CELL COMPLEXES, VALUATIONS, AND THE EULER RELATION

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1. Recently a number of functions have been shown to satisfy relations on polytopes similar to the classic Euler relation. Much of this work has been done by Shephard, and an excellent summary of results of this type may be found in [**11**]. For such functions, only continuity (with respect to the Hausdorff metric) is required to assure that it is a valuation, and the relationship between these two concepts was explored in [**8**]. It is our aim in this paper to extend the results obtained there to illustrate the relationship between valuations and the Euler relation on cell complexes.

To fix our notions, we will suppose that everything takes place in a given finite-dimensional Euclidean space X.

A *polytope* is the convex hull of a finite set of points and will be referred to as a *d-polytope* if it has dimension d. Polytopes have *faces* of all dimensions from 0 to $d-1$ and each of these is in turn a polytope. A k-dimensional face will be termed simply a k-face. (For details in these matters, see [**2**].) The class of all polytopes will be denoted by $\mathscr{P}$.

A (*polyhedral*) *cell complex*, C, is a finite union of polytopes (*cells*) together with all of their faces, such that two cells meet only on a face of each. A *maximal cell* of C is a cell which is properly contained in no other and the *dimension* of C, $d(C)$, is the maximal dimension of a cell of C. We denote the set of all points belonging to some cell of C by $|C|$. A cell $F \in C$ is an *interior cell* if $\operatorname{rel\,int}(F) \subset \operatorname{int}(|C|)$. Otherwise, we say that F is a *boundary cell* of C.

On any class $\mathscr{A}$ of subsets of X, we say that a function φ is a *valuation* on $\mathscr{A}$ if

$$\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B) \tag{1.1}$$

whenever A, B, $A \cup B$, and $A \cap B$ all belong to $\mathscr{A}$. For convenience, we set $\varphi(\emptyset) = 0$. Valuations, especially motion-invariant valuations, have also been extensively considered by Hadwiger. See in particular [**3**, pp. 236–243; **4**].

We say that a function φ defined on $\mathscr{P}$ *satisfies an Euler relation* $E(\epsilon)$ on a d-polytope P if

$$\sum_{i=0}^{d} (-1)^i \sum \varphi(P^i) = \epsilon\varphi(P), \tag{1.2}$$

where the inner summation on the left is taken over all i-faces of P. We will often denote the left-hand side of (1.2) by $\varphi^*(P)$.

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In considering how to extend such relations to (polyhedral) cell complexes, we were led to a more abstract point of view and obtained somewhat more general results than needed. However, from a geometric viewpoint, our most interesting theorems are probably the following.

THEOREM 1.1. *Let φ be any valuation on $\mathscr{P}$. Then we may extend φ in a unique way to a valuation φ^- on $\mathscr{U}$, the class of finite unions of polytopes.*

THEOREM 1.2. *Suppose that φ is a valuation satisfying an Euler relation $E(\epsilon)$ on $\mathscr{P}$ and that φ^- is its extension to $\mathscr{U}$. If C is any cell complex with maximal cells of dimension d, then*

$$\sum_{i=0}^{d} (-1)^i \sum \varphi(C^i) = \epsilon\varphi^-(|C|), \tag{1.3}$$

$$\sum_{i=0}^{d-1} (-1)^i \sum^{\text{bd}} \varphi(C^i) = (\epsilon - (-1)^d)\varphi^-(|C|), \tag{1.4}$$

$$\sum_{i=0}^{d} (-1)^i \sum^{\text{int}} \varphi(C^i) = (-1)^d\varphi^-(|C|). \tag{1.5}$$

The inner summations on the left-hand side of (1.3)–(1.5) *are taken over all i-cells, boundary i-cells, and interior i-cells, respectively.*

Theorem 1.1 is proved in § 2 by means of a lemma which may be useful in other problems of this type. In § 3 we prove Theorem 1.2, while § 4 is devoted to applications of the theorem to some specific cases.

