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On a localization-in-frequency approach for a class of elliptic problems with singular boundary data

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We consider a class of nonhomogeneous elliptic equations in the half-space with critical singular boundary potentials and nonlinear fractional derivative terms. The forcing terms are considered on the boundary and can be taken as singular measure. Employing a functional setting and approach based on localization-in-frequency and Littlewood–Paley decomposition, we obtain results on solvability, regularity, and symmetry of solutions.

Keywords: nonlinear elliptic boundary value problems; localization-in-frequency techniques; Littlewood–Paley decomposition; solvability; symmetry; singular potentials; measure data

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1. Introduction

Nonlinear boundary value problems (NBVPs) for elliptic partial differential equations (PDEs) are widely studied due to the great mathematical interest in themselves and their applications in various areas of science. For example, they arise in the modelling of nonlinear diffusion phenomena and in the theory of nuclear and chemical reactors (see e.g. [**2**, **15**, **30**]). This class of problems has been addressed through different techniques and approaches, such as variational, penalty, and maximum principle-based methods that have prominent historical roles (see e.g. [**27**, **42**, **45**]).

In the present work, we are concerned with a class of NBVPs for elliptic equations with singular boundary potentials and nonlinear derivative terms in the half-space $\mathbb{R}^n_+$. More precisely, we consider the following nonhomogeneous elliptic problem:

$$\begin{cases} -\Delta u = K_1(\partial^\beta u)^{\rho_1}, & \text{in } \mathbb{R}^n_+ \\ \partial_\eta u = V(x')u + K_2 u^{\rho_2} + f(x'), & \text{in } \mathbb{R}^{n-1} \end{cases}, \tag{1.1}$$

where $n \geqslant 3$, $\rho_1, \rho_2 \geqslant 2$, $u = u(x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n > 0$, $\partial_\eta = \partial/\partial\eta$, η is the normal unit outward vector on $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, K_1, K_2 are constants, $0 \leqslant \beta < 2/\rho_1$, and the fractional derivative ∂^β is defined via the Fourier transform on the

$(n-1)$-first variables as

$$(\partial^\beta u)^\wedge(\xi', x_n) = (2\pi|\xi'|)^\beta(\widehat{u})(\xi', x_n). \tag{1.2}$$

The case $\beta = 0$ corresponds to the power-type nonlinearity u^{ρ_1}. Moreover, we can treat doubly supercritical variational cases such as $\rho_1 > 2^* - 1$ and $\rho_2 > 2_* - 1$. However, due to technical issues in our approach, the powers ρ_1, ρ_2 have to be positive integers as well as they and the order β of the derivative present a certain relation between them. The boundary potentials V and forcing terms f can be singular such as critical multipolar potentials and Radon measures, respectively.

Our intent is to analyse problem (1.1) via a different approach based on localization-in-frequency arguments and the Littlewood–Paley decomposition. To handle the influences of different frequency bands on each of the terms of (1.1), especially on those coming from singular potentials and forces, we consider a frequency-based setting, namely the Fourier–Besov space $\mathcal{FB}^s_{p,\infty}$ (FB-space, for short), whose elements h are such that $\widehat{h} \in L^1_{loc}(\mathbb{R}^n)$ and present the control in frequency:

$$\|\widehat{\Delta_j h}\|_{L^p(\mathbb{R}^n)} \leqslant C2^{-sj}, \quad \text{for all } j \in \mathbb{Z},$$

where the Littlewood–Paley operator Δ_j works as a filter in the frequency domain with corresponding passband $\mathcal{A}_j = \{\xi \in \mathbb{R}^n; 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\}$. The parameters $s \in \mathbb{R}$ and $p \in [1, \infty]$ stand for the regularity and integrability indexes of the space, respectively. For more details, see (2.8) and (2.9) in § 2.2. This kind of framework, as well as some of its extensions, has been successfully employed in the analysis of the well-posedness of parabolic problems, see e.g. [**1**, **40**] and references therein.

