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A TEST FUNCTION THEOREM AND APPROXIMATION BY PSEUDOPOLYNOMIALS

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We prove a Korovkin-type theorem on approximation of bivariate functions in the space of B-continuous functions (introduced by K. Bögel in 1934). As consequences, some sequences of uniformly approximating pseudopolynomials are obtained.

1. Introduction

We first recall some notations and definitions. Following Bögel [6,7,8], we say that a real-valued function f on the square $I^2 = [0,1] \times [0,1]$ is $\Delta_{u,v}$-continuous (*B-continuous*) if for every $(x,y) \in I^2$ we have

$$\lim_{\substack{u \to x \\ v \to y}} \Delta_{u,v} f(x,y) = 0,$$

where $\Delta_{u,v} f(x,y) = f(x,y) - f(x,v) - f(u,y) + f(u,v)$.

Let $B(I^2)$ denote the space of all B-continuous and real-valued functions on I^2 and, as usual, $C(I^2)$ the space of continuous and real-valued functions on I^2 .

A B-continuous function is not necessarily continuous (in the usual sense), but the converse is true. Moreover, there is an unbounded B-continuous function, which follows from the fact that for any function of the type $f(x,y) = g(x) + h(y)$ one has $\Delta_{u,v} f(x,y) = 0$.

We shall call *Marchaud pseudopolynomials* the functions introduced in Marchaud's fundamental papers [15,16], that is functions of the type

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$$p : I^2 \ni (x,y) \mapsto \sum_{i=0}^{m} x^i A_i(y) + \sum_{j=0}^{n} B_j(x) y^j \in \mathrm{R} ,$$

where A_i and B_j are bounded real-valued functions on $I = [0,1]$ and m and n are non-negative integers. *Pseudopolynomials* are functions of the above form, where A_i and B_j are arbitrary univariate functions on $I = [0,1]$. We note that a bounded pseudopolynomial is a Marchaud pseudopolynomial (see Popoviciu [19]). Note also that this definition differs slightly from the one used in the recent paper by Gonska and Jetter [13].

There are various methods of constructing pseudopolynomial approximants defined on I^2 . A very effective approach is to use the Boolean sum of the parametric extensions of two univariate polynomial operators. See the papers of Brudnyĭ [9], Delvos and Schempp [10], and of Gonska and Jetter [13], where various aspects of this technique are discussed (see also the references cited in these articles). If the method is applied to two copies B_m and B_n of the well-known Bernstein operators, then the operators

$$H_{m,n}(f;x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} (f(x,j/n) + f(i/m,y) - f(i/m,j/n)) p_{m,i}(x) \cdot p_{n,j}(y) ,$$

$$p_{r,\zeta}(z) = \binom{r}{\zeta} z^{\zeta} \cdot (1-z)^{r-\zeta} , \ 0 \leqslant \zeta \leqslant r, \ 0 \leqslant z \leqslant 1 ,$$

are generated. This particular instance was studied by Badea [1] and by Gordon and Riesenfeld [14], among others. Stancu [21] investigated the corresponding bivariate operator based upon a certain generalization of the classical Bernstein operators (see Section 4 of this paper). Clearly, if the function f in the above representation is bounded, then $H_{m,n}(f;\cdot,*)$ will be a Marchaud pseudopolynomial.

Another method was introduced by I. Badea [2]. His method, as applied to Bernstein operators, yields the pseudopolynomials

$$p_n(f;x,y) = \tfrac{1}{2} \sum_{i=0}^{n} (f(x,i/n) + f(i/n,y) - f(i/n,i/n)) \cdot (p_{n,i}(x) + p_{n,i}(y)).$$

Further contributions to the approximation by this special sequence are due to Badea and Oprea [5], and to Gonska [12]. There are other general methods

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of constructing meaningful pseudopolynomial approximants which we shall not discuss here.

If one is interested only in the uniform approximation of functions in $C(I^2)$ by pseudopolynomials, then there is no problem at all. This follows from the fact that the space of bivariate polynomials is dense in $C(I^2)$ with respect to the uniform norm, and that any bivariate polynomial may be written as a Marchaud pseudopolynomial. However, an *unbounded* B-continuous function cannot be uniformly approximated by Marchaud pseudopolynomials.

