ON A QUASI-LINEAR EQUATION

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1. Introduction. The purpose of this note is to establish some limit theorems for the non-linear recurrence relations

1.1 $$x_i(n+1) = \operatorname{Max}_q \sum_{j=1}^{N} a_{ij}(q)\, x_j(n), \qquad i = 1, 2, \ldots, N;\ n \geqslant 0,$$

under certain assumptions concerning the initial values $c_i = x_i(0)$, and the coefficient matrices $A(q) = (a_{ij}(q))$.

Equations of this type occur in various parts of the theory of dynamic programming, as we shall indicate below, and are, in addition, of interest in furnishing a link between the theory of linear and non-linear operations, as we have discussed elsewhere **(1)**.

Generally speaking, these equations arise in the consideration of processes of Markoff type, see **(2)**, in which decisions are made at various stages of the process.

Results corresponding to those obtained below hold for the more general equations of the form

1.2 $$x_i(n+1) = \begin{cases} \operatorname{Max}_q \sum_{j=1}^{N} a_{ij}(q)\, x_j(n), & i = 1, 2, \ldots, K < N, \\ \sum_{j=1}^{N} a_{ij}(q^*)\, x_j(n), & i = K+1, \ldots, N, \end{cases}$$

where q^* in the lower equations is determined by the upper equations.

2. The homogeneous equation. Let us consider the equation

2.1 $$\lambda y_i = \operatorname{Max}_q \sum_{j=1}^{N} a_{ij}(q)\, y_j \qquad (i = 1, 2, \ldots, N),$$

where we impose the following conditions:

2.2 (a) $q = (q_1, q_2, \ldots, q_N)$ runs over some set of values, S, with the property that the maximum is attained in (1),

(b) $\infty > m \geqslant a_{ij}(q) > 0$ $(i, j = 1, 2, \ldots, N)$ for $q \in S$,

(c) for any q, let $\phi(q)$ denote the characteristic root of $A(q) = (a_{ij}(q))$ of largest absolute value, the Perron root, known to be positive. We assume that there exists at least one value of q for which $\phi(q)$ assumes its maximum for $q \in S$.

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We shall now prove

THEOREM 1. *Under these conditions, there exists a unique positive* λ *with the property that* 2.1 *has a positive solution,* $y_i > 0$ $(i = 1, 2, \ldots, N)$. *This solution is unique up to a multiplicative constant, and*

2.3 $$\lambda = \operatorname*{Max}_{q \in S} \phi(q).$$

Proof. We begin by showing the existence of a positive λ and a positive set of solutions $\{y_i\}$. Consider the region defined by

$$y_i \geqslant 0, \sum_{i=1}^{N} y_i = 1.$$

The normalized transformation

2.4 $$y'_i = \left[\operatorname*{Max}_q \sum_{j=1}^{N} a_{ij}(q)\, y_j\right] \Big/ \left[\sum_{i=1}^{N} \operatorname*{Max}_q \sum_{j=1}^{N} a_{ij}(q)\, y_j\right],$$

is a continuous mapping of this region into itself. Hence there exists a fixed point, $\{y_i\}$. This fixed point is a solution of 2.1, with λ the denominator in 2.4. Each component y_i is positive because of the positivity of $a_{ij}(q)$.

To show that this solution is unique up to a multiplicative constant, let $[\mu, z]$ be another solution of 2.1 with $\mu > 0$ and z a positive vector. Let $\{q\}$ be the set of values for which the maximum is attained in 2.1 and $\{\bar{q}\}$ the similar set associated with z. Observe that we may have different sets for each i. We have then

2.5 $$\lambda y_i = \sum_j a_{ij}(q)\, y_j \geqslant \sum a_{ij}(\bar{q})\, y_j, \qquad i = 1, 2, \ldots, N,$$
$$\mu z_i = \sum_j a_{ij}(\bar{q})\, z_j.$$

