From (2) above $(y+1)^n - y^n - 1$ is exactly divisible by n;
i.e. $(\overline{y-1+1}+1)^n - (\overline{y-1}+1)^n - 1$ is exactly divisible by n;
but $(\overline{y-1}+1)^n - \overline{y-1}^n - 1$ is exactly divisible by n (from (2));
$\therefore$ the sum of these two expressions is exactly divisible by n;
i.e. $(y-1+1+1)^n - (y-1)^n - 2$ is exactly divisible by n;
i.e. $(y-1+2)^n - (y-1)^n - 2$ is exactly divisible by n.

In the same way, by putting for $y-1$, $y-2+1$, we deduce that $(y-2+3)^n - (y-2)^n - 3$ is exactly divisible by n, and so on; deducing ultimately that $(y-\overline{y-1}+y)^n - (y-\overline{y-1})^n - y$,
i.e. $(1+y)^n - 1^n - y$,
i.e. $a^n - a$ is exactly divisible by n.

W. A. LINDSAY

Fermat's Theorem deduced from the theory of circulating Radix Fractions:

$\dfrac{x^n-1}{n+1}$ is an integer, if x is an integer, and $n+1$ is a prime integer which is not a factor of x.

This is not a neat proof of Fermat's theorem, but, as far as I know, it is a new proof, and it may have some little interest, as it seems very possible that the theorem may have been suggested to Fermat in this way. Fermat published the theorem without any demonstration and without indicating what had suggested it, and any proofs, that I have seen, give no indication why one should look for such a theorem, and would very possibly never have been given if the theorem had not been already known.

It was only after I had written this proof that Professor Chrystal pointed out to me that it was a known theorem.

The conversion of recurring decimals into vulgar fractions led me to try to prove that $10^n - 1$ was divisible by $n+1$, $(n+1)$ being a prime integer to which 10 is prime.

e.g. $\frac{1}{7} = \cdot\dot{1}4285\dot{7} = \frac{142857}{999999}$.

$\therefore$ 999999 or $10^6 - 1$ is divisible by 7.

$\frac{1}{13} = \cdot\dot{0}7692\dot{3}$

$\therefore$ $10^6 - 1$ is divisible by 13.

$\therefore$ $10^{12} - 1$ is divisible by 13.

After proving this, I noticed that the theorem held more generally for any scale of notation x.

When Fermat's theorem is proved independently most of the theorems on circulating radix fractions, which I use in the proof, can be at once very neatly deduced from it.

Notation.

(i) I call $\frac{1}{n+1}, \frac{2}{n+1}, \frac{3}{n+1}, \ldots \frac{n}{n+1}$ the n reciprocals of $n+1$.

(ii) In what follows $abc\ldots k$ is an arithmetical notation, and does not mean $a \times b \times c \ldots \times k$.

Thus $abcd \equiv d + cx + bx^2 + ax^3$ where x is the scale of notation.

Also 10 means x (the scale of notation) and not "ten."

I use the symbol g for $x-1$, *i.e.* when x is ten, g is 9.

(iii) To avoid a new term, I use the name "decimal fraction" for a radix fraction with scale x, though it only applies to scale "ten."

I. $\cdot a_1 a_2 \ldots a_s \dot{b}_1 b_2 \ldots \dot{b}_r$ (1) where the a's and b's are integers from zero to g, is any recurring decimal in which there are "s" digits after the decimal point which do not recur, and "r" digits which recur.

Theorem.

