $\times(q^4+q^2+1)$
Orcmp 2	$q^8+q^7-q^5-q^4-q^3+q+1$
Orcmp 3	$q^8-q^7+q^5-q^4+q^3-q+1$
Orcmp 4	q^8-q^4+1

Since K/H is a non-Abelian simple group with $t(K/H) \geqslant 2$, then K/H must be isomorphic to one of the simple groups in Tables I, II or III.

If K/H is isomorphic to the alternating groups, by Table I and II and Lemma 2.10, we must have $m_2(q) = p-2$ or p, where $p \neq 5$ is an odd prime number. If $m_2(q) = p-2$, then $m_2(q)+1 = p-1$ divides $m_1(q)$ which contradicts Lemma 2.8(d), and, if $m_2(q) = p$, then $m_2(q) - 2 = p - 2 \mid m_1(q)$, which contradicts Lemma 2.8(d).

If K/H is isomorphic to one of the sporadic simple groups, $A_2(2)$, $A_2(4)$, ${}^2A_3(2)$, ${}^2A_5(2)$, $E_7(2)$, $E_7(3)$, ${}^2E_6(2)$ or ${}^2F_4(2)'$, by Lemma 2.10 we must have $m_2(q) = 3$, 5, 7, 9, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 67, 71, 73, 127, 757 or 1093. This equation has only one solution $m_2(q) = 31$ and in this case $q = 2$. Thus K/H can be isomorphic to J_4, ON, Ly, $F_2 = B$ or $F_3 = Th$. But in all the above cases $11 \in \pi(K/H)$ and $|G| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$ which is a contradiction because $|K/H| \mid |G|$.

If K/H is isomorphic to one the simple groups ${}^2A_n(q')$, $B_n(q')$ where $n = 2^m \geqslant 4$ and q' is odd, $C_n(q')$ where $n = 2^m \geqslant 2$, ${}^2D_n(q')$, $G_2(q')$ where $q' \equiv 1(\mathrm{mod}\ 3)$, ${}^3D_4(q')$, $F_4(q')$, ${}^2E_6(q')$ where $q' > 2$, ${}^2F_4(q')$ where $q' = 2^{2m+1} > 2$ or ${}^2G_2(q')$ where $q' = 3^{2m+1}$, using Tables I,II and Lemma 2.8(c) we can get a contradiction. For example, if $K/H \cong B_n(q')$ where $n = 2^m \geqslant 4$ and q' is odd, by Lemma 2.10 one of the odd order components of K/H must be $m_2(q) = (q^5-1)/\big((q-1)(5,q-1)\big)$. But by Table I, the odd order component of $B_n(q')$ where $n = 2^m \geqslant 4$ and q' is odd, is $({q'}^n+1)/2$ and thus ${q'}^n + 1 \equiv 0\big(\mathrm{mod}\ m_2(q)\big)$ which contradicts Lemma 2.8(c).

If K/H is isomorphic to $B_n(q')$ where $q' = 3$ and $n = p$ is an odd prime, $C_n(q')$ where $q' = 2$ or 3 and $n = p$ is an odd prime, $D_n(q')$ where $n = p+1$ and $q' = 2$ or 3 or $G_2(q')$ where $q' \equiv -1(\mathrm{mod}\ 3)$, then by Lemma 2.8(d) we get a contradiction. For example, if $K/H \cong B_p(3)$, then by Lemma 2.10, $3^p - 1 \equiv 0\big(\mathrm{mod}\ m_2(q)\big)$. Hence by Lemma 2.8(b), $3^p = q^5$ or q^{10}. Since $3^p + 1 \mid |K/H|$ then $q^5 + 1$ or $q^{10} + 1$ must divide $m_1(q)$ and this contradicts Lemma 2.8(d).

If K/H is isomorphic to $D_p(q')$ where $p \geqslant 5$ is an odd prime number and $q' = 2, 3$ or 5, $E_6(q')$ or $E_8(q')$, by Lemma 2.8(a) we get a contradiction. For example, if $K/H \cong D_p(q')$, then by Table I and Lemma 2.10, $m_2(q) = ({q'}^p - 1)/(q'-1)$, so ${q'}^p = q^5$ or q^{10} by Lemma 2.8(b). Since $p \geqslant 5$, then ${q'}^{p(p-1)} \geqslant {q'}^{4p} > q^{10}$ which contradicts Lemma 2.8(a).

If K/H is isomorphic to ${}^2B_2(q')$ where $q' = 2^{2m+1} > 2$ or $G_2(q')$ where $3 \mid q'$ we can get a contradiction by Lemma 2.8. For example, if $K/H \cong {}^2B_2(q')$, then $m_2(q) = q' - \sqrt{2q'} + 1$, $q' + \sqrt{2q'} + 1$ or $q' - 1$ by Table II and Lemma 2.10. If $m_2(q) = q' - \sqrt{2q'} + 1$ or $q' + \sqrt{2q'} + 1$, then ${q'}^2 + 1 \equiv 0\big(\mathrm{mod}\ m_2(q)\big)$ which contradicts Lemma 2.8(c). If $m_2(q) = q' - 1$, by Lemma 2.8(b) we must have $q' = q^5$ or q^{10}. If $q' = q^{10}$, then ${q'}^2 > q^{10}$ which contradicts Lemma 2.8(a). If $q' = q^5$, then $(q-1)(5,q-1) = 1$ which is impossible.