Let us now assume, without loss of generality that $\lambda < \mu$. Let ϵ be a positive constant chosen so that one, at least, of the components $y_i - \epsilon z_i$ is zero, one at least is positive, and the others are non-negative. This can always be accomplished if y and z are not proportional. If i is an index for which $y_i - \epsilon z_i$ is zero, we have

2.6 $$0 = \mu(y_i - \epsilon z_i) > \lambda y_i - \epsilon\mu z_i \geqslant \sum_{j=1}^{N} a_{ij}(\bar{q})(y_j - \epsilon z_j) > 0,$$

since $a_{ij}(\bar{q}) > 0$, a contradiction. Hence $\lambda = \mu$, and y and z are proportional.

To show that $\lambda = \operatorname{Max} \phi(q)$, we proceed as follows. It is clear that λ, as the characteristic root of some $A(q)$, satisfies the inequality $\lambda \leqslant \mu$, where $\mu = \operatorname{Max} \phi(q)$. Assume that actually $\lambda < \mu$. Let $z = (z_1, z_2, \ldots, z_n)$ be a positive characteristic vector associated with μ and $\bar{q}$ a set of q-values which yield $\mu = \phi(\bar{q})$. Then we have

2.7 $$\mu z_i = \sum_{j=1}^{N} a_{ij}(\bar{q})\, z_j \leqslant \operatorname*{Max}_q \sum_{j=1}^{N} a_{ij}(q)\, z_j.$$

Since y_i is positive, we can find a positive constant m such that $z_i \leqslant my_i$ for $i = 1, 2, \ldots, N$. Hence 2.1 yields

2.8 $$\mu z_i \leqslant m \operatorname*{Max}_q \sum_{j=1}^{N} a_{ij}(q)\, y_j = m\lambda y_i.$$

Thus $z_i \leqslant my_i \lambda/\mu$. Iterating this, we obtain $z_i \leqslant my_i\, (\lambda/\mu)^k$, for arbitrary k. Since $\lambda/\mu < 1$, by assumption, this yields $z_i = 0$, a contradiction. Hence $\lambda = \mu$.

3. The recurrence relation. Let us now return to the recurrence relation of 1.1 and prove

THEOREM 2. *If, in addition to the conditions of* 2.2, *we assume that there is a unique q for which the maximum value of $\phi(q)$ is attained and that $c_i \geqslant 0$, then*

3.1 $$x_i(n) \sim ay_i\, \lambda^n,$$

as $n \to \infty$, where a is a constant dependent upon the initial values c_i.

Proof. Let us take $c_i > 0$ without loss of generality. There are then two positive constants k and K such that $ky_i \leqslant c_i \leqslant Ky_i$ $(i = 1, 2, \ldots, N)$. Let us show inductively that

3.2 $$ky_i\, \lambda^n \leqslant x_i(n) \leqslant Ky_i\, \lambda^n.$$

Assume that we have the result for n, then

3.3 $$\begin{aligned} x_i(n+1) &\leqslant K\lambda^n \operatorname*{Max}_q \sum_{j=1}^{N} a_{ij}(q)\, y_j = K\lambda^{n+1} y_i \\ &\geqslant k\lambda^n \operatorname*{Max}_q \sum_{j=1}^{N} a_{ij}(q)\, y_j = k\lambda^{n+1} y_i. \end{aligned}$$

To establish the asymptotic behavior we show that for n sufficiently large the set of q's which furnish the maximum in 1.1 is precisely the set which yields $\lambda = \operatorname{Max} \phi(q)$.

Assume the contrary. This means that infiitely often we employ a set $\{\bar{q}\}$ which is not identical with the q which furnishes the maximum in $\phi(q)$.

