# ON MEASURE OF SUMSETS
## III. THE CONTINUOUS ($\alpha+\beta$)-THEOREM

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### 1. Introduction

In the second paper under this general title,† it was shown how a theorem about the torus could be deduced by a limiting process from a theorem on finite abelian groups. The object of this paper is to prove a similar continuous analogue of H. B. Mann's ($\alpha+\beta$)-theorem.‡ It was found that the limiting process used in the second paper could not easily be modified to apply to the present problem, and an alternative method had to be found. The method is, roughly, to prove the result first for open sets satisfying certain conditions, then for closed sets by taking intersections of open sets, and finally for arbitrary measurable sets, since every measurable set contains a closed set of almost equal measure.

We consider the space $P$ of positive reals with Lebesgue measure $\mu$. Let $A$, $B$ be measurable subsets of $P$ and let $C = A+B$ be the set consisting of all sums $x+y$ with $x$ in $A$ and $y$ in $B$. If $K$ is any subset of $P$, we denote by $K(t)$ the subset of $K$ consisting of those numbers which do not exceed $t$. Let $\alpha(t) = \mu(A(t))$, $\beta(t) = \mu(B(t))$, and let $\gamma(t)$ denote the inner measure of $C(t)$. The theorem which it is proposed to prove is as follows.

**Theorem.** *If* $\inf A = \inf B = 0$ *and if*

$$\alpha(t)+\beta(t) \geqq kt \text{ for } 0<t<x, \text{ where } k \leqq 1,$$

*then*

$$\gamma(x) \geqq kx. \qquad (1)$$

Throughout the paper $A$, $B$ will denote sets satisfying the hypotheses of the theorem.

### 2. Some Properties of Open Sets

Let $K$ be any open subset of $P$. For each positive $\zeta$, $K[\zeta]$ denotes the set of all positive integers $n$ such that the open interval $(n\zeta, (n+1)\zeta)$ is contained in $K$, and $K\{\zeta\}$ denotes the union of all such intervals $(n\zeta, (n+1)\zeta)$. The absolute value sign denotes the number of elements in a set and the round bracket notation is extended to sets of integers also. Thus, for instance, $|K[\zeta](n)|$ denotes the number of integers in $K[\zeta]$ not exceeding $n$.

† *Proc. Cambridge Phil. Soc.*, 49 (1953), 40-41. See also M. Kneser, *Math. Zeitschrift*, 66 (1958), 88-110.
‡ *Annals of Math.* (2), 43 (1942), 523-527.

**Lemma 1.** *If* $\mu(K)$ *is finite, then* $\lim_{\zeta \to 0} \zeta \mid K[\zeta] \mid = \mu(K)$.

**Proof.** Since $K \supset K\{\zeta\}$, we have

$$\zeta \mid K[\zeta] \mid = \mu(K\{\zeta\}) \leqq \mu(K). \qquad \text{.........................(2)}$$

Now $K$, being open, is a countable disjoint union of intervals, so there is a finite class of disjoint intervals $I_1, \ldots, I_m$, each contained in $K$, such that

$$\mu(K) < \sum_{e=1}^{m} \mu(I_e) + \tfrac{1}{2}\varepsilon.$$

For each interval $I_t$, it is clear that $\mu(I_t\{\zeta\}) > \mu(I_t) - 2\zeta$. If $\zeta < \varepsilon/4m$, we deduce that

$$\mu(K\{\zeta\}) \geqq \sum_{e=1}^{m} \mu(I_e\{\zeta\}) > \sum_{e=1}^{m} \mu(I_e) - \tfrac{1}{2}\varepsilon > \mu(K) - \varepsilon.$$

The lemma follows from this relation and (2).

**Lemma 2.** *If* $A$, $B$ *are open sets, then* $C \supset A \cup B$.

**Proof.** Let $a \,\varepsilon\, A$. Since $A$ is open, there is an $\varepsilon > 0$ such that, if $\mid a - x \mid < \varepsilon$, then $x \,\varepsilon\, A$. Since $\inf B = 0$, there is a $z \,\varepsilon\, B$ such that $0 < z < \varepsilon$. Then $a - z \,\varepsilon\, A$ so that $a = (a-z) + z \,\varepsilon\, A + B$. Thus $A \subset C$ and similarly $B \subset C$.

From now until the end of § 3, $m$, $n$ denote integers and $\zeta = x/n$.

**Lemma 3.** *If* $m < n$, *then* $K(m\zeta)\backslash K(m\zeta)\{\zeta\} \subset K(n\zeta)\backslash K(n\zeta)\{\zeta\}$.

**Proof.** Let $a > 0$, and let $q$ be the integral part of $a/\zeta$. If $a/\zeta$ is not an integer, a necessary and sufficient set of conditions for $a$ to belong to the left-hand set above is: $a \,\varepsilon\, K$, $q < m$, and the interval $(q\zeta, (q+1)\zeta)$ is not contained in $K(m\zeta)$.