By the above argument we deduce that K/H must be isomorphic to a simple group of Lie type A_n. Now we claim that $K/H \cong A_4(q) = PSL(5,q)$ and therefore $H = 1$ and since $|K| = |G|$, we must have $G \cong M$ where $M = PSL(5,q)$. To prove this claim,

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we assume that $K/H \cong A_1(q')$. If $2 \mid q'$, then $m_2(q) = q'-1$ or $q'+1$ by Table II and Lemma 2.10. By Lemma 2.8(c) we must have $m_2(q) = q'-1$, and hence $q' = q^5$ or q^{10} by Lemma 2.8(b). Hence, q^5+1 or $q^{10}+1$ divide $m_1(q)$ which contradicts Lemma 2.8(d). If $q' \equiv \varepsilon (\mathrm{mod}\ 4)$ where $\varepsilon = \pm 1$, then by Table II and Lemma 2.10 we must have $m_2(q) = q'$, $(q'-1)/2$ or $(q'+1)/2$. But from Lemma 2.8(c) we have $m_2(q) = q'$ or $(q'-1)/2$. If $m_1(q) = q'$, then $q'+1 = m_2(q)+1 \mid m_1(q)$ which is impossible. If $m_2(q) = (q'-1)/2$, then $q' = q^5$ or q^{10}, since $q'+1 \mid m_1(q)$, we get a contradiction.

Now we claim that $K/H \not\cong A_p(q')$ where $q'-1 \mid p+1$ and p is an odd prime number. Because if $K/H \cong A_pq'$, then by Lemma 2.10 and Table I, we must have $m_2(q) = (q'^p-1)/(q'-1)$. Lemma 2.8(b) yields $q'^p = q^5$ or q^{10}. If $q'^p = q^{10}$, then $q'^{(p(p+1)/2)} > q^{10}$ which contradicts Lemma 2.8(a). If $q'^p = q^5$ and $p \geqslant 5$, then $q'^{(p(p+1)/2)} \geqslant q'^{3p} > q^{10}$ and again we get a contradiction by Lemma 2.8(a). If $p = 3$ and $q'^3 = q^5$, then $q'-1 = d(q-1)$ where $d = (5, q-1)$. If $d = 1$, then $q' = q$ which is impossible, and if $d = 5$, then $q^5 = (5q-4)^3$ which is a contradiction.

Therefore, $K/H \cong A_{p-1}(q')$ where $(p, q') \neq (3,2), (3,4)$. Then by Lemma 2.10 and Table I, $m_2(q) = (q'^p-1)/\big((q'-1)(p, q'-1)\big)$, $q'^p = q^5$ or q^{10} by Lemma 2.8(b). If $q' = q^{10}$ and $p > 3$, then $q'^{(p(p-1)/2)} \geqslant q'^{2p} > q^{10}$ which contradicts Lemma 2.8(a). If $p = 3$ and $q'^3 = q^{10}$, then

$$(1) \qquad d(q-1)(q^5-1) = d'(q'-1)\,; \; d = (5, q-1) \text{ and } d' = (3, q'-1).$$

If $d = 5$ or $d = d' = 1$, by (1) we must have $q'+1 > q'-1 \geqslant q^5+1$, thus $q' > q^5$, so $q'^3 > q^{10}$ which contradicts Lemma 2.8(a). If $d = 1$ and $d' = 3$, by (1) $q'-1 = \big((q-1)/3\big)(q^5+1) \geqslant q^5+1$ for $q \geqslant 4$ and again we get a contradiction by Lemma 2.8(a). If $q = 2$ or 3, then $q' = q^m$ for some positive integer m, then $q^{3m} = q^{10}$, that is, $3m = 10$ which is impossible. Therefore, $q'^p = q^5$. If $p > 5$, then $q^{p(p-1)/2} > q^{10}$ which contradicts Lemma 2.8(a). If $p = 3$ and $q'^3 = q^5$, then

$$(2) \qquad d(q-1) = d'(q'-1)\,; \; d = (5, q-1) \text{ and } d' = (3, q'-1).$$

If $d = 5$ and $d' = 1$ or 3, by (2) we obtain the equations $q^5 = (5q-4)^3$ or $27q^5 = (5q-2)^3$, respectively, but both of them do not have suitable solutions. By (2), if $d = 1$ and $d' = 3$, $q^3 > q^5$ which are impossible, therefore, $d = d' = 1$ and thus $q = q'$.

Therefore, $q'^p = q^5$ and $p = 5$, that is, $q = q'$ and $p = 5$, thus $K/H \cong A_4(q) = PSL(5, q)$ and the proof is completed. ∎

REMARK 3.1. It is a well known conjecture of J.G.Thompson that if G is a finite group with $Z(G) = 1$ and M is a non-Abelian simple group satisfying $N(G) = N(M)$ where $N(G) = \{n \mid G \text{ has a conjugacy class of size } n\}$, then $G \cong M$. We can give positive answer to this conjecture by our characterisation of the groups under discussion.

COROLLARY 3.2. *Let G be a finite group with $Z(G) = 1$, $M = PSL(5, q)$ and $N(G) = N(M)$, then $G \cong M$.*

PROOF: By [**4**, Lemma 1.5] if G and M are two finite groups satisfying the conditions of Corollary 3.2, then $OC(G) = OC(M)$. So the main theorem implies the corollary. □

Shi and Jianxing in [**16**] put forward the following conjecture.

CONJECTURE. Let G be a group and M a finite simple group, then $G \cong M$ if and only if

(i) $|G| = |M|$

(ii) $\pi_e(G) = \pi_e(M)$, where $\pi_e(G)$ denotes the the set of orders of elements in G.

This conjecture is correct for all groups of alternating type ([**17**]), Sporadic simple groups ([**14**]), and some simple groups of Lie types ([**15, 16**]). As a consequence of the main theorem, we prove the validity of this conjecture for the groups under discussion.

COROLLARY 3.3. *Let G be a finite group and $M = PSL(5,q)$. If $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, then $G \cong M$.*

PROOF: By the assumption we must have $OC(G) = OC(M)$. Thus the corollary follows by the main theorem. □

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