2. Extending valuations. Before stating our principal lemma, which generalizes a result of Klee [**5**, p. 123, (1.7)], it is useful to introduce some additional notation. We let $N = \{1, \ldots, n\}$, and let $\mathscr{S}(N)$ be the collection of all non-empty subsets of N. If N and M are sets of integers, then $N \times M$ is the usual set of ordered pairs which we will identify in an obvious way with $\{1, \ldots, mn\}$. In addition, we define projections, $\pi_1(n, m) = n$ and $\pi_2(n, m) = m$.

For each $v \in \mathscr{S}(N)$, let $|v|$ be the cardinality of v. If $X_1, \ldots, X_n$ is an indexed family of sets, and $v = (i_1, \ldots, i_j) \in \mathscr{S}(N)$, we let $X(v)$ denote $X_{i_1} \cap \ldots \cap X_{i_j}$. We will sometimes write $\varphi(X_1, \ldots, X_n)$ in place of

$$\sum_{v \in \mathscr{S}(N)} (-1)^{|v|-1} \varphi(X(v)).$$

Finally, $\mathscr{F}$ is an *intersectional family* of sets if $F, G \in \mathscr{F}$ implies $F \cap G \in \mathscr{F}$.

LEMMA 2.1. *Let $\mathscr{F}$ be an intersectional family of sets, $V\mathscr{F}$ the family of all finite unions of members of $\mathscr{F}$, and φ a function on $\mathscr{F}$. Then the following statements are equivalent:*

(i) *If $F = F_1 \cup \ldots \cup F_n$, where $F_1, \ldots, F_n$ and $F \in \mathscr{F}$, then*

$$\varphi(F_1, \ldots, F_n) = \varphi(F);$$

(ii) *We may extend* φ *to a valuation* φ^- *on* $V\mathscr{F}$, *where*

$$\varphi^-(F_1 \cup \ldots \cup F_n) = \varphi(F_1, \ldots, F_n);$$

(iii) *If* $V_1, \ldots, V_n \in V\mathscr{F}$, *then* $\varphi^-(V_1, \ldots, V_n) = \varphi^-(V_1 \cup \ldots \cup V_n)$.

Proof (i) $\Rightarrow$ (ii). Let φ^- be the function defined in (ii) above. We will first show that it is well-defined on $V\mathscr{F}$ and then that it is a valuation there.

Suppose that $U = F_1 \cup \ldots \cup F_n = G_1 \cup \ldots \cup G_m$, where $F_i, G_j \in \mathscr{F}$. We have:

$$\sum_{u \in \mathscr{S}(N)} (-1)^{|u|-1}\varphi(F(u)) = \sum_{u \in \mathscr{S}(N)} (-1)^{|u|-1}\left\{\sum_{v \in \mathscr{S}(M)} (-1)^{|v|-1}\varphi(F(u) \cap G(v))\right\}$$
$$= \sum_{v \in \mathscr{S}(M)} (-1)^{|v|-1}\left\{\sum_{u \in \mathscr{S}(N)} (-1)^{|u|-1}\varphi(F(u) \cap G(v))\right\}$$
$$= \sum_{v \in \mathscr{S}(M)} (-1)^{|v|-1}\varphi(G(v)),$$

where the first and last equalities follow from (i). Thus φ^- is well-defined.

To see that φ^- is a valuation on $V\mathscr{F}$, let $S = F_1 \cup \ldots \cup F_n$ and $T = G_1 \cup \ldots \cup G_m$. Then we have:

$$(2.1)\quad \begin{cases} \varphi^-(S) = \displaystyle\sum_{u \in \mathscr{S}(N)} (-1)^{|u|-1}\varphi(F(u)); \\ \varphi^-(T) = \displaystyle\sum_{v \in \mathscr{S}(M)} (-1)^{|v|-1}\varphi(G(v_,)); \\ \varphi^-(S \cup T) = \displaystyle\sum_{u \in \mathscr{S}(N)} (-_1)^{|u|-1}\varphi(F(u)) + \sum_{v \in \mathscr{S}(M)} (-1)^{|v|-1}\varphi(G(v)) \\ \qquad\qquad + \displaystyle\sum_{u \in \mathscr{S}(N)} \sum_{v \in \mathscr{S}(M)} (-1)^{|u|+|v|-1}\varphi(F(u) \cap G(v)); \\ \varphi^-(S \cap T) = \displaystyle\sum_{w \in \mathscr{S}(M \times N)} (-1)^{|w|-1}\varphi(F(\pi_1(w)) \cap G(\pi_2(w))). \end{cases}$$