Varying the levels of regularity and integrability, we are able to cover singular classes of boundary potentials V and forcing terms f as well as obtain properties for solutions such as axial symmetry, positivity, and homogeneity. Of particular interest, we have the critical boundary potential $V(x') = C\,|x'|^{-1}$ as well as its multipolar versions (even infinitely many poles):

$$V(x') = \sum_{j=1}^{l} \frac{\lambda_j}{|x' - x^j|} \quad \text{and} \quad V(x') = \sum_{j=1}^{l} \frac{(x' - x^j) \cdot d^j}{|x' - x^j|^2}, \tag{1.3}$$

where $x^j \in \partial\mathbb{R}^n_+$ are the poles, $d^j \in \partial\mathbb{R}^n_+$ are constant vectors, λ_j are real constants, $j = 1, \ldots, l$, and $l \in \mathbb{N} \cup \{\infty\}$. Indeed, for $0 < \sigma < n-1$ and $1 \leqslant p \leqslant \infty$, a simple computation yields that the potentials $|x'|^{-\sigma}$ and $x'\,|x'|^{-(\sigma+1)}$ belong to $\mathcal{FB}^{n-1-((n-1)/p)-\sigma}_{p,\infty}(\mathbb{R}^{n-1})$ as well as their translations and (for $\sigma = 1$) those in (1.3). These critical potentials can be regarded as boundary versions of the so-called Hardy-type potentials in the whole space $\mathbb{R}^n$. The latter has been the object of study in a number of works mainly by combining variational methods, Hardy-type inequalities, and Sobolev spaces (see e.g. [**17**, **18**] and references therein). For a study via a contraction argument and a sum of weighted L^∞-spaces, see [**21**].

In what follows, we review some works on NBVPs. Chipot *et al.* [**13**] described the non-trivial and non-negative solutions of the NBVP $-\Delta u = au^{\rho_1}$ in $\mathbb{R}^n_+$ and $\partial_\eta u = bu^{\rho_2}$ on $\mathbb{R}^{n-1}$ with $n \geqslant 3$, a, $b \geqslant 0$, $\rho_1 = (n+2)/(n-2)$ and $\rho_2 = n/(n-2)$. For $a \geqslant 0$ and $b = 1$, the existence part was extended by [**14**] to the case $\rho_1 \geqslant$

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$(n+2)/(n-2)$ and $\rho_2 \geqslant n/(n-2)$. Harada [39] analysed the same problem with $a=0$ (Laplace equation), $b=1$, and $\rho_2 > n/(n-2)$ obtaining results on x_n-axial symmetry and asymptotic expansion for positive solutions. For $a=0$, $b=1$, and $1 < \rho_2 < n/(n-2)$, Hu [28] proved the non-existence of non-negative classical solutions. By means of a variational approach and the method of invariant sets, Liu and Liu [31] studied the existence of positive solutions and sign-changing solutions for the Laplace equation in $\mathbb{R}^n_+$ with the nonlinear boundary condition $\partial_\eta u = \lambda V(x')u + g(u)$, where the potential $V \in L^\infty(\mathbb{R}^{n-1})$, $0 \leqslant V \leqslant 1$, $\lim_{|x'|\to\infty} V(x') = 1$, λ is a negative parameter, and g is superlinear at zero and asymptotically linear at infinity. Linked to the self-similarity problem for the semilinear heat equation in $\mathbb{R}^n_+$, the authors of [20, 22] analysed the elliptic PDE with drift $-\Delta u = (1/2)x \cdot \nabla u + cu + g_1(u)$ in $\mathbb{R}^n_+$ with $\partial_\eta u = g_2(u)$ on $\mathbb{R}^{n-1}$ by employing variational techniques along with weighted Sobolev spaces. See also [23, 47] for further related results on NBVPs in the half-space and/or bounded domains.