To convert (1) into a vulgar fraction, we must take

$(a_1 a_2 \ldots a_s b_1 \ldots b_r) - (a_1 \ldots a_s)$ as numerator,

and as denominator r g's followed by s zeros.

$$\text{For } \cdot a_1 \ldots a_s \dot{b}_1 \ldots \dot{b}_r = \frac{a_1 \ldots a_s}{10^s} + \frac{b_1 \ldots b_r}{10^{r+s}} + \frac{b_1 \ldots b_r}{10^{2r+s}} \ldots ad \infty.$$

$$= \frac{a_1 \ldots a_s}{10^s} + \frac{(b_1 \ldots b_r)/10^{r+s}}{1 - \frac{1}{10^r}} = \frac{a_1 \ldots a_s}{10^s} + \frac{b_1 \ldots b_r}{(10^r - 1)10^s}$$

$$= \frac{(a_1 \ldots a_s)(10^r - 1) + (b_1 \ldots b_r)}{(10^r - 1)10^s}$$

$$= \frac{(a_1 \ldots a_s b_1 \ldots b_r) - (a_1 \ldots a_s)}{(10^r - 1)10^s}$$

and $10^r - 1 = gg \ldots (r \text{ digits})$.

(i) If we express $\frac{m}{n+1}$ [where m is one of the integers $1, 2, \ldots n$, and $n+1$ is any prime integer to which x is prime] as a decimal fraction (scale x), we must get a recurring decimal, for we can get no integer a which, when multiplied by $n+1$, will give us a power of 10 (*i.e.* of x).

(ii) The period must begin immediately after the decimal point for the following reason:—If it does not, suppose that

$$\frac{m}{n+1} = \cdot a_1 \ldots a_s \dot{b}_1 \ldots \dot{b}_r = \frac{q}{(10^r-1)10^s} = \frac{m \times a}{(n+1) \times a}$$

where a is an integer.

10^s does not contain a factor $n+1$,

$\therefore$ 10^s is a factor of a,

$\therefore$ there are as many zeros in numerator as in denominator; *i.e.* the last s digits of $a_1 \ldots a_s b_1 \ldots b_r$ must be $a_1 \ldots a_s$.

$\therefore$ the period begins immediately after the decimal point.

II. All the n reciprocals of $n+1$ must have the same number of digits in their periods as recurring decimals (same restrictions on $n+1$ as before).

For suppose that $\dfrac{m}{n+1}$ has s digits in its period; then $gg \ldots$(s digits) is divisible by $n+1$, and no smaller number of g's is divisible by $n+1$, as conversion into decimal fraction is unique for any given scale of notation.

Let $gg \ldots (s \text{ digits}) = k \times (n+1)$ where k is an integer.

Then $\dfrac{r}{n+1} = \dfrac{r \times k}{(n+1)k} = \dfrac{r \times k}{gg \ldots (s \text{ digits})}$,

and $r \times k < gg \ldots (s \text{ digits})$ if $r < n+1$.

$\therefore$ $\dfrac{r}{n+1}$ has s digits in its period (the period is in fact $a_1 \ldots a_s \equiv r \times k$ where $a_1 \ldots$, etc., may be zeros).

This is where theorem breaks down if $n+1$ is not prime, for $\dfrac{r}{n+1}$ is in that case reducible for certain values of r.

III. Process of finding the recurring decimal for $\dfrac{r_1}{n+1}$ with scale of notation x [where r_1 has any of the values $1, 2, \ldots n$, and $n+1$ is same as before].

The process is naturally that of division as with scale "ten," and $a0$ means $a \times 10$, *i.e.* $a \times x$.

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$$
\begin{array}{l}
\dfrac{r_1}{n+1}\Big)\; r_1 0 \qquad \underline{|\cdot \dot{a}_1 a_2 a_3 \ldots \dot{a}_s} \\
\qquad \underline{a_1(n+1)} \\
\qquad\quad r_2 0 \\
\qquad\quad \underline{a_2(n+1)} \\
\qquad\qquad r_3 0 \\
\qquad\qquad \underline{a_3(n+1)} \\
\qquad\qquad\quad r_4 0 \\
\qquad\qquad\quad \cdot \\
\qquad\qquad\quad \cdot \\
\qquad\qquad\quad \underline{\cdot\qquad} \\
\qquad\qquad\qquad r_s 0 \\
\qquad\qquad\qquad \underline{a_s(n+1)} \\
\qquad\qquad\qquad\quad r_1 \\
\qquad\qquad\qquad\quad \cdot \\
\qquad\qquad\qquad\quad \cdot \\
\qquad\qquad\qquad\quad \cdot \; ad\,\infty.
\end{array}
$$

All the r's are less than $n+1$, and none of them can be zero.

