THE SOLUTIONS OF SOME FUNCTIONAL EQUATIONS

T. D. Howroyd

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In this paper we are concerned with functional equations of the type

$$(1) \qquad f(x+y) = F[f(x), f(y), f(x-y)]$$

in which x, y do not appear explicitly. J. Aczel [1] has given a method for finding real solutions of some of these equations. We prove a theorem which can sometimes be used to solve problems concerning the uniqueness of solutions of such equations.

In the following x, y are real variables, C is the complex plane, and $\overline{C}$ is the extended complex plane. Our solution space is $\overline{C}$ and F is defined on $\overline{C} \times \overline{C} \times \overline{C}$ in the usual way. Then, for example, $f(x) = \cos x$ is a solution of

$$(2) \qquad f(x+y) + f(x-y) = 2f(x)f(y) ,$$

and $f(x) = \tan x$ is a solution of

$$(3) \qquad f(x+y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}$$

provided we define $\tan x = \infty$ when $\cos x = 0$. Both of these equations appear in [1].

THEOREM. Let $D \subset C$. Let the functional equation

$$(4) \qquad g(x+y, Z) = F[g(x, Z), g(y, Z), g(x-y, Z)]$$

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have for each $Z \in D$ a solution $g(x, Z)$, which defines for each $x \neq 0$ an inverse function of Z, and is independent of Z at $x = 0$. Then if $f(x)$ is a solution of (1) such that

$$(5) \qquad \begin{cases} f(0) = g(0, Z), & Z \in D \\ f(x) = g(x, Z) & \text{for some } Z \in D, \end{cases}$$

there exists $u(x) \in D$ such that

$$(6) \qquad f(x) = g(x, u(x)) = g(x, u(x/n)), \quad n = 2, 3, \ldots, \; x \neq 0.$$

Proof. Let $h(x, Z)$ be an inverse of $g(x, Z)$ for each $x \neq 0$, and $f(x)$ be a solution of (1) satisfying (5). Then

$$u(x) = h(x, f(x))$$

satisfies the first part of (6). From (1) and (4)

$$f(2x) = F[f(x), f(x), f(0)] = F[g(x, u(x)), g(x, u(x))] = g(2x, u(x)),$$

and by induction

$$f(nx) = g(nx, u(x)), \qquad n = 2, 3, \ldots;$$

which gives the second part of (6).

Remark 1. For practical purposes the application of this theorem is only successful when the range of $g(x, Z)$, $Z \in D$, is the same for each $x \neq 0$.

Remark 2. If we can deduce from (6) the equations

$$(7) \qquad u(x) = u(x/n), \qquad n = 2, 3, \ldots,$$

then for any $c \neq 0$ and rational m/n

$$u(cm/n) = u(c/n) = u(c)$$

$$f(cm/n) = g(cm/n, u(c)).$$

Thus if $f(x)$ and $g(x, K)$ $(K \in D)$ are continuous in some neighbourhood of 0, it follows that there exists $K \in D$ with

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$$f(x) = g(x, K)$$

in this neighbourhood, and hence for all x.

Example 1. We can apply the above to (3) with

$$g(x, Z) = \tan xZ \ ;$$

the range of this function of Z is $\overline{C} - [\pm i]$ for each $x \neq 0$. Let $f(x)$ be a solution of (3) such that $f(x) \in \overline{C} - [\pm i]$; then $f(0) = 0$ and there exists $u(x) \in C$ such that

$$f(x) = \tan xu(x) = \tan xu(x/n) , \qquad n = 2, 3, \ldots .$$

Let there exist $\delta > 0$ such that $f(x)$ is continuous and

(8) $$f(x) = f(0) + O(x)$$

for $|x| < \delta$. Obviously we may assume $xu(x)$ is in the fundamental region of the Z-plane, $-\pi/2 \leqq \text{Re } Z < \pi/2$, so that $xu(x)$ is also continuous for $|x| < \delta$ and $xu(x) \to 0$ as $x \to 0$. Hence for $|x|$ sufficiently small

$$|xu(x)|/2 < |\tan xu(x)| < c|x|$$

$$|xu(x/n)| < 2c|x| , \qquad n = 2, 3, \ldots ,$$

so that $xu(x)$ and $xu(x/n)$ are both in the above fundamental region of the complex plane, and so are equal. It follows from Remark 2 that there exists $K \in C$ such that

$$f(x) = \tan Kx .$$

It is immediately evident that the only solutions of (3) not entirely in $\overline{C} - [\pm i]$ are $f(x) = i$ and $f(x) = -i$.

Thus if $f(x)$ is a solution of (3) which, in some neighbourhood of 0, is continuous and satisfies the Lipschitz condition (8), then

$$f(x) = \tan Kx, \ i, \text{ or } -i \quad (K \in C) .$$

Example 2. Similarly we find that $f(x) = 0$, $\cos Kx$, $(K \in C)$ are the only solutions of (2) which, in some neighbourhood of 0, are continuous and satisfy the Lipschitz condition

$$f(x) = f(0) + O(x^2) .$$

REFERENCE

1. J. Aczel, Vorlesungen Uber Funktionalgleichungen und Ihre Anwendungen, Chap. 2 (Basel 1961).

University of New Brunswick

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