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Entropy of expansive flows

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Abstract Let $h(\phi)$ be the topological entropy of a real continuous flow ϕ on a compact metric space X Introducing an equivalent definition for the topological entropy on an expansive real flow enables us to investigate the topological entropies of mutually conjugate expansive flows and estimate the periodic orbits of an expansive flow which has the pseudo-orbit tracing property

Introduction

In this paper we assume that the spaces are compact metric spaces, and (X, ϕ) denotes a continuous real flow [i e ϕ $X\times\mathbb{R}\to X$ continuous and $\phi(x, t+s)=\phi(\phi(x, t), s)$] Write ϕ_t for the homeomorphism of X defined by $\phi_t(x)=\phi(x, t)$ ϕ is called *h-expansive* if there is an $\varepsilon>0$ so that the set

$$\phi_\varepsilon(x)=\{y\in X,\ d(\phi_s y, \phi_s x)\le\varepsilon \text{ for all } s\ge 0\}$$

has zero topological entropy for each $x\in X$ It is obvious that every expansive flow is h-expansive [**4**]

For $E, F\subset X$ we say that E (t, δ)-*spans* F (with respect to ϕ), if for each $x\in F$, there is an $e\in E$ so that $d(\phi_s e, \phi_s x)\le\delta$ for all $0\le s\le t$ Let $r_t(F, \delta)=r_t(F, \delta, \phi)$ denote the minimum cardinality of a set which (t, δ)-spans F If F is compact, then the continuity of ϕ guarantees $r_t(F, \delta)<\infty$ We define

$$\bar r_\phi(F, \delta)=\limsup_{t\to\infty}\frac{1}{t}\log r_t(F, \delta)$$

For $E, F\subset X$ we say also that E is a (t, δ)-*separate subset of* F (with respect to ϕ), if for every $x, y\in E$ with $x\neq y$ we have $\max_{0\le s\le t} d(\phi_s x, \phi_s y)>\delta$ Let $s_t(F, \delta)=s_t(F, \delta, \phi)$ denote the maximum cardinality of a set which is a (t, δ)-separated subset of F If F is compact, then Theorem 6 4 in [**10**] shows that $s_t(F, \delta)<\infty$ We define

$$\bar s_\phi(F, \delta)=\limsup_{t\to\infty}\frac{1}{t}\log s_t(F, \delta)$$

and topological entropy by

$$h(\phi, F)=\lim_{\delta\to 0}\bar r_\phi(F, \delta)=\lim_{\delta\to 0}\bar s_\phi(F, \delta)$$

By Lemma 1 in [**2**] these limits exist and are equal

In fact the topological entropy of a flow ϕ equals the topological entropy of the homeomorphism ϕ_1, and more generally $h(\phi_t)=|t|h(\phi_1)$ For more details see [**2**]

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(0 1) *Standing hypothesis* We shall assume throughout the remainder of the paper that ϕ is a continuous real flow on a compact metric space X without fixed points

Let I be any interval of real numbers containing the origin A reparametrization of I is an orientation-preserving homeomorphism (increasing) from I onto its image fixing the origin Define Rep (I) to be the set of all reparametrizations of I

Given a continuous real flow (X, ϕ) and $\varepsilon > 0$ For $x \in X$ and $\gamma > \varepsilon$ define

$$U(t, x, \gamma) = \{y \in X, d(\phi_{\alpha(s)}y, \phi_s x) \leq \gamma \text{ for some } \alpha \in \operatorname{Rep}(I) \text{ and all } 0 \leq s \leq t\}$$

Let

$$\bar{U}(t, x, \varepsilon) = \bigcap_{\gamma > \varepsilon} U(t, x, \gamma)$$

We will show later that $\bar{U}(t, x, \varepsilon)$ is closed in X

For $E \subset X$ and $\delta > 0$ we say that E (t, δ)-*weakly spans* X (with respect to ϕ), if for each $x \in X$, there exist $e \in E$ and $\alpha \in \operatorname{Rep}[0, t]$ such that

$$d(\phi_{\alpha(s)}x, \phi_s e) \leq \delta \qquad \text{for all } 0 \leq s \leq t$$

Let $R_t(X, \delta) = R_t(X, \delta, \phi)$ be the smallest cardinality of any (t, δ)-weakly spanning set for X Compactness of X guarantees $R_t(X, \delta) < \infty$ Define

$$\bar{R}_\phi(X, \delta) = \limsup_{t \to \infty} \frac{1}{t} \log R_t(X, \delta)$$

(notice that $\bar{R}_\phi(X, \delta)$ increases as δ decreases)

For $E \subset X$ and $\delta > 0$ we say that E *is a* (t, δ)-*strongly separated set in* X if for every $x, y \in E$, $x \neq y$ and for every $\alpha, \beta \in \operatorname{Rep}[0, t]$

$$d(\phi_{\alpha(s)}x, \phi_s y) > \delta \qquad \text{for some } s \in [0, t]$$

or

$$d(\phi_{\beta(s)}y, \phi_s x) > \delta \qquad \text{for some } s \in [0, t]$$

Let $S_t(X, \delta) = S_t(X, \delta, \phi)$ be the largest cardinality of any (t, δ)-strongly separated subset of X We will show later that $S_t(X, \delta) < \infty$ Define

$$\bar{S}_\phi(X, \delta) = \limsup_{t \to \infty} \frac{1}{t} \log S_t(X, \delta)$$

We now define

$$H(\phi) = \lim_{\delta \to 0} \bar{R}_\phi(X, \delta) = \lim_{\delta \to 0} \bar{S}_\phi(X, \delta)$$

Later we will show also that these limits exist and are equal Note that $H(\phi) \leq h(\phi)$

We would like to raise the following

Conjecture If (X, ϕ) is a continuous real flow (without fixed points), then $H(\phi) = h(\phi)$.

In this paper (§ 2) we will use an adaptation of work by Bowen [**4**] involving certain complications to prove this conjecture under certain additional assumptions

A flow (X, ϕ) is said to be *strongly h-expansive* if there is an $\varepsilon > 0$ called the *h-expansive constant,* so that for every $x \in X$ the set $\xi_\varepsilon(x) = \bigcap_{t \geq 0} \bar{U}(t, x, \varepsilon)$ has zero topological entropy (i e $h(\phi, \xi_\varepsilon(x)) = 0$)

THEOREM A *If ϕ is a strongly h-expansive flow on a compact metric space X (without fixed points), then $H(\phi)=h(\phi)$*

Using this theorem we can investigate the topological entropies of mutually conjugate expansive flows as Theorem B in § 3

A flow (X, ϕ) is said to be expansive if for every $\varepsilon>0$ there exists $\delta>0$ with the property that if $d(\phi_s x, \phi_{\alpha(s)}y)<\delta$ for all $s\in\mathbb{R}=(-\infty,\infty)$ and a pair of points $x, y\in X$ and a continuous map α $\mathbb{R}\to\mathbb{R}$ with $\alpha(0)=0$, then $y=\phi_s x$, where $|s|<\varepsilon$