In another branch of research, we have the study of boundary value problems (BVPs) with singular data which have been a subject of great interest to elliptic PDEs community, see e.g. [3, 33] and references therein. As a matter of fact, there exists a rich literature about the analysis of such problems with measure as forcing terms and boundary data. By employing comparison principles, monotonicity arguments, Kato inequality, weak compactness in weighted L^1-spaces, or suitable capacity-based characterizations, we would like to mention the works [4–6, 9, 11, 12, 25, 26, 34, 35, 37, 48], where the reader can find results on solvability and qualitative properties for BVPs of coercive type in smooth bounded domains Ω of $\mathbb{R}^n$ (see also the book [33] for a nice review). Gmira and Véron [25] considered the problem $-\Delta u + g(u) = 0$ in Ω with $u = f$ on $\partial\Omega$, where the boundary data f is a measure and $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that $\int_1^\infty (|g(s)| + |g(-s)|)s^{-(2n/(n-1))} \mathrm{d}s < \infty$. They proved existence of a unique solution $u \in L^1(\Omega)$ such that $\rho(x)g(u) \in L^1(\Omega)$ where $\rho(x) = d(x, \partial\Omega)$. For related results involving the nonlinearity $g(s) = s\,|s|^{\rho-1}$ and positive measures f, see [34, 35]. Brézis and Ponce [4] studied the same problem for a bounded measure f and $g : \mathbb{R} \to \mathbb{R}$ being a continuous nondecreasing function satisfying $g(s) = 0$ for $s \leqslant 0$. They developed a programme in the spirit of [5, 6] by introducing a concept of reduced measure f^* and showing that f^* is the largest measure such that $f^* \leqslant f$ and the problem has $L^1(\Omega)$-solution with boundary data f^* (good measure), among other properties. In the case of boundary nonlinearities, Boukarabila and Véron [9] showed the solvability of the NBVP $-\Delta u = 0$ in Ω with $\partial_\eta u + g(u) = f$ on $\partial\Omega$, for Radon measures f and $g : \mathbb{R} \to \mathbb{R}$ a continuous nondecreasing function satisfying $g(0) = 0$ and an integral subcritical condition. In the case of problems involving potentials V and Radon measures f, we highlight [48] where the authors studied nonnegative $L^1(\Omega)$-solutions and reduced measure for the BVP $-\Delta u + Vu = 0$ in Ω with $\partial_\eta u = f$ on $\partial\Omega$ by means of an approach with capacity depending on the locally bounded potential $V \geqslant 0$ (then, it can be singular near $\partial\Omega$), the Poisson kernel and the first positive eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$. For results on semilinear problems considering an interplay between measure data and Hardy-type potentials, see [11, 12, 26, 37].

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In [3], Amann and Quittner considered the doubly nonlinear problem $-\Delta u = g_1(x, u) + \varepsilon f_1$ in Ω with $\partial_\eta u = g_2(x, u) + \varepsilon f_2$ on $\partial\Omega$ in the noncoercive case (i.e. $g_i(x, z)$ nondecreasing in z), where f_1, f_2 are finite Radon measures and $\varepsilon > 0$. Among others, they obtained existence and multiplicity results by assuming suitable smallness conditions on ε and employing a mix of sub-super solution method, Sobolev–Slobodeckij spaces, and techniques of fixed points in ordered Banach spaces. In [7], Bidaut-Véron *et al.* treated the problem $-\Delta u = g(u, \nabla u)$ in $\mathbb{R}^n_+$ with the Dirichlet condition $u = \varepsilon f$ on $\mathbb{R}^{n-1}$, where $n \geqslant 3$ and f is a finite Radon measure. For $g(u, \nabla u) = u^\rho$ with $\rho > 1$, they proved existence of positive solution for small $\varepsilon > 0$, as well as some sharp pointwise estimates of the solutions, by assuming suitable conditions involving the Riesz capacity on $\mathbb{R}^{n-1}$ and employing some ideas by Kalton and Verbitsky [29] who developed an extensive study about a class of integral equations with measure data. The authors of [7] also analysed the case of smooth bounded domains Ω (see [8] for related results) as well as the mixed gradient-power case $g(u, \nabla u) = u^{\rho_1} |\nabla u|^{\rho_2}$, where $\rho_1, \rho_2 \geqslant 0$, $\rho_1 + \rho_2 > 1$, and $\rho_2 < 2$, both considering small boundary data εf.

In [10], the authors considered a class of weighted L^∞-spaces in Fourier variables, namely the pseudomeasure spaces $\mathcal{PM}^a$ in the half-space $\mathbb{R}^n_+$, and obtained results on solvability and regularity for (1.1) with $\beta = 0$ (nonlinearity independent of derivatives) and Robin boundary conditions by means of suitable weighted-type estimates and convolution properties of homogeneous functions. Their approach in $\mathcal{PM}^a$ is also based on Fourier analysis and employs an integral formulation similar to ours, nevertheless without using localization arguments and the Littlewood–Paley decomposition as in the case of $\mathcal{FB}^s_{p,\infty}$-spaces. Moreover, we have that $\mathcal{PM}^a \subset \mathcal{FB}^s_{p,\infty}$ for $s = a - n/p$ and $1 \leqslant p < \infty$, and then our results allow more singular potentials and forcing terms. For a Fourier analysis approach and an application of the $\mathcal{PM}^a$-framework in the study of elliptic problems in the whole space $\mathbb{R}^n$ with nonlinear derivative terms, see [19]. In this context, difficulties related to the trace and boundary terms are not present in the integral formulation of the problem, and handling the Fourier transform is relatively simpler as the transform can be applied to the whole $\mathbb{R}^n$ and not just to some components of $x = (x_1, x_2, \ldots, x_n)$.