This last equality arises as follows. Note that $S \cap T = \bigcup_i \bigcup_j (F_i \cap G_j)$, and hence

$$\varphi^-(S \cap T) = \sum_{w \in \mathscr{S}(M \times N)} (-1)^{|w|-1}\varphi(H(w)),$$

where $H_{ij} = F_i \cap G_j$. But if $w = (i_1, j_1) \cup \ldots \cup (i_k, j_k) \in \mathscr{S}(M \times N)$, then

$$H(w) = F_{i_1} \cap \ldots \cap F_{i_k} \cap G_{j_1} \cap \ldots \cap G_{j_k} = F(\pi_1(w)) \cap G(\pi_2(w)).$$

From (2.1) we see then that φ^- satisfies the valuation property if and only if

$$(2.2)\quad \sum_{w \in \mathscr{S}(M \times N)} (-1)^{|w|-1}\varphi(F(\pi_1(w)) \cap G(\pi_2(w)))$$
$$= \sum_{u \in \mathscr{S}(N)} \sum_{v \in \mathscr{S}(M)} (-1)^{|u|+|v|}\varphi(F(u) \cap G(v)).$$

This in turn reduces to showing that for all $u \in \mathscr{S}(N)$, $v \in \mathscr{S}(M)$,

$$(2.3)\quad \sum_{w \in \mathscr{S}(u, v, M \times N)} (-1)^{|w|-1} = (-1)^{|u|+|v|},$$

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where $\mathscr{S}(u, v, M \times N)$ denotes all $w \in \mathscr{S}(M \times N)$ such that $\pi_1(w) = u$, $\pi_2(w) = v$. But this identity has been verified by Klee (see [**5**, p. 123, proof of (1.7)]).

(ii) $\Rightarrow$ (i). Suppose that $F_1, \ldots, F_n$ and F are as in (i). Since φ^- is a valuation on $V\mathscr{F}$, the result follows for $n = 2$. Then it is easily verified for larger values of n by induction.

(i) and (ii) $\Rightarrow$ (iii). Let $\mathscr{G} = V\mathscr{F}$ and let $\psi = \varphi^-$ on $\mathscr{G}$. Assume that $G_1 \cup \ldots \cup G_n = G$, where $G_1, \ldots, G_n, G \in \mathscr{G}$. Then by (i) and (ii), $\psi(G_1, \ldots, G_n) = \psi(G)$ if and only if ψ^- is a valuation on $V\mathscr{G}$. But $V\mathscr{G} = V\mathscr{F}$ and on $V\mathscr{F}$ we know that $\psi^- = \varphi^-$ which is assumed to be a valuation. Hence ψ^- is a valuation on $V\mathscr{G}$, and the desired result follows.

That (iii) $\Rightarrow$ (i) is immediate.

We observe that φ^- is the only valuation on $V\mathscr{F}$ which extends φ.

LEMMA 2.2. *Let $\mathscr{F}$ be an intersectional family and let ξ, ψ be two valuations on $V\mathscr{F}$ which are equal on $\mathscr{F}$. Then $\xi \equiv \psi$.*

Proof. Let $F_1 \cup \ldots \cup F_n = U$ be a typical member of $V\mathscr{F}$.
Then by Lemma 2.1 (iii),

$$\xi(U) = \sum_{u \in \mathscr{S}(N)} (-1)^{|u|-1}\xi(F(u)),$$

and similarly for $\psi(U)$. But since $\mathscr{F}$ is an intersectional family, $F(u) \in \mathscr{F}$ and thus $\xi(F(u)) = \psi(F(u))$ for all $u \in \mathscr{S}(N)$. Since the sums are equal term by term, $\xi(U) = \psi(U)$. Our principal lemma will now be used to prove Theorem 1.1. All that is necessary to do is to verify that Lemma 2.1 (i) holds for valuations on $\mathscr{P}$.