Lemma 1 in [5] shows that the study of flows with the expansive property can be reduced to those without fixed points

Let (X, ϕ) be a continuous real flow Given δ, $a>0$, a (δ, a)-*chain* is a collection of sequences $(\{x_i\},\{t_i\})$ so that $t_i\geq a$ and $d(\phi_{t_i}x_i, x_{i+1})<\delta$ for all integer i. The definition of a (ϕ, a)-*pseudo-orbit* is the same as that of a (δ, a)-chain [8], [9]

Let $(\{x_i\},\{t_i\})$ be a (δ, a)-pseudo orbit The following notation will be standard throughout this paper $s_0=0$, $s_n=\sum_{i=0}^{n-1} t_i$ and $s_{-n}=\sum_{i=-n}^{-1} t_i$ We always assume $\sum_{i=j}^{k}(\quad)_i=0$ if $k<j$ In particular $\sum_0^{-1} t_i=0$

A (δ, a)-pseudo orbit $(\{x_n\},\{t_n\})$ is ε-*traced by an orbit* $(\phi_t z)_{t\in\mathbb{R}}$ if there exists an $\alpha\in\mathrm{Rep}\,(\mathbb{R})$ such that

$$d(\phi_{\alpha(t)}z, \phi_{t-s_n}x_n)<\varepsilon \qquad \text{whenever } s_n\leq t<s_{n+1} \text{ for } n=0,1,2,$$

and

$$d(\phi_{\alpha(t)}z, \phi_{t+s_{-n}}x_{-n})<\varepsilon \qquad \text{whenever } -s_{-n}\leq t<s_{-n+1} \text{ for } n=1,2,3,$$

We say that a flow (X, ϕ) has the *pseudo-orbit tracing property* (POTP) if for all $\varepsilon>0$, there exists $\delta>0$ such that every $(\delta, 1)$-pseudo orbit is ε-traced by an orbit of ϕ

Now using Theorem A we can show

THEOREM C *If (X, ϕ) is an expansive flow and has the* POTP, *then*

$$h(\phi)=\lim_{t\to\infty}\frac{1}{t}\log v(t),$$

where $v(t)$ is the number of closed orbits in X with period $\leq t$

This result is known if ϕ is a continuous flow on a compact manifold M which satisfies Axiom A [3]

1 *Preparatory lemmas*

Let (X, ϕ) be a continuous real flow (no fixed points)

LEMMA 1 1 (cf [5, Lemma 2]) *There exists $T_0>0$ such that for all λ satisfying $0<\lambda<T_0$ there exists $\gamma>0$ with $d(\phi_\lambda x, y)>\gamma$ provided that $x, y\in X$ and $d(x,y)<\gamma$*

Now let us introduce our basic lemma

LEMMA 1 2 *For all $\lambda>0$, there exists $\varepsilon>0$ such that for every $x, y\in X$ and for every $[T_1, T_2]$ containing the origin and for every $\alpha\in\mathrm{Rep}\,[T_1, T_2]$, if $d(\phi_{\alpha(s)}x, \phi_s y)\leq\varepsilon$ for all $s\in[T_1, T_2]$, then $|\alpha(s)-s|<\lambda$ for $|s|<1$ in $[T_1, T_2]$ and $|\alpha(s)-s|<|s|\lambda$ for $|s|\geq 1$ in $[T_1, T_2]$*

Proof Suppose $\lambda > 0$ Without loss of generality let $\lambda < T_0$ (see Lemma 1 1) and also take δ' small enough that it satisfies Lemma 1 1 with respect to λ Let $0 < \varepsilon < \delta'$ with the property that $d(\phi_s x, \phi_s y) < \delta'$ for $0 \leq s \leq 2$ whenever $d(x, y) < \varepsilon$ Suppose for $x, y \in X$, $d(\phi_{\alpha(s)}x, \phi_s y) < \varepsilon$ for $0 \leq s \leq 2$ Then $d(\phi_{\alpha(s)-s}\phi_s x, \phi_s y) < \varepsilon$ for $0 \leq x \leq 2$ Thus by the continuity of α and by Lemma 1 1, $|\alpha(s) - s| < \lambda$ for $0 \leq s \leq 2$ For the case $2 \leq s \leq 4$, let $d(\phi_{\alpha(s)}x, \phi_s y) < \varepsilon$ for some α Then letting $u = s - 2$, we get

$$d(\phi_{\alpha(u+1)}x, \phi_u \phi_1 y) = d(\phi_{\alpha(s)}x, \phi_{s-1}\phi_1 y) < \varepsilon \qquad \text{for } 0 \leq u \leq 2$$

Let $\gamma(u) = \alpha(u+1) - \alpha(1)$ Then γ is increasing continuous with $\gamma(0) = 0$ and

$$d(\phi_{\gamma(u)}\phi_{\alpha(1)}x, \phi_u \phi_1 y) < \varepsilon \qquad \text{for } 0 \leq u \leq 2$$

Thus $|\gamma(u) - u| < \lambda$ for $0 \leq u \leq 2$, and so $|\alpha(u+1) - \alpha(1) - (u+1) + 1| < \lambda$ It follows that $|\alpha(s) - s| < 2\lambda$ for $2 \leq s \leq 4$ Using a similar argument one can show inductively that for $2n - 2 \leq s \leq 2n$,

$$|a(s) - s| < n\lambda \qquad \text{for } n = 2, 3, 4,$$

since $n/(2n-2) \leq 1$ for $n = 2, 3,$ Thus for all $s \geq 1$ in $[T_1, T_2]$ we have

$$|\alpha(s) - s| < n\varepsilon = \frac{n}{s} s\lambda \leq s\lambda$$

For negative s we can use a similar process and the proof is finished

LEMMA 1 3 (i) *For all* $\lambda > 0$, *there exists* $\delta > 0$ *such that* $S_{(1-\lambda)t}(X, \delta) \leq R_t(X, \delta/2) < \infty$

(ii) $R_t(X, \delta) \leq S_t(X, \delta)$

(iii) *For* $\delta_1 < \delta_2$, $\bar{R}_\phi(X, \delta_2) \leq \bar{R}_\phi(X, \delta_1)$ *and* $\bar{S}_\phi(X, \delta_2) \leq \bar{S}_\phi(X, \delta_1)$

Proof Given $\lambda > 0$, choose $\delta > 0$ satisfying Lemma 1 2 with respect to λ Let E be a $((1-\lambda)t, \delta)$-strongly separated set in X with the largest cardinality and let F be a $(t, \delta/2)$-weakly spanning set of X Define $f \; E \to F$ by choosing for each $x \in E$ some point $f(x) \in F$ and some $\alpha \in \operatorname{Rep}[0, t]$ such that

$$d(\phi_{\alpha(s)}x, \phi_s f(x)) \leq \delta/2 \qquad \text{for all } 0 \leq s \leq t$$