In [41], Quittner and Reichel addressed the problem $-\Delta u = 0$ in Ω with $\partial_\eta u + u = g(x, u)$ on $\partial\Omega$, where $n \geqslant 3$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain. Considering the growth condition $|g(x, s)| \lesssim (1 + |s|^p)$ for some $p \in (1, (n-1)/(n-2))$, and developing suitable *a priori* estimates, they proved that all positive very weak solution belongs to $L^\infty(\Omega)$ (see also [46] for related results). In addition, they provided examples showing that $\bar{p} = (n-1)/(n-2)$ is a sharp critical exponent. In fact, for $n = 3, 4$, some exponents $p > \bar{p}$ and $g(x, u) = u^p + f$ with some $f \in L^\infty(\partial\Omega)$, they constructed two unbounded very weak solutions blowing-up at a prescribed point on $\partial\Omega$, where Ω is taken within a half-space and with a flat boundary piece. In turn, Merker and Rakotoson [36] analysed very weak solutions of the Poisson equation $-\Delta u = h$ in a bounded domain Ω for singular forcing terms h and singular Neumann boundary conditions, by means of a framework based on Lorentz-spaces $L^{p,q}(\Omega)$, with $p \in (1, \infty)$ and $q \in [1, \infty)$, and an approach relying on a suitable duality formulation for the BVP. They proved an existence and uniqueness result covering the following classes of forces: (i) $h \in L^1(\Omega)$ with $\partial_n u = (-\int_\Omega h \mathrm{d}x)\delta_{x_0}$ and (ii)

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non-integrable h with $h \cdot |x - x_0| \in L^1(\Omega)$ and $x_0 \in \partial\Omega$. Moreover, a generalization for finite Radon measures μ in place of δ_{x_0} was also discussed by them.

We analyse the unique solvability of (1.1) in a setting based on Fourier–Besov spaces $\mathcal{FB}^s_{p,\infty}$ in which localization-in-frequency arguments play a key role (see theorem 3.2). Moreover, the regularity of solutions is investigated with the help of Fourier–Sobolev spaces which naturally provide further decay in Fourier variables for them (see theorem 3.5). Due to the scaling analysis, the power ρ_2 is connected to ρ_1 and β via the relation (see (3.13))

$$(\rho_2 - \rho_1)\gamma = \rho_1\beta - 1 \text{ with } \gamma = \frac{2 - \rho_1\beta}{\rho_1 - 1}. \tag{1.4}$$

So, we can think that ρ_1 and β are free and determine ρ_2. Or, alternatively, that ρ_1 and ρ_2 are free and determine β. However, in the case $K_2 = 0$, BVP (1.1) does not depend on ρ_2 and then we no longer have condition (1.4), and thus ρ_1 and β are free from each other (except for natural conditions involving the parameter ranges). Assuming that V and f are radially symmetric in $\mathbb{R}^{n-1}$, we show that the obtained solutions are x_n-axial symmetric (see theorem 3.7). Our solvability result can also be adapted to the case of (1.1) with Robin boundary conditions in place of the Neumann one (see remark 3.1). With a slight modification in statements and proofs, our results work well for (1.1) with an additional forcing term h acting within the domain $\mathbb{R}^n_+$, namely considering $-\Delta u = K_1(\partial^\beta u)^{\rho_1} + h$ as the first equation in (1.1) (see remark 3.3(iii)).