Thus, the following natural problem arose: Does the Weierstrass approximation theorem still hold for B-continuous functions and pseudopolynomials? This problem was posed by Nicolescu [17]. Partial contributions to this problem were given by Nicolescu [18] himself, Vaida [22], and by Dobrescu and Matei [11] (see the discussion in [3]). Nicolescu's problem was solved by Badea [1] in 1973 using the Boolean sum approach mentioned and the above operators $H_{m,n}$. We note further that the result is, indeed, one in terms of Marchaud's product-type modulus (see Marchaud's paper [16] for details).

The purpose of this article is to prove a Korovkin-type theorem for convergence in the space $B(I^2)$, and thus to provide the reader with a variety of approximation processes which may be used to approximate B-continuous functions uniformly and arbitrarily well (for a Korovkin-type theorem in the smaller space $C(I^2)$, see for example Volkov [23]).

The method of constructing these operators is to start off with suitable positive linear mappings defined on $\mathbb{R}^{I^2}$ and to transform them into a sequence of operators on $B(I^2)$ which does the job. Our technique is similar to that of replacing the tensor product of two operators by their Boolean sum. As applications, further solutions for Nicolescu's problem are given, among others.

2. A Lemma

We shall need the following auxiliary result.

LEMMA. *Let* $f \in B(I^2)$ *be arbitrarily chosen. For every positive number* ε *there are two positive numbers* $A(\varepsilon) = A(\varepsilon,f)$ *and* $B(\varepsilon) = B(\varepsilon,f)$

such that for every $(x,y),(s,t) \in I^2$ *we have*

$$|\Delta_{s,t} f(x,y)| \leqslant \varepsilon/3 + A(\varepsilon)(x-s)^2 + B(\varepsilon)(y-t)^2 .$$

Proof. Because f is B-continuous on I^2, the function f is also *uniformly* B-*continuous* ([8, Satz 7]), that is for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that for every $(x_1,y_1),(x_2,y_2) \in I^2$ with $|x_1-x_2| \leqslant \delta(\varepsilon)$ and $|y_1-y_2| \leqslant \delta(\varepsilon)$ we have:

$$|\Delta_{x_2,y_2} f(x_1,y_1)| \leqslant \varepsilon/3 . \tag{2.1}$$

Let ε be a given positive real number and $(x,y),(s,t) \in I^2$. We shall investigate the following four situations:

(i) $|x-s| \leqslant \delta(\varepsilon)$, $|y-t| \leqslant \delta(\varepsilon)$;

(ii) $|x-s| \leqslant \delta(\varepsilon)$, $|y-t| > \delta(\varepsilon)$;

(iii) $|x-s| > \delta(\varepsilon)$, $|y-t| \leqslant \delta(\varepsilon)$;

(iv) $|x-s| > \delta(\varepsilon)$, $|y-t| > \delta(\varepsilon)$.

In case (i), using (2.1) we have

$$|\Delta_{s,t} f(x,y)| \leqslant \varepsilon/3 . \tag{2.2}$$

Now we consider case (ii). Because f is B-continuous, there is (see [1 Lema]) a positive number M such that

$$|\Delta_{s,t} f(x,y)| \leqslant M . \tag{2.3}$$

From (2.3) and the second inequality of (ii) it follows that

$$|\Delta_{s,t} f(x,y)| \leqslant M(y-t)^2[\delta(\varepsilon)]^{-2} . \tag{2.4}$$

In case (iii) we obtain in a similar manner

$$|\Delta_{s,t} f(x,y)| \leqslant M(x-s)^2[\delta(\varepsilon)]^{-2} . \tag{2.5}$$

From the two inequalities in (iv) and from (2.3) we get

$$|\Delta_{s,t} f(x,y)| \leqslant M(x-s)^2(y-t)^2[\delta(\varepsilon)]^{-4} . \tag{2.6}$$

Consequently, employing (2.2), (2.4), (2.5) and (2.6), we have the following inequality

$$(2.7)\qquad |\Delta_{s,t}f(x,y)| \leqslant \varepsilon/3 + M(x-s)^2[\delta(\varepsilon)]^{-2} + M(y-t)^2[\delta(\varepsilon)]^{-2} + M(x-s)^2(y-t)^2[\delta(\varepsilon)]^{-4} .$$