If $f(x) = f(x')$ for $x, x' \in E$, the triangle inequality implies that

$$d(\phi_{\alpha(s)}x, \phi_{\gamma(s)}x') \leq \delta$$

for some $\alpha, \gamma \in \operatorname{Rep}[0, t]$ and for all $0 \leq s \leq t$ By taking $u = \gamma(s)$, we get

$$d(\phi_{\alpha\gamma^{-1}(u)}x, \phi_u x') \leq \delta \qquad \text{for all } 0 \leq u \leq (1-\lambda)t,$$

and by taking $u = \alpha(s)$, we get

$$d(\phi_u x, \phi_{\gamma\alpha^{-1}(u)}x') \leq \delta \qquad \text{for all } 0 \leq u \leq (1-\lambda)t$$

As E is a $((1-\lambda)t, \delta)$ - strongly separated set, we clearly have $x = x'$ Thus cardinality of E is less than or equal to the cardinality of F, and so $S_{(1-\lambda)t}(X, \delta) \leq R_t(X, \delta/2)$

Since this lemma is a version of Lemma 1 in [**2**], the rest follows by a slightly modified version of the proof of that lemma

This lemma shows these limits

$$H(\phi) = \lim_{\delta \to 0} \bar{R}_\phi(x, \delta) = \lim_{\delta \to 0} \bar{S}_\phi(X, \delta)$$

exist and are equal

PROPOSITION 1 *For small $\varepsilon > 0$, $\bar{U}(t, x, \varepsilon)$ is a closed subset of X for every $x \in X$ and $t \geq 0$*

Proof Given $\lambda > 0$ choose γ satisfying Lemma 1 2 with respect to λ Take any $0 < \varepsilon < \gamma$ Now for $t \geq 0$ and $x \in X$ we want to show $\bar{U}(t, x, \varepsilon)$ is a closed subset of X Let $\{y_n\}$ be any sequence in $\bar{U}(t, x, \varepsilon)$ and assume $\{y_n\}$ converges to y in X Then there exists a sequence $\{\alpha_n\}$ of reparametrizations on $[0, t]$ such that

$$d(\phi_{\alpha_n(s)}y_n, \phi_s x) \leq \gamma \qquad \text{for all } 0 \leq s \leq t, \text{ and } \varepsilon < \lambda \leq \gamma$$

Using Lemma 1 2 we know that $(1-\lambda)s \leq \alpha_n(s) \leq (1+\lambda)S$ Therefore for any $\xi > 0$, there exists a positive integer M such that

$$d(\phi_{\alpha_m(s)}y_m, \phi_{\alpha_m(s)}y) \leq \xi \qquad \text{for all } 0 \leq s \leq t, \text{ and } m \geq M$$

Hence

$$d(\phi_{\alpha_m(s)}y, \phi_s x) \leq \gamma + \xi \qquad \text{for all } 0 \leq s \leq t, \gamma > \varepsilon, \text{ and } \xi > 0$$

This means that $y \in \bar{U}(t, x, \varepsilon)$

For $x \in X$ and $\gamma > 0$, define

$$W(t, x, \gamma) = \{y \in X, d(\phi_{\alpha(s)}y, \phi_s x) \leq \gamma \text{ for some } \alpha \in \operatorname{Rep}[-t, t) \text{ and for all } -t \leq s \leq t\},$$

and let

$$\bar{W}(t, x, \varepsilon) = \bigcap_{\gamma > \varepsilon} W(t, x, \gamma)$$

Using a similar argument as above (Proposition 1) one can show that $\bar{W}(t, x, \varepsilon)$ is a closed subset of X This means that

$$\Gamma_\varepsilon(x) = \bigcap_{t \geq 0} \bar{W}(t, x, \varepsilon)$$

is also a closed subset of X

The following lemma is also essentially Theorem 3 of [**5**]

LEMMA 1 4 (cf [**9**, Lemma 8]) *Let ϕ be an expansive flow on X Then for all $\varepsilon > 0$, there exists $\delta > 0$ with the property that for all $\varepsilon_0 > 0$, there exists $T > 0$ such that for every x, $y \in X$ and for every continuous and increasing real valued function s on a closed interval $[-T, T]$ with $s(0) = 0$ if $d(\phi_{s(t)}x, \phi_t y) < \delta$ for all $t \in [-T, T]$, then $d(\phi_r x, y) < \varepsilon_0$ for some $r \in [-\varepsilon, \varepsilon]$*

Using this lemma one can show the following

LEMMA 1 5 *If (X, ϕ) is an expansive flow, then there exists $\lambda > 0$ such that $h(\phi, \Gamma_\lambda(x)) = 0$ for every $x \in X$*

Proof For $\varepsilon > 0$, take δ satisfying the above lemma Let $\lambda = \delta/2$ If $y \in \mu_\lambda(x)$, then $y \in \bar{W}(t, x, \lambda)$ for all $t \geq 0$ This means that $y \in W(t, x, \gamma)$ for some γ, $\lambda < \gamma \leq \delta$ and for all $t \geq 0$ Given $\varepsilon_0 = 1/n$ Lemma 1 4 implies that $d(\phi_{r_n}y, x) < 1/n$ for some $r_n \in [-\varepsilon, \varepsilon]$ Compactness of $[-\varepsilon, \varepsilon]$ implies that there exists $r \in [-\varepsilon, \varepsilon]$ such that $\phi_r y = x$ This means that $\Gamma_\lambda(x) \subset \phi_{[-\varepsilon\, \varepsilon]}x$ Therefore $h(\phi, \Gamma_\lambda(x)) = 0$

2 *Strongly H-expansive flows*

This section is an adaptation of work by Bowen [**4**] involving certain technical complications

LEMMA 2 1 *For $\varepsilon > 0$, let $a = \sup_{x \in X} h(\phi, \xi_\varepsilon(x))$, and suppose $\delta, \beta > 0$ are given Then there exist t, $T > 0$, $t \le T$ such that for every $x \in X$, there exists a set E_x which (t, δ)-spans $\bar{U}(s, x, \varepsilon)$ for every $s \ge T$ and*

$$\text{card } E_x \le \operatorname{Exp}[(\alpha + \beta)t]$$

Proof For $y \in X$, let $t_y > 0$ with the property that for every $t \ge t_y$ there is a set E_y which $(t, \delta/2)$-spans $\xi_\varepsilon(y)$ and

$$\frac{1}{t} \log \text{card } E_y \le a + \beta$$

Let

$$N(y) = \{w \in X, \text{ there exists } z \in E_y \text{ such that } d(\phi_s x, \phi_s z) \le \delta \text{ for } 0 \le s \le t\}$$

