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A BROUWER TYPE COINCIDENCE THEOREM AND THE FUNDAMENTAL THEOREM OF ALGEBRA

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ABSTRACT. It is shown that a coincidence theorem which is a natural generalisation of Brouwer's fixed point theorem also gives a short and simple proof of the fundamental theorem of algebra.

Schirmer [6] has proved the following interesting coincidence theorem for the k-dimensional closed unit disc D^k.

THEOREM 1. *Suppose that the continuous function $f:D^k \to D^k$ maps the boundary S^{k-1} essentially onto itself. Then for any continuous function $g:D^k \to D^k$, the equation*

$$g(x)=f(x)$$

has a solution in D^k.

This result is closely related to Brouwer's fixed point theorem that for each continuous function $g:D^k \to D^k$ there is a point x_0 in D^k such that $g(x_0)=x_0$. Clearly Brouwer's theorem follows immediately on putting $f(x) = x$; and by considering the intersection of the directed ray from $g(x)$ through $f(x)$ with the boundary S^{k-1}, a standard proof of Brouwer's theorem (for example [8], p. 194) can be adapted readily to give a proof of Theorem 1. Theorem 1 has been generalised in a number of ways by Schirmer and others ([1], [2], [4], [5], [7]); a particularly elegant and simple extension to a Poincare type of coincidence theorem is due to Reich [5]. Schirmer's proof of Theorem 1 is somewhat complicated and involves higher homotopy groups but her object was to provide a new proof of Brouwer's theorem. The object of this note is to show that Theorem 1 also gives a simple and apparently new proof of the fundamental theorem of algebra. This and the other applications to complex analysis arose out of conversations with Dr T. B. Sheil-Small.

THEOREM 2. *Every complex, non-constant polynomial has a zero.*

Proof. To solve the equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0, \tag{1}$$

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where $n \geq 1$ and $a_n \neq 0$, put $z = Rw$, where

$$R = \max\left\{1, \frac{1}{|a_n|}(|a_{n-1}| + \cdots + |a_0|)\right\}.$$

Then on rearranging, (1) becomes

$$(2)\qquad \begin{aligned} w^n &= -\frac{1}{Ra_n}\left(a_{n-1}w^{n-1} + \cdots + \frac{a_1 w}{R^{n-2}} + \frac{a_0}{R^{n-1}}\right) \\ &= q(w) \text{ say.} \end{aligned}$$

The function $q : \mathbb{C} \to \mathbb{C}$ is continuous and when $|w| \leq 1$,

$$\begin{aligned} |q(w)| &\leq \frac{1}{R\,|a_n|}\left(|a_{n-1}| + \cdots + \frac{|a_1|\,|w|}{R^{n-2}} + \frac{|a_0|}{R^{n-1}}\right) \\ &\leq \frac{1}{R\,|a_n|}(|a_{n-1}| + \cdots + |a_1| + |a_0|) \\ &\leq 1, \end{aligned}$$

so that $q(D^2) \subseteq D^2$. As is well known, the nth power map $P_n : S^1 \to S^1 : w \mapsto w^n$ is of degree $n \geq 1$ and so is essential. Thus the conditions of Theorem 1 are satisfied and (2) has a solution in D^2, whence (1) has a solution in the disc $\{z : |z| \leq R\}$.

This proof relies on the polynomial being dominated by z^n and can be extended to show that the harmonic polynomial

$$a_n z^n + \cdots + a_1 z + a_0 + a_{-1}\bar{z} + \cdots + a_{-n}\bar{z}^n,$$

where $|a_n| \neq |a_{-n}|$, has a zero with modulus at most

$$\max\{1, \|a_n| - |a_{-n}\|^{-1}(|a_{n-1}| + \cdots + |a_{-n+1}|)\}.$$

More general versions of the fundamental theorem can be proved using similar arguments. For instance it can be shown that a continuous function $f : \mathbb{C} \to \mathbb{C}$ which satisfies

$$\lim_{z \to \infty} \frac{f(z)}{az^n + b\bar{z}^n} \neq 0$$

for some positive integer n and complex numbers a, b, is surjective.

Eilenberg and Niven [3] have proved a fundamental theorem of algebra for quaternions by showing that a polynomial of degree n of the form

$$p(x) = a_0 x a_1 x \cdots x a_n + r(x),$$

where $a_0, \ldots, a_n$ are non-zero quaternions and $r(x)$ is a sum of a finite number of monomials of the form $b_0 x b_1 x \cdots x b_m$, $m < n$, always has a zero. Their proof is also topological and they prove that the map $p : S^4 \to S^4$, where S^4 is the one point compactification of the quaternions $\mathbb{H}$, is homotopic to the map $P_n : S^4 \to S^4 : z \mapsto z^n$ of degree n. By taking $k=4$ in Theorem 1, Theorem 2 can be modified readily to give another proof of the fundamental theorem for quaternions. The main change needed is that the map $P_n : S^3 \to S^3$ is replaced by the homotopic map $h : S^3 \to S^3$, given by

$$h(x) = a_0 x a_1 x \cdots x a_n;$$

the fact that h is essential follows from Lemmas 1 and 2 of [3].

Providing that the polynomial is parenthesised in a definite manner, a fundamental theorem can also be proved for Cayley numbers (corresponding to $k=8$ in Theorem 1) by using either the approach of Eilenberg and Niven or that of Theorem 2.

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