Because $y,t \in [0,1]$, it follows that $(y-t)^2 \leqslant 1$. Hence from (2.7) we conclude that the following inequality

$$(2.8)\quad |\Delta_{s,t}f(x,y)| \leqslant \varepsilon/3 + M[\delta(\varepsilon)]^{-2}\{1 + [\delta(\varepsilon)]^{-2}\}(x-s)^2 + M[\delta(\varepsilon)]^{-2}(y-t)^2$$

holds, and the lemma is proved. □

3. A Korovkin-Type Theorem for Approximation in $B(I^2)$

We are now ready to prove the main result of this paper. Let us consider the following real-valued functions on I^2 :

$$e(s,t) = 1,\quad \phi(s,t) = s,\quad \psi(s,t) = t .$$

THEOREM. *Let* $\{L_{m,n}\}, (m,n) \in \mathbb{N}^2$ *, be a sequence of positive linear operators transforming functions of* $\mathbb{R}^{I^2}$ *into functions of* $\mathbb{R}^{I^2}$ *such that for all* $(x,y) \in I^2$ *one has*

(i) $L_{m,n}(e;x,y) = 1$,

(ii) $L_{m,n}(\phi;x,y) = x + u_{m,n}(x,y)$,

(iii) $L_{m,n}(\psi;x,y) = y + v_{m,n}(x,y)$,

(iv) $L_{m,n}(\phi^2 + \psi^2;x,y) = x^2 + y^2 + w_{m,n}(x,y)$,

where $u_{m,n}(x,y)$, $v_{m,n}(x,y)$ *and* $w_{m,n}(x,y)$ *converge to zero uniformly on* I^2 *as* m,n *approach infinity in any manner whatsoever. If* $f(\cdot,*) \in B(I^2)$ *and* $(x,y) \in I^2$ *we put*

$$U_{m,n}(f;x,y) = L_{m,n}(f(\cdot,y) + f(x,*) - f(\cdot,*);x,y) .$$

Under these conditions, for every $f \in B(I^2)$ *, the sequence* $\{U_{m,n}(f)\}$ *converges uniformly to* f *on* I^2 *.*

Proof. If $(x,y) \in I^2$ is fixed, then the B-continuity of f implies that of the function

$$F(\cdot,*) = f(\cdot,y) + f(x,*) - f(\cdot,*) .$$

This is a consequence of the fact that, for all $(u,v),(s,t) \in I^2$, one has

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$$\Delta_{u,v}F(s,t) = -\Delta_{u,v}f(s,t) \text{ , independent of } (x,y) \in I^2 .$$

Hence $U_{m,n}$ is a well-defined linear operator on $B(I^2)$.

Now let $f \in B(I^2)$ be arbitrarily chosen, and let $(x,y) \in I^2$ and $\varepsilon > 0$ be given. Because $L_{m,n}$ is a linear operator reproducing constant functions we have

$$(3.1) \qquad f(x,y) - U_{m,n}(f;x,y) = L_{m,n}(\Delta_{x,y}f(\cdot,*);x,y) .$$

Furthermore,

$$(3.2) \qquad |L_{m,n}(g;x,y)| = \max\{L_{m,n}(g;x,y),\ L_{m,n}(-g;x,y)\}$$

for every function $g \in B(I^2)$.

Applying this equality to $g(s,t) = \Delta_{x,y} f(s,t)$ and, further using the monotonicity of $L_{m,n}$ and the lemma, we find (with $C(\varepsilon) = \max\{A(\varepsilon), B(\varepsilon)\}$) the inequality

$$(3.3) \quad |f(x,y) - U_{m,n}(f;x,y)| \leq L_{m,n}(\varepsilon/3 + C(\varepsilon)(x-\cdot)^2 + C(\varepsilon)(y-*)^2;x,y) .$$

After some manipulation of (3.3) we arrive at the inequality

$$(3.4) \quad |f(x,y) - U_{m,n}(f;x,y)| \leq \varepsilon/3 + C(\varepsilon)L_{m,n}(\phi^2 + \psi^2;\ x,y) -$$
$$- 2C(\varepsilon)[x\ L_{m,n}(\phi;x,y) + y\ L_{m,n}(\psi;x,y)] +$$
$$+ C(\varepsilon)(x^2+y^2)L_{m,n}(e;x,y) .$$