Then $N(y)$ is a neighbourhood of $\xi_\varepsilon(y)$ and E_y is a (t, δ)-spanning set of $N(y)$ Since $\bigcap_{t \ge 0} \bar{U}(t, y, \varepsilon) = \xi_\varepsilon(y)$, we may choose a real number T_y such that $\bar{U}(s, y, \varepsilon) \subset N(y)$ for all $s \ge T_y$ For $\gamma > \varepsilon$ we have $\bigcap_{\gamma > \varepsilon} \bar{U}(s, y, \gamma) = \bar{U}(s, y, \varepsilon)$ Thus $\bar{U}(s, y, \gamma) \subset N(y)$ for some $\gamma > \varepsilon$ Let

$$v(y) = \{u \in X, d(\phi_r u, \phi_r y) < \gamma - \varepsilon \text{ for } 0 \le r \le s\}$$

Then $V(y)$ is a neighbourhood of y and $\bar{U}(s, u, \varepsilon) \subset \bar{U}(s, y, \gamma) \subset N(y)$ for every $u \in V(y)$ Let $V(y_1)$, $V(y_2)$, , $V(y_n)$ cover the compact space X and take $T \ge \max\{T_{y_1}, T_{y_2}, \quad, T_{y_n}, t_{y_1}, t_{y_2}, \quad, t_{y_n}\}$ This finishes the proof of this lemma

LEMMA 2 2 *For $\varepsilon > 0$, let $a = \sup_{x \in X} h(\phi, \Gamma_\varepsilon(x))$, and suppose δ, $\beta > 0$ are given Then there exist t, $T > 0$, $t \le T$ such that for every $x \in X$, there exists a set E_x which (t, δ)-spans $\bar{W}(s, x, \varepsilon)$ for every $s \ge T$ and*

$$\text{card } E_x \le \operatorname{Exp}[(\alpha + \beta)t]$$

Proof Exactly similar to the proof of Lemma 2 1

LEMMA 2 3 (cf [4, Lemma 2 1] *Suppose E_1 (t_1, δ)-spans F and E_i (t_i, δ)-spans $\phi_{s_{i-1}} F$ for $i = 2, 3, \quad, n$ Then there exists a set Q which $(s_n, 2\delta)$-spans F and* card $Q \le$ $\prod_{i=1}^n$ card E_i, *where $s_n = \sum_{i=1}^k t_i$*

LEMMA 2 4 *For all $\lambda > 0$, there exists $\varepsilon > 0$, such that if $y \in \bar{U}(s, x, \varepsilon)$, then the time distance between y and $\bar{U}(s - t, \phi_t x, \varepsilon)$ is between $t - \lambda t$ and $t + \lambda t$ for every $t \le s$*

Proof Given $\lambda > 0$, choose ε satisfying Lemma 1 2 and take $y \in \bar{U}(s, x, \varepsilon)$ Then there exists $\alpha \in \operatorname{Rep}[0, s]$ such that

$$d(\phi_{\alpha(u)} y, \phi_u x) \ge \gamma$$

for $0 \le u \le s$, and $\gamma > \varepsilon$ Therefore

$$d(\phi_{\alpha(u)} y, \phi_{u-t} \phi_t x) \le \gamma$$

for $0 \le u \le s$ Now let $w = u - t$ and $\gamma(w) = \alpha(w + t) - \alpha(t)$ It is obvious that $\gamma \in \operatorname{Rep}[0, s - t]$ and for $0 \le w \le s - t$ we have

$$d(\phi_{\gamma(w)} \phi_{\alpha(t)} y, \phi_w \phi_t x) \le \gamma$$

This means that $\phi_{\alpha(t)} y \in \bar{U}(s - t, \phi_t x, \varepsilon)$ Using Lemma 1 2 we know that $(1 - \lambda)t \le \alpha(t) \le (1 + \lambda)t$ and this finishes the proof of this lemma

The above lemma is also true for the case when we have $\bar{W}(s, x, \varepsilon)$ and $\bar{W}(s-t, \phi_t x, \varepsilon)$ instead of $\bar{U}(s, x, \varepsilon)$ and $\bar{U}(s-t, \phi_t x, \varepsilon)$ respectively

LEMMA 2 5 *Given $\lambda > 0$, there exists $\varepsilon > 0$ such that if E_i is a $(t+\lambda t, \delta)$-spanning set of $\bar{U}(s-it, \phi_{it}x, \varepsilon)$ for $i = 1, 2, \ldots, m-1$ and E_m is any (T, δ)-spanning set of $\bar{U}(s-mt, \phi_{mt}x, \varepsilon)$, then there exists a set Q which $[m(t-\lambda t)+T, 2\delta]$-spans $\bar{U}(s, x, \varepsilon)$ and*

$$\operatorname{card} Q \leq \prod_{i=1}^{m} \operatorname{card} E_i$$

Proof Choose ε satisfying Lemma 1 2 with respect to λ Lemma 2 4 implies that the arc $\phi_{[t-\lambda t, t+\lambda t]}y$ meets $\bar{U}(s-it, \phi_{it}x, \varepsilon)$ for every point y in $\bar{U}(s-(i-1)t, \phi_{(i-1)t}x, \varepsilon)$ So the rest of the proof of this lemma is just exactly as the proof of Lemma 2 1 in [4]

The above lemma is also true when we have $\bar{W}(s-it, \phi_{it}x, \varepsilon)$ instead of $\bar{U}(s-it, \phi_{it}x, \varepsilon)$ for $i = 1, 2, \ldots, m$

PROPOSITION 2 *For all $\lambda > 0$, there exists $\varepsilon > 0$ such that if $a = \sup\{h(\phi, \xi_\varepsilon(x)), x \geq X\}$ and $\delta, \beta > 0$ are given, then there exists $T > 0$ such that for every $s \gtrsim T$ and $x \in X$, there exists a set Q which $[(s-T)(1-\lambda)+T, \delta]$-spans $\bar{U}(s, x, \varepsilon)$ and*

$$\operatorname{card} Q \leq c \exp[(\alpha+\beta)s(1+\lambda)]$$

Proof For $\lambda > 0$, choose ε satisfying Lemma 1 2 and t, T satisfying Lemma 2 1 with respect to δ, β Without loss of generality fix T large enough such that for every $s \geq T$, $\bar{U}(s, x, \varepsilon)$ is $(t+\lambda t, \delta/2)$-spanned by a set E with

$$\operatorname{card} E \leq \exp[(a+\beta)t(1+\lambda)]$$

Also without loss of generality assume $s = mt + T$ for some positive integer m It is obvious that each $\bar{U}(s-it, \phi_{it}x, \varepsilon)$ is $(t+\lambda t, \delta/2)$-spanned by a set, say E_i, with

$$\operatorname{card} E_i \leq \exp[(a+\beta)t(1+\lambda)]$$

for $i = 1, 2, \ldots, m-1$ Let E_m be any $(T, \delta/2)$-spanning set of X with minimum cardinality Then E_m is a $(T, \delta/2)$-spanning set of $\bar{U}(s-mt, \phi_{mt}x, \varepsilon) \subset X$ Using the above lemma there exists a set Q which $[mt(1-\lambda)+T, \delta]$-spans $\bar{U}(s, x, \varepsilon)$ and

$$\operatorname{card} Q \leq \operatorname{card} E_m \exp[(\alpha+\beta)mt(1+\lambda)]$$

