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Finiteness: Another aspect of topoi

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In this thesis I examine one way of generalizing the notion of finiteness from the category of sets to an arbitrary topos. I have chosen to use 'finite' to mean doubly well ordered since it is meaningful even in the absence of a natural numbers object, it requires no external notion of finiteness, and it gives the right kind of result. For example, the main result of the thesis is that

the full subcategory $\mathrm{Fin}\ E$ *of finite objects in a topos* E *is a topos with logical embedding* $\mathrm{Fin}\ E \rightarrowtail E$ *precisely when* E *is boolean. Moreover, if* E *is non-degenerate and boolean, then* $\mathrm{Fin}\ E$ *cannot satisfy the Axiom of Infinity.*

A theory of well order and recursion is developed in order to obtain this result. The topos analogue of the statement 'every non-empty upward-closed subset of an ordered set A is of the form $\{b \mid b \geq a\}$ for some a in A ' yields a notion of well order for topoi which is shown to inherit several properties from the category of sets. In particular, it has links with the Axiom of Choice. These properties can show that the booleanness condition on the internal logic of a topos is itself a local version of the Axiom of Choice.

There is no immediate connection between well order and transfinite recursion. Instead, the recursive properties of well ordered sets are inherited by Street ordered objects. Their definition is obtained from the topos analogue of 'every downward-closed proper subset of an ordered set A is of the form $\{b \mid b < a\}$ for some a in A '; they coincide with well ordered objects only in the boolean case. It is therefore in

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boolean topoi that finite objects have the finite recursion property required in the proof of the main theorem.