Since $m < n$, these conditions also imply that $a$ belongs to the set on the right. The case where $a$ is an integral multiple of $\zeta$ is equally easy.

**Lemma 4.** $K((m+1)\zeta)[\zeta] = K[\zeta](m)$.

The proof is left to the reader.

## 3. Proof of the Theorem when A, B are Open Sets

**Lemma 5.** *If* $A$, $B$ *are open and each contains the interval* $(0, \xi)$, *where* $\xi > 0$, *then* (1) *holds.*

**Proof.** In the proof of this lemma we assume Mann's theorem in the following form (easily seen to be equivalent to the usual statement).

Let $X$, $Y$ be sets of positive integers, and let $Z$ denote the set $X \cup Y \cup (X + Y)$. If, for $0 < m < n$, $\mid X(m) \mid + \mid Y(m) \mid \geqq km$, where $k \leqq 1$, then $\mid Z(n) \mid \geqq kn$.

We assume, without loss of generality, that $\xi < x$. Since $\zeta = x/n$, we may, by Lemma 1, choose $n$ so large that

$$\zeta \mid A(x)[\zeta] \mid > \alpha(x) - \xi\varepsilon, \quad \zeta \mid B(x)[\zeta] \mid > \beta(x) - \zeta\varepsilon. \qquad \text{..............(3)}$$

We shall show that (3) implies, for $0<m\leqq n-1$,

$$|A[\zeta](m)|+|B[\zeta](m)|>(k-2\varepsilon)m. \qquad (4)$$

Write $t=(m+1)\zeta$. If $t<\xi$, (4) is clear, so we assume that $t\geqq\xi$. By Lemma 3, (2) and (3) we have

$$\mu(A(t)\backslash A(t)\{\zeta\})<\xi\varepsilon.$$

Hence, using Lemma 4,

$$\zeta\,|A[\zeta](m)|=\zeta\,|A(t)[\zeta]|=\mu(A(t)\{\zeta\})>\alpha(t)-\xi\varepsilon$$

and similarly

$$\zeta\,|B[\zeta](m)|>\beta(t)-\xi\varepsilon.$$

Hence

$$\zeta\,|A[\zeta](m)|+\zeta\,|B[\zeta](m)|>\alpha(t)+\beta(t)-2\xi\varepsilon\geqq kt-2\xi\varepsilon\geqq(k-2\varepsilon)t>(k-2\varepsilon)m\zeta.$$

This proves (4).

Now, by Lemma 2, $C[\zeta]\supset A[\zeta]\cup B[\zeta]\cup(A[\zeta]+B[\zeta])$. Hence, applying Mann's theorem to the sets $X=A[\zeta]$, $Y=B[\zeta]$, we deduce that

$$\mu(C(x))\geqq\zeta\,|C[\zeta](n-1)|\geqq\zeta(k-2\varepsilon)(n-1)=(k-2\varepsilon)(x-\zeta).$$

Letting $\varepsilon$, $\zeta$ tend to zero, we derive (1). This completes the proof of Lemma 5.

## 4. Proof of the Theorem in General

**Lemma 6.** *If $A$, $B$ are closed sets (in the relative topology as subsets of $P$), then* (1) *holds.*

**Proof.** Let $A_\xi$ denote the set of all points $x\,\varepsilon\,P$ such that

$$\inf\{|x-a|:a\,\varepsilon\,A\}<\xi.$$

Then $A_\xi+B_\xi=C_{2\xi}$, and $A_\xi$, $B_\xi$ are open sets satisfying the requirements of Lemma 5. Hence $\mu(C_{2\xi}(x))\geqq kx$ for every $\xi>0$. Since $C$ is closed,

$$C(x)=\bigcap_{\xi>0}C_{2\xi}(x), \quad \text{so} \quad \gamma(x)=\lim_{\xi\to 0}\mu(C_\xi(x))\geqq kx.$$

This proves Lemma 6.

To prove the theorem in its most general form, choose two positive constants $p>1$, $q<1$. Let $I_n$ denote the interval $(p^n, p^{n+1})$ for integer $n$. Choose a closed set $A'\subset A$ such that $\mu(A'\cap I_n)>q\mu(A\cap I_n)$ for all $n$, and choose $B'$ similarly. Let $C'=A'+B'$, which is contained in $C$. Then if $t>0$, we have $p^r\leqq t<p^{r+1}$ for some $r$, and $\mu(A'(t))\geqq\mu(A'(p^r))>q\mu(A(p^r))$ and similarly $\mu(B'(t))>q\mu(B(p^r))$. Hence

$$\mu(A'(t))+\mu(B'(t))>q\mu(A(p^r))+q\mu(B(p^r))\geqq kqp^r\geqq kqt/p.$$

By Lemma 6, $\gamma(x)\geqq\mu(C'(x))\geqq pkx/p$, and the theorem follows on letting $q/p$ tend to 1.

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