But $mt = s - T$, so Q is a set which $[(s-T)(1-\lambda)+T, \delta]$-spans $\bar{U}(s, x, \varepsilon)$ and

$$\operatorname{card} Q \leq \operatorname{card} E_m \exp[(a+\beta)(s-T)(1+\lambda)]$$

Taking

$$c = \operatorname{card} E_m \exp[(a+\beta)(-T)(1+\lambda)],$$

finishes the proof of this proposition

PROPOSITION 3 *For all $\lambda > 0$, there exists $\varepsilon > 0$ such that if $a = \sup\{h(\phi, \Gamma_\varepsilon(x)), x \in X\}$ and $\delta, \beta > 0$ are given, then there exists $T > 0$ such that for every $s \gtrsim T$ and $x \in X$, there exists a set Q which $[(s-T)(1-\lambda)+T, \delta]$-spans $\bar{W}(s, x, \varepsilon)$ and*

$$\operatorname{card} Q \leq c \exp[(a+\beta)s(1+\lambda)]$$

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Proof Using Lemma 2 2 we can obtain an exactly similar proof of the above proposition

LEMMA 2 6 *For all $\lambda > 0$, there exists $\varepsilon > 0$ such that for large s and $x \in X$*

$$\phi_{s+\lambda s}(\xi_\varepsilon(x)) \subset \phi_{[0,2\lambda s]} \bar{W}(s, \phi_s x, \varepsilon)$$

Proof For $\lambda > 0$, choose ε satisfying Lemma 1 2 Let z be an element in $\xi_\varepsilon(x)$ Then z is an element in $\bar{U}(2s, x, \varepsilon)$ for every $s \geq 0$ In other words

$$d(\phi_{\alpha(r)} z, \phi_r x) \leq \gamma,$$

for $0 \leq r \leq s$ and for all $\gamma > \varepsilon$ and for some $\alpha \in \text{Rep}\,[0, 2s]$ Now assume $u = r - s$ and $\beta(u) = \alpha(u+s) - \alpha(s)$ Then $\beta \in \text{Rep}\,[-s, s]$ and

$$d(\phi_{\beta(u)} \phi_{\alpha(s)} z, \phi_u \phi_s x) \leq \gamma \qquad \text{for } -s \leq u \leq s$$

Thus $\phi_{\alpha(s)} z$ is an element of $w(s, \phi_s x, \gamma)$, so $\phi_{\alpha(s)-s} \phi_s z = \phi_{\alpha(s)} z$ is an element of $\bar{W}(s, \phi_s x, \varepsilon)$ Lemma 1 2 implies that $\phi_s z$ is an element of $\phi_{[-\lambda s\, \lambda s]} \bar{W}(s, \phi_s x, \varepsilon)$ It follows that

$$\phi_{s+\lambda s}(\xi_\varepsilon(x)) = \phi_{\lambda s} \phi_s \xi_\varepsilon(x) \subset \phi_{\lambda s} \phi_{[-\lambda s\, \lambda s]} \bar{W}(s, \phi_s x, \varepsilon)$$

Therefore

$$\phi_{s+\lambda s}(\xi_\varepsilon(x)) \subset \phi_{[0\, 2\lambda s]} \bar{W}(s, \phi_s x, \varepsilon)$$

LEMMA 2 7 *If E (t, δ)-spans W, then there exists $\delta' > 0$ (depending only on δ) such that for every λ, $\delta' \leq \lambda < t$, there is a set Q which $(t - \lambda, 2\delta)$-spans $\phi_{[0,\lambda]} W$ and*

$$\text{card } Q \leq \left(\frac{\lambda}{\delta'}\right) \text{card } E$$

Proof Given $\delta > 0$, choose $\delta' > 0$ small enough such that $d(\phi_t x, \phi_t y) \leq \delta$ for all $t \in \mathbb{R}$ whenever $x = \phi_\varepsilon y$, where $|\varepsilon| \leq \delta'$ For $x \in X$ define a set

$$\Sigma_x = \{\phi_{n\delta}\, x,\ n = 0, 1, \quad , m\} \cup \{\phi_\lambda x\},$$

where m is the largest integer less than λ / δ' Take

$$Q = \bigcup \{\Sigma_x,\ x \in E\}$$

Then Q is a (t, δ)-spanning set of W and

$$\text{card } Q \leq \left(\frac{\lambda}{\delta'}\right) \text{card } E$$

In order to prove that Q is a $(t - \lambda, 2\delta)$-spanning set of $\phi_{[0,\lambda]} W$, let x be any element in $\phi_{[0,\lambda]} W$ If x is an element of W or an element of $\phi_{[0\,\lambda]} E$, then the rest follows easily If $\phi_{-r} x$ is an element of W for $0 \leq r \leq \lambda$, then there exists a point e in E such that

$$d(\phi_s \phi_{-r} x, \phi_s e) \leq \delta \qquad \text{for } 0 \leq s \leq t$$

Let $u = s - r$ Then

$$d(\phi_u x, \phi_u \phi_r e) \leq \delta \qquad \text{for } 0 \leq u \leq t - r$$

Pick a point z in Σ_x so that

$$d(\phi_u \phi_r e, \phi_u z) \leq \delta \qquad \text{for all } u \in \mathbb{R}$$

Then

$$d(\phi_u x, \phi_u z) \le d(\phi_u x, \phi_u \phi_r e) + d(\phi_u \phi_r e, \phi_u z) < 2\delta$$

for all $0 \le u \le t-r$ and this finishes the proof of this lemma

In order to show that an expansive flow is strongly h-expansive we need the following which is a version of corollary 2 3 in [4]

PROPOSITION 4 *Every expansive flow* (X, ϕ) *is strongly h-expansive*

Proof For $\lambda > 0$, choose $\varepsilon > 0$ satisfying Lemma 1 2 and let

$$a = \sup \{h(\phi, \Gamma_\varepsilon(x)), x \in X\}$$

Then there exists $T > 0$ satisfying Proposition 3 For $s \ge T$, let E_1 be an $(s+\lambda s, \delta/2)$-spanning set of $\xi_\varepsilon(x)$ For $\beta > 0$, Proposition 3 implies that there is a set Q which $[(s-T)(1-\lambda)+T, \delta/4]$-spans $\bar{W}(s, \phi_s x, \varepsilon)$ and

$$\operatorname{card} Q \le c \quad \exp[(a+\beta)s(1+\lambda)]$$

Using the above lemma, there exists $\delta' \ge 0$ and a set E_2 which $[(s-T)(1-\lambda)+T-2\lambda s, \delta/2]$-spans $\phi_{[0,2\lambda s]}\bar{W}(s, \phi_s x, \varepsilon)$ and

$$\operatorname{card} E_2 \le c\left(\frac{2\lambda s}{\delta'}\right) \exp[(a+\beta)s(1+\lambda)]$$