LEMMA 2.3. *Suppose that φ is a valuation on $\mathscr{P}$, the class of polytopes, and that $P = P_1 \cup \ldots \cup P_n$, where $P_1, \ldots, P_n$ and $P \in \mathscr{P}$. Then*

$$\varphi(P) = \varphi(P_1, \ldots, P_n).$$

The proof of this assertion is exactly the same as the corresponding statement for Steiner points which is proved in [**7**, p. 78].

Proof of Theorem 1.1. By Lemma 2.2, any valuation φ on $\mathscr{P}$ satisfies Lemma 2.1 (i). Hence φ^- is a valuation on $\mathscr{U} = V\mathscr{P}$, and by Lemma 2.2, φ^- is the only valuation on U which agrees with φ on $\mathscr{P}$.

3. Proof of Theorem 1.2. We begin by recalling the definition

$$\epsilon\varphi(P) = \varphi^*(P) = \sum_{i=0}^{d} (-1)^i \sum \varphi(P^i),$$

where P is a d-polytope and the inner summation on the right is taken over all i-faces of P. It may be shown [**8**, (4.2)] that φ^* is a valuation if and only if φ is a valuation.

Now suppose that C is a cell complex with n maximal cells, $Q_1, \ldots, Q_n$. Then

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by Theorem 1.1 and Lemma 2.1, we extend φ^* to $\mathscr{U}$ to see that

$$(3.1)\qquad \varphi^*(|C|) = \sum_{v\in\mathscr{S}(N)} (-1)^{|v|-1}\varphi^*(Q(v))$$
$$= \sum_{v\in\mathscr{S}(N)} (-1)^{|v|-1} \sum_{i=0}^{\dim(Q(v))} (-1)^i \sum_{C^i\subseteq C(v)} \varphi(C^i).$$

Then for a fixed i-cell, C^i, $\varphi(C^i)$ has as a coefficient

$$(3.2)\qquad (-1)^i \textstyle\sum' \varphi(C^i),$$

where the prime denotes that the summation is taken over all $v \in \mathscr{S}(N)$ such that $C^i \subseteq Q(v)$. Without loss of generality, suppose that C^i is contained only in $Q_1, \ldots, Q_r$. Then (3.2) equals

$$(3.3)\qquad (-1)^i\left\{\sum_{v\in\mathscr{S}(R)} (-1)^{|v|-1}\right\} = (-1)^i\left\{\binom{r}{1} - \binom{r}{2} + \ldots + (-1)^{r-1}\binom{r}{r}\right\}$$
$$= (-1)^i.$$

Thus, rearranging (3.1) shows that

$$(3.4)\qquad \varphi^*(|C|) = \sum_{i=0}^{d} (-1)^i \sum \varphi(C^i).$$

Since $\varphi^* = \epsilon\varphi$ on $\mathscr{P}$, (1.3) follows immediately.

In order to prove (1.4), we define a function ψ on $\mathscr{U}$ as follows. Let $U \in \mathscr{U}$, and let C be a cell complex such that $U = |C|$. Then set

$$(3.5)\qquad \psi(U) = \sum_{i=0}^{d} (-1)^i \sum^{\mathrm{bd}} \varphi(C^i).$$

ψ is well-defined since the boundary cells of C form a complex to which we may apply (1.3). Moreover, ψ is a valuation on $\mathscr{U}$. For let $U, V \in \mathscr{U}$, and let E be a complex such that $|E| = U \cup V$, and which contains subcomplexes E_1, E_2 with $|E_1| = U$, $|E_2| = V$. Then for each i,

$$(3.6)\qquad \sum^{\mathrm{bd}(E)} \varphi(E^i) = \sum^{\mathrm{bd}(E_1)} \varphi(E^i) + \sum^{\mathrm{bd}(E_2)} \varphi(E^i) - \sum^{\mathrm{bd}(E_1\cap E_2)} \varphi(E^i).$$

Summing (3.6) with alternating signs yields the valuation property. However, on $\mathscr{P}$, $\psi = \varphi^* - (-1)^d\varphi = (\epsilon - (-1)^d)\varphi$. Extending the valuation on the right to $\mathscr{U}$ yields (1.4).