In comparison with previous works, we are treating an NBVP in the half-space $\mathbb{R}^n_+$ with singular boundary potentials as (1.3) (see remark 3.4(i)) and a nonlinearity involving a fractional derivative. The solvability theory is developed via a contraction argument in a new setting for the context of elliptic PDEs providing new classes of solutions, potentials, and forcing terms. Moreover, it covers cases of variational supercritical powers on the boundary and (when $\beta = 0$) within the domain. Note that the results are new even for other relevant subcases of the model problem (1.1) such as the simpler one $\beta = 0$, $K_1 = 1$, $V = 0$, and $K_2 = 0$, that is, $-\Delta u = u^{\rho_1}$ in $\mathbb{R}^n_+$ with $\partial_\eta u = f$ on $\mathbb{R}^{n-1}$. Another feature is that $\mathcal{FB}^s_{p,\infty}$-spaces lack of good compactness properties and they are non-reflexive (consequently, neither uniformly convex nor q-convex spaces), making it very difficult to employ capacity approaches, variational techniques, Leray–Schauder theory, among others, and thus motivating an analysis based on a non-topological fixed point argument.

In view of the strict continuous inclusions (see property (2.11) in § 2.2)

$$\mathcal{FB}^{s+(n/p_1)}_{\infty,\infty} \subset \mathcal{FB}^{s+((n/p_1)-(n/p_2))}_{p_2,\infty} \subset \mathcal{FB}^s_{p_1,\infty} \subset \mathcal{FB}^{s-(n-(n/p_1))}_{1,\infty}, \tag{1.5}$$

where $s \in \mathbb{R}$ and $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, we can feel the breadth of the family of spaces $\mathcal{FB}^s_{p,\infty}$, especially for negative regularity indexes s. By Hausdorff–Young inequality and (1.5), we can see that $\dot{H}^s \subset \mathcal{FB}^s_{2,\infty} \subset \mathcal{FB}^s_{p,\infty}$ for $1 \leqslant p \leqslant 2$ and $s \in \mathbb{R}$, where $\dot{H}^s$ stands for the homogeneous Sobolev spaces. Also, denoting the space of finite Radon measures in $\mathbb{R}^n$ by $\mathcal{M} = \mathcal{M}(\mathbb{R}^n)$ and taking $s = -(n/p_1)$ in (1.5), we arrive

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at

$$\mathcal{M} \subset \mathcal{FB}^0_{\infty,\infty} \subset \mathcal{FB}^{-n/p_1}_{p_1,\infty} \subset \mathcal{FB}^{-n}_{1,\infty}, \tag{1.6}$$

which allows us to cover measure data by considering the space $\mathcal{FB}^s_{p,\infty}$ with $s = -(n/p)$ and $1 \leqslant p \leqslant \infty$ (see remark 3.4(ii)). Moreover, if $f \in \mathcal{M}$ with $\text{supp}(f)$ contained in a set of Hausdorff dimension $s \in [0, n)$, it follows that $|\hat{f}(\xi)| \lesssim |\xi|^{-s/2}$ and $f \in \mathcal{FB}^{s/2}_{\infty,\infty}$ (see [**38**, p. 40]). By considering suitable indexes, we point out that smallness conditions involving $\mathcal{FB}^s_{p,\infty}$-norms allow us to consider some functions with large L^p and H^s-norms, as well as large Radon measures.

This paper is organized as follows. Section 2 is devoted to some preliminaries by recalling basic notations of Fourier analysis as well as reviewing basic definitions and properties on Littlewood–Paley decomposition, Fourier–Sobolev spaces, and Fourier–Besov spaces. In § 3, we state our results on solvability, regularity, and symmetry for (1.1). The purpose of § 4 is to develop key estimates for the terms of the integral formulation associated with (1.1). In § 5, with the estimates in hand, we show the proofs of our results.

2. Preliminaries

2.1. Basic definitions and notations

In this section, we collect some notations that will be used throughout this paper. We denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^n$ by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ and its dual, the space of tempered distributions, by $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$. In both of them, the Fourier transform of f is an isomorphism and denoted by $\hat{f}(\xi)$ or $\mathcal{F}(f)$. For its inverse, we use the notation $f^{\vee}(\xi)$ or $\mathcal{F}^{-1}(f)$. In the case of $\mathcal{S}$, their actions can be represented in an integral form by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi i x\cdot\xi} f(x)\mathrm{d}x \quad \text{and} \quad \check{f}(\xi) = \hat{f}(-\xi), \quad \forall \xi \in \mathbb{R}^n. \tag{2.1}$$

Also, the operators in (2.1) in the $\mathcal{S}'$-setting are defined via the pair duality between $\mathcal{S}'$ and $\mathcal{S}$.