Using Lemma 2 6 we have

$$\phi_{s+\lambda s}\xi_\varepsilon(x) \subset \phi_{[0,2\lambda s]}\bar{W}(s, \phi_s x, \varepsilon)$$

Therefore the set E_2 is a $[s-3\lambda s+\lambda T, \delta/2]$-spanning set of $\phi_{s+\lambda s}\xi_\varepsilon(x)$ Lemma 2 3 implies that there exists a set Σ which $[2s(1-\lambda)+T\lambda, \delta]$-spans $\xi_\varepsilon(x)$ and $\operatorname{card} \Sigma \le \operatorname{card} E_1 \quad \operatorname{card} E_2$ This implies that

$$\frac{1}{2s}\log \operatorname{card} \Sigma \le \frac{1}{2s}\log \operatorname{card} E_1 + \frac{1}{2s}\log \operatorname{card} E_2$$

Since $(1/2s)\log(2\lambda sc/\delta') \to 0$ as $s \to \infty$, then it is obvious that

$$(1-\lambda)h(\phi, \xi_\varepsilon(x)) \le \frac{1+\lambda}{2}h(\phi, \xi_\varepsilon(x)) + \frac{(a+\beta)(1+\lambda)}{2}$$

Hence

$$(1-3\lambda)h(\phi, \xi_\varepsilon(x)) \le (a+\beta)(1+\lambda)$$

for every λ, $\beta > 0$ Expansiveness and Lemma 1 5 imply that $a = 0$ Since $\beta > 0$ is arbitrarily small, therefore $h(\phi, \xi_\varepsilon(x)) = 0$ for all $x \in X$ This completes the proof

PROPOSITION 5 *For all* $\lambda > 0$, *there exists* $\varepsilon > 0$ *such that*

$$(1-\lambda)h(\phi) \le H(\phi) + (1+\lambda)\sup_{x \in X} h(\phi, \xi_\varepsilon(x))$$

Proof For $\lambda > 0$, choose $\varepsilon > 0$ satisfying Lemma 1 2 and let $T > 0$ satisfy Proposition 2 For $s \ge T$, let E be any (s, ε)-weakly spanning set of X For β, $\delta > 0$ and for every $x \in E$ there exists a set E_x which $[(s-T)(1-\lambda)+T, \delta]$-spans $\bar{U}(s, x, \varepsilon)$ and

$$\operatorname{card} E_x \le c \quad \exp[(a+\beta)s(1+\lambda)],$$

where

$$a = \sup\{h(\phi, \xi_\varepsilon(x)), x \in X\}$$

Since

$$\bigcup \{\bar{U}(s, x, \varepsilon), x \in E\} = X$$

take

$$W = \bigcup \{E_x, x \in E\}$$

Therefore W $[(s-T)(1-\lambda)+T, \delta]$-spans X and

$$\text{card } W \leq \text{card } E \quad c \quad \exp[(a+\beta)s(1+\lambda)]$$

Therefore

$$\frac{1}{s}\log \text{card } W \leq \frac{1}{s}\log \text{card } E + \frac{1}{s}\log c + (a+\beta)(1+\lambda)$$

As $s \to \infty$ we have

$$(1-\lambda)h(\phi) \leq \bar{R}_\phi(x, \varepsilon) + (a+\beta)(1+\lambda)$$

Using Lemma 1 3(iii) we have $\bar{R}_\phi(X, \varepsilon) \leq H(\phi)$ This implies that

$$(1-\lambda)h(\phi) \leq H(\phi) + (a+\beta)(1+\lambda)$$

for every $\beta > 0$ and the proof is completed

Proof of Theorem A If (X, ϕ) is a strongly h-expansive flow, then $\sup_{x \in X} h(\phi, \xi_\varepsilon(x)) = 0$ Proposition 5 implies that $(1-\lambda)h(\phi) \leq H(\phi)$ for every $\lambda > 0$ This means that $h(\phi) \leq H(\phi)$ and the proof is finished using the fact that $H(\phi) \leq h(\phi)$

3 *Entropy and conjugacy*

In [7] Ohno investigated topological entropies of mutually conjugate flows as Theorem 1 This theorem is proved in [7] using a measure theoretical point of view As an application of Theorem A one can introduce a different and easier proof for the following theorem which is stronger than Theorem 1 in [7], but under an extra assumption

We recall that the flows (X, ϕ) and (Y, ψ) are conjugate (topological conjugate) if there is a homeomorphism γ from X onto Y mapping orbits of ϕ onto orbits of ψ with preserved orientation

LEMMA 3 1 (cf [**8**, Lemma 4]) *If (X, ϕ) and (Y, ψ) are conjugate flows with a conjugate homeomorphism γ $X \to Y$ and have no fixed points, then there exists a unique continuous function σ $X \times \mathbb{R} \to X$ and a unique continuous function β $Y \times \mathbb{R} \to Y$ such that*

(1) $\sigma_x(0) = 0$ *and* σ_x $\mathbb{R} \to \mathbb{R}$ *is a strictly increasing homeomorphism for every x in X*
(2) $\gamma\phi_t x = \psi_{\sigma_x(t)}\gamma x$ *for every $x \in X$, and $t \in \mathbb{R}$*
(3) $\sigma_x(s+t) = \sigma_{\phi_t(x)}(s) + \sigma_x(t)$ *for every s, $t \in \mathbb{R}$ and $x \in X$*
(4) $\beta_y(0) = 0$ *and* β_y $\mathbb{R} \to \mathbb{R}$ *is a strictly increasing homeomorphism for every y in Y*
(5) $\beta_y(s+t) = \beta_{\psi_t(y)}(s) + \beta_y(t)$ *for every s, $t \in \mathbb{R}$ and $y \in Y$*
(6) $\beta_y = \sigma_x^{-1}$ *whenever $\gamma x = y$*
(7) $\gamma^{-1}(\psi_t y) = \phi_{\beta_y(t)}\gamma^{-1}y = \phi_{\sigma_x^{-1}(t)}x$ *for $\gamma x = y$ and $t \in \mathbb{R}$*

σ and β are called the cocycles of ϕ and ψ with values in $\mathbb{R}$ respectively

LEMMA 3 2 *If σ is the cocycle of the flow (X, ϕ) with values in $\mathbb{R}$, then there exist $m, M>0$ such that*

$$mt \le \sigma_x(t) \le Mt \qquad \text{for all } |t| \ge 1 \text{ and } x \in X$$

Proof Continuity of σ and compactness of X imply that there exist $m', M'>0$ such that $m' \le \sigma_x(t) \le M'$ for all $1 \le t \le 2$ and $x \in X$ Since (X, ϕ) has no fixed points (standing hypothesis), Property 3 of Lemma 3 1 implies that $mt \le \sigma_x(t) \le Mt$ for all $t \ge 1$ and $x \in X$, where $m = m'/2$ and $M = M'$ Similar arguments can be used for the case when $t \le -1$