The above process needs no explanation. It is simply that of ordinary division with addition of a zero to every remainder to get the next digit in decimal.

(i) From what has been proved before, none of the remainders $r_2 r_3 \ldots$, etc., can occur twice before getting a remainder r_1; otherwise the period would not begin immediately after the decimal point.

(ii) The maximum number of digits in the period is n, for there are only n possible remainders when dividing by $n+1$, viz. $1, 2, \ldots n$.

(iii) The same digits $a_1 \ldots a_s$ in the same cyclic order must give us the recurring decimals for the s reciprocals $\dfrac{r_1}{n+1}, \dfrac{r_2}{n+1}, \ldots \dfrac{r_s}{n+1}$ and for no more.

Thus $\dfrac{r_l}{n+1} = \cdot \dot{a}_l a_{l+1} \ldots a_s a_1 \ldots \dot{a}_{l-1}$.

(iv) These s reciprocals are all distinct, as no two of the r's are the same.

Hence we see that if $\dfrac{r_1}{n+1}$ have s digits in its period, there are s distinct reciprocals $\left(\text{including } \dfrac{r_1}{n+1}\right)$, and no more than s, which have as their periods the same digits in the same cyclic order.

IV. Consider the case in which $s<n$; let us choose a reciprocal, say $\frac{t_1}{n+1}$, which is not among $\frac{r_1}{n+1}, \ldots \frac{r_s}{n+1}$.

Suppose $\frac{t_1}{n+1} = \cdot\dot{b}_1 b_2 \ldots \dot{b}_s$; the digits $b_1 \ldots b_s$ in the same cyclic order give us the periods of s and only s distinct reciprocals of $n+1$, and none of these can be $\frac{r_1}{n+1} \ldots \frac{r_s}{n+1}$; for suppose $\frac{r_s}{n+1}$ occurred among them, that means that in the process of finding the period of $\frac{t_1}{n+1}$, we get a remainder r_s; but after getting a remainder r_s we can never get a remainder t_1, as division in III. shows. Now this is impossible, as the period of $\frac{t_1}{n+1}$ must begin immediately after the decimal point.

Hence combining III. and IV., we see that the reciprocals of $n+1$ can be divided into groups of s distinct reciprocals, and that two distinct groups are mutually exclusive.

V. The number of digits "s" in the above periods, if not equal to n, must be a submultiple of n.

For if s be not a submultiple of n (and not equal to n) let $n = p \times s + q$ where p and q are integers and $q<s$.

Then if $\frac{r_1}{n+1} = \cdot\dot{a}_1 \ldots \dot{a}_s$, we must have s corresponding reciprocals which are distinct. A.

Suppose now that we take $\frac{t_1}{n+1}$, which is not in group A, we will get another group of s distinct reciprocals. B.

Suppose that, continuing in this manner, we have already got p groups; then we have already accounted for $p \times s$ distinct reciprocals since these groups are mutually exclusive.

Let $\frac{w_1}{n+1}$ be a reciprocal which is not accounted for, and let $\frac{w_1}{n+1} = \cdot\dot{c}_1 c_2 \ldots \dot{c}_s$; then these digits in the same cyclic order are the periods of s reciprocals which are all distinct and not contained in the groups above and which *must* exist; but this is impossible, as we have only q reciprocals left and $q<s$.

$\therefore$ s must be a factor of n.

VI. Since $\dfrac{m}{n+1}$ [$m<n+1$ and other conditions same as before] has s digits in its period,

$\therefore$ $gg\ldots$(s digits) is divisible by $n+1$, *i.e.* x^s-1 is divisible by $n+1$. But since s is a factor of n, $\therefore$ $\dfrac{x^n-1}{x^s-1}$ is an integer.

$\therefore$ We have Fermat's theorem: x^n-1 is divisible by $n+1$, provided x is an integer and $n+1$ is a prime integer to which x is prime.

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