We then have, for $i = 1, 2, \ldots, N$,

3.4 $$x_i(n+1) = \sum_{j=1}^{N} a_{ij}(\bar{q})\, x_j(n) \leqslant \left(\sum_{j=1}^{N} a_{ij}(\bar{q})\, y_j \right) K\lambda^n.$$

For some index i we must have

3.5 $$\sum_{j=1}^{N} a_{ij}(\bar{q})\, y_j < \lambda y_i,$$

with strict inequality. For if

$$\sum_{j=1}^{N} a_{ij}(\bar{q})\, y_j \geqslant \lambda y_i$$

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for all i, the characteristic root of $A(\bar{q}) = (a_{ij}(\bar{q}))$ of largest absolute value, $\phi(\bar{q})$, would at least equal $\lambda = \text{Max}\, \phi(q)$, which would contradict the assumption concerning the uniqueness of the maximum of $\phi(q)$.

Hence, for some component, say the first, we have

3.6 $$x_1(n+1) \leqslant \theta K \lambda^{n+1} y_1, \qquad 0 < \theta < 1.$$

Since $a_{1j}(q^*) > 0$ for i, j, where q^* is the value of q for which $\lambda = \phi(q^*)$, we see that, for $i = 1, 2, \ldots, N$,

3.7 $$x_i(n+2) \leqslant K\lambda^{n+1}\left[\sum_{j=2}^{N} a_{ij}(q^*)\, y_i + \theta a_{1j}(q^*)\, y_1\right] \leqslant \theta_1 K \lambda^{n+2} y_i,$$

where $\theta < 1$.

If therefore a set of q's distinct from q^* are used R times, we obtain

3.8 $$x_i(n) \leqslant \theta_1{}^R K \lambda^n y_i,$$

for n sufficiently large. Since $0 < \theta_1 < 1$, if R is too large we eventually contradict the lower bound for $x_i(n)$.

Hence for $n \geqslant n_0 = n_0(c_i)$, we have

3.9 $$x(n+1) = A(q^*)\, x(n),$$

whence the asymptotic statement of 3.1 follows.

4. A dynamic programming problem. Suppose that we are engaged in a multi-stage decision process of the following type. At each stage we have our choice of various operations, which we number $i = 1, 2, \ldots, K$. The ith operation has a probability distribution attached with the following properties:

4.11 There is a probability p_{ik} that we receive k units and the process continues, $k = 1, 2, \ldots, R$;

4.12 There is a probability p_{i0} that we receive nothing and the process terminates.

How do we proceed so as to maximize the probability that we receive at least n units before the process terminates?

Let us define the sequence

4.2 $u(n)$ = the probability of attaining at least n units before the termination of the process using an optimal procedure.

Then using the intuitive "principle of optimality" (1), we see that $u(n)$ satisfies the recurrence relation

4.3 $$u(n) = \begin{cases} \text{Max}_i \left[\sum_{k=1}^{R} p_{ik} u(n-k)\right], & n > 0, \\ 1, & n \leqslant 0. \end{cases}$$

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Using methods similar to those above, we see that for large n,

4.4 $$u(n) \sim c\rho^n,$$

where ρ is the root of largest absolute value, necessarily positive, of

4.5 $$1 = \sum_{k=1}^{R} p_{ik}\rho^{-k},$$

for the value of i which maximizes ρ.

5. An analogue of a result of Markoff.

Markoff showed that if

5.1 $$x_i(n+1) = \sum_{j=1}^{N} a_{ij}x_j(n) \qquad (n = 0, 1, \ldots)$$

and $x_i(0) > 0$, with the conditions

5.2 $$a_{ij} > 0, \ \sum_j a_{ij} = 1, \qquad (i = 1, 2, \ldots, N),$$

then

5.3 $$\lim_{n\to\infty} x_i(n) = c, \qquad (i = 1, 2, \ldots, N),$$

where c depends on the initial values.

The same proof shows that the same result holds for the sequence defined by

5.4 $$x_i(n+1) = \operatorname*{Max}_q \sum_{j=1}^{N} a_{ij}(q)\, x_j(n),$$

provided that the conditions in 5.2 hold uniformly in q. The constant will, of course, in general, be different from that above.

References

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