We will call m and M the lower and upper bounds of σ respectively

THEOREM B *Suppose a flow (X, ϕ) is topologically conjugate to an expansive flow (Y, ψ) with a conjugate homeomorphism γ $X \to Y$ and the cocycle σ of the flow (X, ϕ) with values in $\mathbb{R}$ Then*

$$mh(\psi) \le h(\phi) \le Mh(\psi),$$

where m and M are the lower and upper bounds of σ

Proof As expansiveness is a conjugacy invariant [**5**], clearly (X, ϕ) and (Y, ψ) are strongly h-expansive flows Given $\varepsilon>0$ smaller than the strongly h-expansive constant of ψ, choose $\delta>0$ which is also smaller than the strongly h-expansive constant of ϕ and with the property that $d(\gamma a, \gamma b) \le \varepsilon$ whenever $d(a, b) < \delta$ in X For large t, let E be a set which is (t, δ)-weakly spanning set of X with minimum cardinality Thus for every y in Y with $\gamma x = y$ there exist a point $e \in E$ and $\alpha \in \text{Rep}\,[0, t]$ such that

$$d(\phi_{\alpha(s)}x, \phi_s e) \le \delta \qquad \text{for all } 0 \le s \le t$$

so

$$d(\psi_{\sigma_x(\alpha(s))}\gamma x, \psi_{\sigma_e(s)}\gamma e) = d(\gamma\phi_{\alpha(s)}x, \gamma\phi_s e) \le \varepsilon,$$

for $0 \le s \le t$ Now taking $u = \sigma_e(s)$ and $\beta(u) = \sigma_x \alpha \sigma_e^{-1}(u)$, we have

$$d(\psi_{\beta(u)}y, \psi_u \gamma e) \le \varepsilon \qquad \text{for } 0 \le u \le mt$$

Thus γE is a (mt, ε)-weakly spanning set of Y, and this means $R_{mt}(Y, \varepsilon) \le R_t(X, \delta)$ which implies that $mh(\psi) \le h(\phi)$ Since $t/M \le \sigma_x^{-1}(t)$ for every $t \in \mathbb{R}$ and $x \in X$ and suppose E (t, δ)-weakly spans Y with minimum cardinality Then clearly $\gamma^{-1}E$ is $(t/M, \varepsilon)$-weakly spanning set of X Hence $R_{t/M}(X, \varepsilon) \le R_t(X, \delta)$ which implies that $h(\phi) \le Mh(\psi)$

The following is a direct consequence of Theorem B and Lemma 1 in [**6**]

COROLLARY 1 *If ψ is a flow obtained from an expansive flow ϕ on X by a positive continuous change of velocity λ $X \to \mathbb{R}$, then*

$$mh(\phi) \le h(\psi) \le Mh(\phi),$$

where $m = \inf\{1/\lambda(x), x \in X\}$ and $M = \sup\{1/\lambda(x), x \in X\}$

COROLLARY 2 *If* (Y, ϕ) *is the suspension flow of an expansive homeomorphism* $T\ X \to X$ *under a positive continuous function* $f\ X \to \mathbb{R}$, *then*

$$\frac{1}{M} h(T) \le h(\phi) \le \frac{1}{m} h(T),$$

where $m = \inf\{f(x), x \in X\}$ *and* $M = \sup\{f(x), x \in X\}$

Proof Let (Σ, ψ) be the suspension flow of (X, T) under the constant 1 Then it is obvious that $h(\psi) = h(T)$ and (Y, ϕ) is conjugate to (Σ, ψ) with the cocycle $\sigma_x(s) = s/f(x)$ for every $x \in X$ Let $m = \inf\{f(x), x \in X\}$ and $M = \sup\{f(x), x \in X\}$ Then it is obvious that $(1/M)t \le \sigma_x(t) \le (1/m)t$ for every $t \in \mathbb{R}$ and $x \in X$ Theorem B finishes the proof

4 *Entropy and chain recurrence*

Let ϕ be an expansive flow which has the POTP on a compact metric space X Given $x, y \in X$, a (δ, a)-*chain from* x *to* y is a collection

$$\{x = x_0, x_1, \quad , x_k = y, t_0, t_1, \quad , t_{k-1}\}$$

so that $t_i \ge a$ and $d(\phi_{t_i} x_i, x_{i+1}) < \delta$

A point x is *chain equivalent* to y (written $x \sim y$ if for every δ, $a > 0$, there is a (δ, a)-chain from x to y and from y to x The *chain recurrent* set of ϕ is

$$\mathrm{CR}(\phi) = \{x \in X, x \sim x\}$$

In this section we give some standard results as Lemma 4 1

LEMMA 4 1 (a) $\Omega = \mathrm{CR}$

(b) *For all* $r > 0$, $x \sim \phi_r x$ *for every* $x \in \Omega$

(c) *The set of periodic points is dense in* Ω

(d) *Let* Ω_λ *be an equivalence class of* Ω *under the relation* $\sim$ *Then* Ω_λ *is invariant, closed, and open in* Ω

Proof Is an easy exercise for the reader

Since Ω is compact, therefore Ω is uniquely expressed as a disjoint union $\Omega = \bigcup_{i=1}^m \Omega_i$ where Ω_i $(1 \le i \le m)$ is an equivalence class under $\sim$ (note that there are no fixed points) Moreover one can show easily that (Ω_i, ϕ) is topologically transitive for all i (i e Ω_i contains a dense orbit)

LEMMA 4 2 *For all* $\delta > 0$, *there exists* $L > 0$ *such that for every* $x, y \in \Omega$ *if* $x \sim y$, *then there exists* $w \in \Omega$ *so that* $d(w, x) < \delta$ *and* $d(\phi_s w, y) < \delta$ *for some* $0 \le s \le L$

Proof $x \sim y$ implies $x, y \in \Omega_i$ for some i, $1 \le i \le m$ Take $\{U_\delta\}_{j=1}^n$ to be a finite cover for Ω_i of open sets each of diameter less than δ Topologically transitive implies that there exists $r_{sk} > 0$ $(1 \le k \le n, 1 \le s \le n)$ such that $\phi_{r_{sk}} U_s \cap U_k \ne \emptyset$ Take $L_i = \max_{s\,k} r_{sk}$ and $L = \max_i L_i$ This finishes the proof

LEMMA 4 3 *For all* $\lambda > 0$, *there exists* $\varepsilon > 0$ *and there exists* $B > 0$ *such that for every* $r > 0$ *and* $x \in \Omega$, *there exists a periodic point* z *of period* $\le (1 + \lambda) r + B$ *and* $\alpha \in \mathrm{Rep}\,[0, r]$ *so that*

$$d(\phi_{\alpha(t)} z, \phi_t x) \le \varepsilon \qquad \text{for all } 0 \le t \le r$$