For $p \in [1, \infty]$ and the Lebesgue measure μ on $\mathbb{R}^n$, we denote by $L^p(\mathbb{R}^n) = L^p(\mathbb{R}^n, \mathrm{d}\mu)$ the usual L^p-space endowed with the norm $\|\cdot\|_p$. In the case of the counting measure μ, we have the sequence Lebesgue space with p-summability $l^p = l^p(\mathbb{Z}^n)$.

Consider the Fourier–Sobolev space

$$H^{1,s} = H^{1,s}(\mathbb{R}^n) = \{f \in \mathcal{S}'; \|f\|_{H^{1,s}} = \|(1 + |\xi|^s)\hat{f}(\xi)\|_1 < \infty\}, \tag{2.2}$$

which is a Banach space with the norm $\|\cdot\|_{H^{1,s}}$. We have the following basic properties:

(i) For a constant $C > 0$, we have the Hölder-type inequality

$$\|u_1 u_2 \dots u_m\|_{H^{1,s}} \leqslant C \|u_1\|_{H^{1,s}} \|u_2\|_{H^{1,s}} \dots \|u_m\|_{H^{1,s}}. \tag{2.3}$$

(ii) The continuous inclusion $H^{1,s} \subset H^{1,t}$ holds true for $s \geqslant t$.

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Now, for each $m \in \mathbb{N}$ we define the space $C_0^m(\mathbb{R}^n)$ as the space of functions u such that $\partial^\alpha u$ is continuous and goes to zero as $|x| \to \infty$, for all multi-index $\alpha \in \mathbb{N}^n$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_n \leqslant m$.

Finally, for $z, w \in \mathbb{C}$ with $\mathrm{Re}(z), \mathrm{Re}(w) > 0$, we recall the Gamma and Beta functions

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} \mathrm{e}^{-t} \mathrm{d}t \quad \text{and} \quad \mathcal{B}(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1} \mathrm{d}t, \tag{2.4}$$

respectively, which verify the relation $\mathcal{B}(z, w) = \Gamma(z)\Gamma(w)/\Gamma(z+w)$.

2.2. Fourier–Besov spaces

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy the following properties

$$0 \leqslant \widehat{\phi}(\xi) \leqslant 1, \quad \forall \xi \in \mathbb{R}^n, \mathrm{supp}(\widehat{\phi}(\xi)) \subset \{\xi \in \mathbb{R}^n; 2^{-1} \leqslant |\xi| \leqslant 2\}, \tag{2.5}$$

and $\sum_{j \in \mathbb{Z}} \widehat{\phi}_j = 1$, $\forall \xi \in \mathbb{R}^n$, where $\phi_j(x) = 2^{jn}\phi(2^j x)$. For each $k \in \mathbb{Z}$, the dyadic k-block Δ_k and the low-frequency operator S_k are defined as

$$\Delta_k f = \phi_k * f \quad \text{and} \quad S_k f = \sum_{j=-\infty}^{k} \Delta_j f, \quad \text{for all } f \in \mathcal{S}'.$$

Let $\mathcal{P}$ stands for the set of all polynomials. For $f \in \mathcal{S}'/\mathcal{P}$, we have the Littlewood–Paley decomposition

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f. \tag{2.6}$$

Moreover, for f, g $\mathcal{S}'/\mathcal{P}$, the Bony paraproduct is given by

$$fg = \sum_{j \in \mathbb{Z}} S_{j-3} f \Delta_j g + \sum_{j \in \mathbb{Z}} S_{j-3} g \Delta_j f + \sum_{j,k \in \mathbb{Z}} \sum_{|j-k| \leqslant 2} \Delta_k f \Delta_j g. \tag{2.7}$$

For $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, the homogeneous Fourier–Besov space (FB-spaces), denoted by $\mathcal{FB}^s_{p,q}$, is the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that $\widehat{f} \in L^1_{loc}(\mathbb{R}^n)$ and the norm

$$\|f\|_{\mathcal{FB}^s_{p,q}} := \left\| \left\{ 2^{sj} \parallel \widehat{\phi}_j \widehat{f} \parallel_{L^p(\mathbb{R}^n)} \right\}_{j \in \mathbb{Z}} \right\|_{l^q(\mathbb{Z})} < \infty. \tag{2.8}$$

The pair $(\mathcal{FB}^s_{p,q}, \|\cdot\|_{\mathcal{FB}^s_{p,q}})$ is a Banach space.