Using the relations (i) through (iv) from the statement of the theorem we can write

$$(3.5) \quad |f(x,y) - U_{m,n}(f;x,y)|$$
$$\leq \varepsilon/3 + C(\varepsilon)\cdot[w_{m,n}(x,y) - 2x\cdot u_{m,n}(x,y) - 2y\cdot v_{m,n}(x,y)] .$$

Letting m and n tend to infinity yields the desired result. □

REMARK. If equality (i) of the theorem does not hold, then equation (3.1) is not true. If one replaces (i) by

$$(\text{i}') \qquad L_{m,n}(f;x,y) = 1 + \alpha_{m,n}(x,y) ,$$

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then it can be shown that the following holds:

$$|f(x,y) - U_{m,n}(f;x,y)| \leqslant |f(x,y)|\cdot|\alpha_{m,n}(x,y)| + \varepsilon/3(1+\alpha_{m,n}(x,y))$$

$$+ C(\varepsilon)\cdot[w_{m,n}(x,y) - 2x\cdot u_{m,n}(x,y) - 2y\cdot v_{m,n}(x,y) + (x^2+y^2)\alpha_{m,n}(x,y)] .$$

Clearly, for $\alpha_{m,n}(x,y) = 0$, this reduces to inequality (3.5). However, as mentioned earlier, there are examples of unbounded B-continuous functions. Hence the term $|f(x,y)|\cdot|\alpha_{m,n}(x,y)|$ is indeed a critical one, and the above inequality allows only the conclusion that we have pointwise convergence to $f(x,y)$ for all $(x,y) \in I^2$, if $\alpha_{m,n}(x,y)$ converges uniformly to zero as m,n tend to infinity. □

4. Applications

In this section we shall show how the Korovkin-type theorem from the above section can be used to obtain pseudopolynomials which approximate B-continuous functions uniformly.

4.1 Boolean Sums of Bernstein-Stancu Operators

In his paper [21 , Theorem 4.1] Stancu studied the operators

$$(4.1)\quad H_{m,n}^{[\alpha,\beta]}(f;x,y) =$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n}[f(x,j/n)+f(i/m,y)-f(i/m,j/n)]\cdot w_{m,i}^{[\alpha]}(x)\cdot w_{n,j}^{[\beta]}(y) ,$$

where

$$(4.2)\qquad w_{p,k}^{[\gamma]}(t) = \frac{\binom{p}{k}\prod_{\nu=0}^{k-1}(t+\nu\gamma)\cdot\prod_{\nu=0}^{p-k-1}(1-t+\nu\gamma)}{\prod_{\nu=0}^{p-1}(1+\nu\gamma)}$$

These linear operators are obtained via our approach if in the definition of $U_{m,n}$ the tensor product operators

$$L_{m,n}^{[\alpha,\beta]}(f;x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} f(i/m,j/n)\cdot w_{m,i}^{[\alpha]}(x)\cdot w_{n,j}^{[\beta]}(y)$$

are used. Clearly, $H_{m,n}^{[0,0]}$ is the above transformation $H_{m,n}$. Because

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$w_{p,k}^{[\gamma]}$ is a polynomial, it is easily seen that $H_{m,n}^{[\alpha,\beta]}f$ is a pseudopolynomial. The following corollary shows that $H_{m,n}^{[\alpha,\beta]}$ provides us with a variety of solutions of Nicolescu's problem, including the one of Badea mentioned earlier.

COROLLARY 1. *If* $f \in B(I^2)$, $0 \leqslant \alpha = \alpha(m) \to 0$ *when* $m \to \infty$ *and* $0 \leqslant \beta = \beta(n) \to 0$ *for* $n \to \infty$, *then* $H_{m,n}^{[\alpha,\beta]}f$ *converges to* f *uniformly on* I^2 .

Proof. The proof is a consequence of the facts that, for $\alpha,\beta \geqslant 0$, the operator $L_{m,n}^{[\alpha,\beta]}$ is positive and satisfies the conditions (i) through (iv) of our above theorem if $\alpha(m)$ and $\beta(n)$ converge to zero (see Stancu [20 , Theorem 2]. □

Keeping in mind the result of Popoviciu mentioned in the first section of this paper, from Corollary 1 we obtain

COROLLARY 2. *If* $f \in B(I^2)$ *is bounded and if* $\alpha = \alpha(m)$ *and* $\beta = \beta(n)$ *tend to zero, then the sequence of Marchaud pseudopolynomials* $H_{m,n}^{[\alpha,\beta]}f$ *converges to* f *uniformly on* I^2 .