Finally, (1.5) follows by subtracting (1.4) from (1.3). This completes the proof.

We observe that we have actually proved the following result.

THEOREM 3.1. *Suppose that φ is a valuation on $\mathscr{P}$ and that $\psi = (\varphi^*)^-$ is the*

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extension of φ^* *to* $\mathscr{U}$. *Then for any cell complex C such that* $d(c) = d$, *we have:*

$$\sum_{i=0}^{d} (-1)^i \sum \varphi(C^i) = \psi(|C|), \tag{3.8}$$

$$\sum_{i=0}^{d} (-1)^i \sum^{\text{bd}} \varphi(C^i) = \psi(|C|) - (-1)^d \varphi^-(|C|), \tag{3.9}$$

$$\sum_{i=0}^{d} (-1)^i \sum^{\text{int}} \varphi(C^i) = (-1)^d \varphi^-(|C|). \tag{3.10}$$

The inner summations are taken over all i-cells, boundary i-cells, and interior i-cells, respectively.

4. Applications. A number of valuations are known to satisfy an Euler relation on $\mathscr{P}$, and thus fulfill the assumptions of Theorem 1.2. The best known of these is, of course, the identity function which gives rise to the usual Euler relation. There are also references to the mean width [**10**], the Steiner point [**8**; **9**], and mixed volumes [**11**] in the literature. Thus we may state the following corollary to Theorem 1.2.

COROLLARY 4.1. *Let* φ *be any of the valuations* (a), (b), (c) *on* $\mathscr{P}$ *and* φ^- *the extension of* φ *to* $\mathscr{U}$. *If C is any polyhedral cell complex, then relations* (1.3)–(1.5) *hold for* φ.

(a) $\varphi(P) \equiv 1$.
(b) $\varphi(P)$ = *mean width of P*.
(c) $\varphi(P)$ = *Steiner point of P*.

The corollary above seems to be new in the case of mean widths. For the identity function we obtain the well-known Euler-Poincaré formula [**1**, p. 214], while the assertion for the Steiner point was first made by Shephard (private communication).

The disadvantage of Corollary 4.1 is having to use the extension of φ to φ^-. For, while some valuations extend nicely to $\mathscr{U}$ (see [**7**, p. 81, **6**] for equivalent definitions extending the notion of the Steiner point to $\mathscr{U}$), others lose all geometric significance. A case-in-point is mean widths. Examples may be given of non-convex sets having arbitrarily small diameters and arbitrarily large "mean widths". (For instance, consider a "fan" consisting of many short line segments joined at a common point.) Such problems may be avoided if $|C|$, the space underlying a complex C, is convex; that is, if C is a cellular decomposition of a polytope. Then we have the following result.

THEOREM 4.2. *Let C be a cell complex such that* $|C|$ *is convex of dimension d and let* φ *denote any one of the following valuations on* $\mathscr{P}$:

(a) $\varphi(P) \equiv 1$;
(b) $\varphi(P)$ = *mean width of P*;
(c) $\varphi(P)$ = *Steiner point of P*.

Then

$$\sum_{i=0}^{d} (-1)^i \sum \varphi(C^i) = \epsilon\varphi(|C|), \tag{4.1}$$

$$\sum_{i=0}^{d-1} (-1)^i \sum^{\text{bd}} \varphi(C^i) = (\epsilon - (-1)^d)\varphi(|C|), \tag{4.2}$$

$$\sum_{i=0}^{d} (-1)^i \sum^{\text{int}} \varphi(C^i) = (-1)^d\varphi(|C|), \tag{4.3}$$

where the inner summations on the left are taken over all i-cells, boundary i-cells, and interior i-cells, respectively.

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