Proof For $\lambda > 0$, choose $\varepsilon > 0$ satisfying Lemma 1 2 Take $\delta > 0$ satisfying the definition of POTP with respect to ε and $\delta < \varepsilon$ For $x \in \Omega$, we have $x \sim x_r$ Lemma 4 2 implies that there exist $L > 0$ and $w \in \Omega$ so that $d(w, \phi_r x) < \delta$ and $d(\phi_s w, x) < \delta$ for some s, $0 \le s \le L$ Take $B = (1+\lambda)L$, and consider the periodic δ-pseudo-orbit $\{x, w, x, r, s\}$ Expansiveness and POTP imply that there is a periodic point z and $\alpha \in \operatorname{Rep}[0, r+s]$ such that

$$d(\phi_{\alpha(t)}z, \phi_t x) \le \varepsilon \qquad \text{for all } 0 \le t \le r,$$

and

$$d(\phi_{\alpha(t)}z, \phi_t w) \le \varepsilon \qquad \text{for all } r \le t \le r+s$$

Using Lemma 1 2, we have

$$(1-\lambda)t \le \alpha(t) \le (1+\lambda)t$$

for all $0 \le t \le r+s$ Therefore the period of $z \le (1+\lambda)r + B$ and the proof is finished

We fix some notation p is the set of all periodic orbits of ϕ, $p(t)$ those with period $\tau \le t$, and $p_\varepsilon(t)$ those with period τ in the interval $[t-\varepsilon, t+\varepsilon]$ Let $v(t)$ and $v_\varepsilon(t)$ be the number of closed orbits with period $\tau \le t$ and $\tau \in [t-\varepsilon, t+\varepsilon]$ respectively Set

$$D(t) = \sum_{\gamma \in p(t)} \tau(\gamma) = \sum_{\gamma \in p(t)} \text{period of } \gamma$$

It is obvious that $D(t) \le tv(t)$

PROPOSITION 6 *Let (X, ϕ) be an expansive flow which has the* POTP *Then*

$$h(\phi) \le \liminf_{t \to \infty} \frac{1}{t} \log v(t)$$

Proof Given $\lambda > 0$, choose β satisfying expansiveness with respect to λ Take $\delta < \min\{\frac{1}{3}\beta, \lambda\}$ and satisfying the above lemma and Lemma 1 2 with respect to λ Let E be a (t, β)-strongly separated set of Ω Take $r = ((1+\lambda)/(1-\lambda))t$ in the above lemma Then for $x \ne y$ in E, there exist z_x and z_y periodic points each of period $\le (1+\lambda)r + B$ (for some B) with $\alpha_x, \alpha_y \in \operatorname{Rep}[0, r]$ and

(i) $d(\phi_{\alpha_x(s)}z_x, \phi_s x) \le \delta$ for all $0 \le s \le r$,

and

(ii) $d(\phi_{\alpha_y(s)}z_y, \phi_s y) \le \delta$ for all $0 \le s \le r$

Choose $\varepsilon > 0$ small enough such that

$$\sup\{d(z, \phi_u z), z \in x, |u| \le 3\varepsilon\} \le \delta$$

Now assume $z_x \in \phi_{[-3\varepsilon\, 3\varepsilon]} z_y$ This means that

$$d(\phi_s z_x, \phi_s z_y) \le \delta \qquad \text{for all } s \in \mathbb{R}$$

Take $u = \alpha_x(s)$ in (i) and $u = \alpha_y(s)$ in (ii) Therefore

$$d(\phi_u z_x, \phi_{\alpha_x^{-1}(u)} x) \le \delta \qquad \text{for all } 0 \le u \le (1-\lambda)r = (1+\lambda)t,$$

and

$$d(\phi_u z_y, \phi_{\alpha_y^{-1}(s)} y) \le \delta \qquad \text{for all } 0 \le u \le (1+\lambda)t$$

The triangle inequality implies that

$$d(\phi_{\alpha_x^{-1}(u)} x, \phi_{\alpha_y^{-1}(u)} y) \le 2\delta \qquad \text{for all } 0 \le u \le (1+\lambda)t$$

Now again let $v=\alpha_x^{-1}(u)$ Then we have

$$d(\phi_v x, \phi_{h(v)}y)\le 2\delta \qquad \text{for all } 0\le v\le t,$$

where $h(v)=\alpha_y^{-1}\alpha_x(v)$

Also if we take $v=\alpha_y^{-1}(u)$, then we have

$$d(\phi_{\gamma(v)}x, \phi_v y)\le 2\delta \qquad \text{for all } 0\le v\le t,$$

where $\gamma(v)=\alpha_x^{-1}\alpha_y(v)$ This contradicts the fact that E is a (t,β)-strongly separated set of Ω Hence $z_x\notin\phi_{[-3\varepsilon,3\varepsilon]}z_y$, and so

$$S_t(\Omega,\beta)\le\frac{1}{2\varepsilon}D[(1+\lambda)r+B]=\frac{1}{2\varepsilon}D\left[\frac{(1+\lambda)^2}{1-\lambda}t+B\right]$$

Using the fact that $D(t)\le tv(t)$, we get

$$S_t(\Omega,\beta)\le\frac{1}{2\varepsilon}\left[\frac{(1+\lambda)^2}{1-\lambda}t+B\right]v\left[\frac{(1+\lambda)^2}{1-\lambda}t+B\right]$$

Therefore

$$\frac{1}{t}\log S_t(\Omega,\beta)\le\frac{1}{t}\log\frac{1}{2\varepsilon}\left[\frac{(1+\lambda)^2}{1-\lambda}t+B\right]+\frac{1}{t}\log v\left[\frac{(1+\lambda)^2}{1-\lambda}t+B\right]$$

This means that

$$\bar{S}_\phi(\Omega,\beta)\le\frac{(1+\lambda)^2}{1-\lambda}\liminf_{t\to\infty}\frac{1}{t}\log v(t)$$

As $\lambda\to 0$, we have

$$H(\phi,\Omega)\le\liminf_{t\to\infty}\frac{1}{t}\log v(t)$$

Using Theorem A we have,

$$h(\phi,\Omega)\le\liminf_{t\to\infty}\frac{1}{t}\log v(t)$$

Theorem 2 4 in [1] implies that

$$h(\phi)\le\liminf_{t\to\infty}\frac{1}{t}\log v(t)$$

Proof of Theorem C Using a similar argument to the proof of the second part of Lemma 4 10 in [3] we can show that for any $\varepsilon>0$, there exists $\lambda>0$ such that $v_\lambda(t)\le S_t(x,\varepsilon)$ As

$$v(t)\le v_\lambda(t)+v_\lambda(t-2\lambda)+\qquad+v_\lambda(0)$$

(there are at most $t/2\lambda$ terms), and $S_t(X,\varepsilon)\le S_{t'}(X,\varepsilon)$ for $t\le t'$, we have

$$v(t)\le\frac{t}{2\lambda}S_t(X,\varepsilon),$$

and so

$$\limsup_{t\to\infty}\frac{1}{t}\log v(t)\le h(\phi)$$

Proposition 6 finishes the proof of this theorem

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