4.2 Boolean Sums of Positive Linear Operators of Discrete Type

The example considered in 4.1 is a special instance of the more general case in which the operator $L_{m,n}$ in the above theorem is the product of the parametric extensions of two univariate positive linear operators L_m and $\overline{L}_n$, both mapping $C(I)$ into itself, say, and given by

$$L_m(f,x) = \sum_{i=0}^{m} f(x_i)\cdot p_{m,i}(x)\,,\ x_i \in I,\ p_{m,i}(x) \geqslant 0 \ \text{ for } 0 \leqslant i \leqslant m \text{ and all } x \in I\ ;$$

$$\overline{L}_n(g,y) = \sum_{j=0}^{n} g(y_i)\cdot q_{n,j}(y)\,,\ y_j \in I,\ q_{n,j}(y) \geqslant 0 \ \text{ for } 0 \leqslant j \leqslant n \text{ and all } y \in I\ .$$

We also assume that

$$\sum_{i=0}^{m} p_{m,i}(x) = \sum_{j=0}^{n} q_{n,j}(y) = 1 \quad \text{for all } m,n \text{ , and all } x,y \in I\ .$$

If we denote the extensions by ${}_xL_m$ and ${}_y\overline{L}_n$ (here ${}_xL_m$ acts on the bivariate functions $f(x,y)$ as if y is a fixed parameter, and ${}_y\overline{L}_n$ is defined similarly), then

$$L_{m,n}(f;x,y) = ({}_xL_m \circ {}_y\overline{L}_n)(f;x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} f(x_i,y_j) \cdot p_{m,i}(x) \cdot q_{n,j}(y)$$

is a positive linear operator defined for any $f \in \mathbb{R}^{I^2}$, and for functions in $B(I^2)$ in particular. Moreover, for the operator $U_{m,n}$ we have

$$U_{m,n} = {}_xL_m \oplus {}_y\overline{L}_n = {}_xL_m + {}_y\overline{L}_n - {}_xL_m \circ {}_y\overline{L}_n ,$$

that is $U_{m,n}$ is the Boolean sum of ${}_xL_m$ and ${}_y\overline{L}_n$. If for the univariate positive linear operators L_m the conditions (here $e_i(x) = x^i$, $i \geqslant 0$)

$$L_m(e_1;x) = x + u_m(x) ,$$

$$L_m(e_2;x) = x^2 + w_m(x) ,$$

are satisfied, and if similar conditions hold for $\overline{L}_n$ with $u_m(x)$ and $w_m(x)$ replaced by $v_n(y)$ and $t_n(y)$, respectively, then it follows for $L_{m,n}$ that

$$L_{m,n}(e;x,y) = 1 ,$$

$$L_{m,n}(\phi;x,y) = x + u_m(x) = x + u_{m,n}(x,y) ,$$

$$L_{m,n}(\psi;x,y) = y + v_n(y) = y + v_{m,n}(x,y) ,$$

$$L_{m,n}(\phi^2+\psi^2;x,y) = x^2 + y^2 + w_m(x) + t_n(y) = x^2 + y^2 + w_{m,n}(x,y).$$

Proper assumptions on u_m, v_n, w_m and t_n guarantee that the functions $u_{m,n}$, $v_{m,n}$ and $w_{m,n}$ converge to zero uniformly as m and n tend to infinity.

This discussion leads to the statement of

COROLLARY 3. *If the sequences* $(L_m)_{m\in\mathbb{N}}$, $(\overline{L}_n)_{n\in\mathbb{N}}$ *of positive linear* *operators are given as above, and if* $L_m e_i \to e_i$, $\overline{L}_n e_i \to e_i$ *uniformly for* $i = 1,2$, *then the operators* $U_{m,n}$ *constructed on the basis of* $L_{m,n} = {}_xL_m \circ {}_y\overline{L}_n$ *have the property that* $U_{m,n}f$ *converges uniformly to* f *for each B-continuous function* f *defined on* I^2 , *as* m,n *tend to infinity.*

5. Concluding Remark

Further material on the subject discussed here, namely the uniform approximation of B-continuous functions (in a more general setting), is contained in the thesis of Badea [4]. It should also be noted that the questions investigated here may be discussed from a quantitative point of view. This will be the topic of a forthcoming paper.

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