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A NOTE ON THE POLIGNAC NUMBERS

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Abstract

Suppose that $k_0 \geq 3.5 \times 10^6$ and $\mathcal{H} = \{h_1, \ldots, h_{k_0}\}$ is admissible. Then, for any $m \geq 1$, the set $\{m(h_j - h_i) : h_i < h_j\}$ contains at least one Polignac number.

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1. Introduction

A recent huge breakthrough in prime number theory is Zhang's brilliant work (see [4]), which asserts that

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 7 \times 10^7,$$

where p_n denotes the nth prime. For a set $\mathcal{H} = \{h_1, h_2, \ldots, h_{k_0}\}$ of positive integers, we say that $\mathcal{H}$ is *admissible* if $\nu_p(\mathcal{H}) < p$ for every prime p, where $\nu_p(\mathcal{H})$ denotes the number of distinct residue classes occupied by those h_i modulo p. Zhang proved that if $k_0 \geq 3.5 \times 10^6$ and $\mathcal{H} = \{h_1, \ldots, h_{k_0}\}$ is admissible, then, for sufficiently large x, there exists $n \in [x, 2x]$ such that $\{n + h_1, n + h_2, \ldots, n + h_{k_0}\}$ contains at least two primes. He also constructed an admissible $\mathcal{H} = \{h_1, \ldots, h_{k_0}\}$ such that $\max_{i,j} |h_j - h_i| \leq 7 \times 10^7$.

In fact, one may have the following 'cheap' extension of Zhang's theorem.

Theorem 1.1. *Let $k_0 \geq 3.5 \times 10^6$ and $A > 0$. Suppose that x is sufficiently large and $1 \leq q \leq (\log x)^A$. If $\mathcal{H} = \{h_1, \ldots, h_{k_0}\}$ is admissible and $(q, h_1 h_2 \cdots h_{k_0}) = 1$, there exists $n \in [x, 2x]$ such that $\{qn + h_1, qn + h_2, \ldots, qn + h_{k_0}\}$ contains at least two primes.*

The proof of Theorem 1.1 is just a copy of Zhang's original one. The only modification is to set

$$P(n) = \prod_{i=1}^{k_0}(qn + h_i) \quad \text{and} \quad \mathfrak{S} = \prod_{\substack{p \text{ prime} \\ p \nmid q}} \left(1 - \frac{\nu_p(\mathcal{H})}{p}\right) \cdot \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{-k_0}.$$

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As an immediate consequence of Theorem 1.1, for $0 \le b < q$ with $(b,q) = 1$,

$$\liminf_{n\to\infty} \frac{p_{n+1}^{(b,q)} - p_n^{(b,q)}}{q} \le 7 \times 10^7, \tag{1.1}$$

where $p_n^{(b,q)}$ denotes the nth prime of the form $qm + b$. In fact, suppose that prime $p \nmid q$ and $\{h_1, \dots, h_{k_0}\}$ does not cover the residue class c modulo p. Then $\{b + qh_1, \dots, b + qh_{k_0}\}$ does not cover $(b + cq)$ modulo p, as well. And if $p \mid q$, then evidently $\{b + qh_1, \dots, b + qh_{k_0}\}$ does not cover 0 modulo p. That is, the admissibility of $\{h_1, \dots, h_{k_0}\}$ always implies that of $\{b + qh_1, \dots, b + qh_{k_0}\}$. Thus (1.1) easily follows from Theorem 1.1.

However, the main purpose of this short note is to give another application of Theorem 1.1, concerning Polignac numbers [3]. A positive even number d is called a Polignac number if there exist infinitely many n such that $p_{n+1} - p_n = d$. Of course, it is believed that every positive even number is a Polignac number. And Zhang's theorem shows that the smallest Polignac number is not greater than 7×10^7.

Recently, combining Zhang's techniques with some lemmas from [1], Pintz [2] proved that the set of all Polignac numbers has a positive lower density. We shall now show that this lower density is at least 2×10^{-21}. In fact, we have the following theorem.

THEOREM 1.2. *Suppose that $k_0 \ge 3.5 \times 10^6$ and $\mathcal{H} = \{h_1, \dots, h_{k_0}\}$ is admissible. Let $\sigma(\mathcal{H}) = \{h_j - h_i : h_i < h_j\}$. Then, for any $m \ge 1$, the set $m \cdot \sigma(\mathcal{H}) = \{md : d \in \sigma(\mathcal{H})\}$ contains at least one Polignac number.*

Evidently, by taking $k_0 = 3.5 \times 10^6$ and $\max_{i,j} |h_i - h_j| \le 7 \times 10^7$ in Theorem 1.2, we can get that the lower density of all Polignac numbers is at least

$$\frac{2}{k_0^2 \cdot \max_{i,j} |h_j - h_i|} \ge 2 \times 10^{-21}.$$

2. Proof of Theorem 1.2

Without loss of generality, assume that $h_1 < h_2 < \cdots < h_{k_0}$. Let

$$X = \{a \in [mh_1, mh_{k_0}] : a \equiv mh_1 \pmod 2, a \notin \{mh_1, \dots, mh_{k_0}\}\}.$$

Assume that $X = \{a_1, a_2, \dots, a_l\}$. Arbitrarily choose distinct primes $p_1, p_2, \dots, p_l > mh_{k_0}$. Let $b > 0$ be an integer such that $b \equiv 1 \pmod m$ and $b \equiv -a_j \pmod{p_j}$ for $1 \le j \le l$. Let $q = mp_1p_2 \cdots p_l$. Since $(b,m) = 1$, $\{b + mh_1, \dots, b + mh_{k_0}\}$ is admissible. And for each j, noting that $p_j \mid b + a_j$ and $p_j > mh_{k_0}$, we must have

$$\prod_{i=1}^{k_0} (b + mh_i) \not\equiv 0 \pmod{p_j}.$$

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That is, q is coprime to $(b + mh_1)(b + mh_2) \cdots (b + mh_{k_0})$. By Theorem 1.1, there exist infinitely many n such that $\{qn + b + mh_1, qn + b + mh_2, \ldots, qn + b + mh_{k_0}\}$ contains at least two primes.

Let $n_1, n_2, n_3, \ldots$ be all such positive integers n. For each $s \geq 1$, noting that $\{qn_s + b + mh_1, \ldots, qn_s + b + mh_{k_0}\}$ contains at least two primes, we may choose a pair (i_s, j_s) with $i_s < j_s$ such that both $qn_s + b + mh_{i_s}$ and $qn_s + b + mh_{j_s}$ are prime, but $qn_s + b + mh_k$ is composite for all $i_s < k < j_s$. Since $1 \leq i_s < j_s \leq k_0$, clearly there exists a pair (i_*, j_*) such that the set $\{s : (i_s, j_s) = (i_*, j_*)\}$ is infinite. That is, $qn + b + mh_{i_*}$ and $qn + b + mh_{j_*}$ are prime for infinitely many n. But according to the definition of q, for any $a_j \in (mh_{i_*}, mh_{j_*})$, $qn + b + a_j \equiv 0 \pmod{p_j}$, that is, $qn + b + a_j$ cannot be prime. So $qn + b + mh_{i_*}$ and $qn + b + mh_{j_*}$ must be two consecutive primes, that is, $m(h_{j_*} - h_{i_*})$ is a Polignac number. We are done.

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