In what follows, we define the functional setting that will be employed in the study of BVP (1.1). For $s \in \mathbb{R}$, $1 \leqslant p, q, r \leqslant \infty$, and $d > 0$, we consider the space $\mathcal{L}^r_d \mathcal{FB}^s_{p,q} = \mathcal{L}^r_d \mathcal{FB}^s_{p,q}(\mathbb{R}^n_+)$ of all Bochner measurable functions $u : (0, \infty) \longrightarrow$

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$\mathcal{FB}^s_{p,q}(\mathbb{R}^{n-1})$ such that the norm $\|\cdot\|_{\mathcal{L}^r_d\mathcal{FB}^s_{p,q}}$ is finite, where

$$\|u\|_{\mathcal{L}^r_d\mathcal{FB}^s_{p,q}} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{sjq} \parallel x_n^d \parallel \widehat{\phi_j}\widehat{u}(\xi', x_n) \parallel_{L^p(\mathbb{R}^{n-1})} \|^q_{L^r(0,\infty)} \right)^{1/q}, & \text{if } q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{sj} \parallel x_n^d \parallel \widehat{\phi_j}\widehat{u}(\xi', x_n) \parallel_{L^p(\mathbb{R}^{n-1})} \|_{L^r(0,\infty)}, & \text{if } q = \infty. \end{cases} \tag{2.9}$$

In the sequel, we recall a Bernstein-type inequality in Fourier variables which is useful for carrying out estimates in the spaces $\mathcal{FB}^s_{p,q}(\mathbb{R}^n)$ and $\mathcal{L}^r_d\mathcal{FB}^s_{p,q}(\mathbb{R}^n_+)$. For $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, a multi-index α of nonnegative real numbers, $j \in \mathbb{Z}$, $R > 0$ and $\text{supp}(\widehat{f}) \subset \{\xi \in \mathbb{R}^n; |\xi| \leqslant R2^j\}$, we have that

$$\|\xi^\alpha \widehat{f}\|_{p_1} \leqslant C 2^{j|\alpha| + j(n/p_2 - n/p_1)} \|\widehat{f}\|_{p_2}, \tag{2.10}$$

where $C > 0$ is a constant independent of n, α, j, p_1, p_2, ξ, and f. Estimate (2.10) yields the continuous inclusion

$$\mathcal{FB}^{s_2}_{p_2,q}(\mathbb{R}^n) \subset \mathcal{FB}^{s_1}_{p_1,q}(\mathbb{R}^n), \tag{2.11}$$

where $1 \leqslant p_1,\ p_2,\ q \leqslant \infty$ and $s_1,\ s_2 \in \mathbb{R}$ satisfy $p_1 < p_2$ and $n/p_1 + s_1 = n/p_2 + s_2$.

The proposition below contains an useful scaling property for the norms of the spaces $\mathcal{FB}^s_{p,q}(\mathbb{R}^n)$ and $\mathcal{L}^r_d\mathcal{FB}^s_{p,q}(\mathbb{R}^n_+)$.

Proposition 2.1 (see [**1**, **40**]). *Let $1 \leqslant p,\ q \leqslant \infty$, $s \in \mathbb{R}$, and $d > 0$.*

(i) *For $u \in \mathcal{FB}^s_{p,q}(\mathbb{R}^n)$, consider the rescaling $u_\lambda = \lambda^\gamma u(\lambda\cdot)$. If*

$$s + \gamma - n + \frac{n}{p} = 0, \tag{2.12}$$

then

$$\|u\|_{\mathcal{FB}^s_{p,q}} \lesssim \| u_\lambda \|_{\mathcal{FB}^s_{p,q}} \lesssim \| u \|_{\mathcal{FB}^s_{p,q}}.$$

(ii) *For $u \in \mathcal{L}^r_d\mathcal{FB}^s_{p,q}(\mathbb{R}^n_+)$, consider the rescaling $u_\lambda = \lambda^\gamma u(\lambda\cdot)$. If*

$$s + \gamma - (n-1) + \frac{(n-1)}{p} - d - \frac{1}{r} = 0, \tag{2.13}$$

then

$$\|u\|_{\mathcal{L}^r_d\mathcal{FB}^s_{p,q}} \lesssim \| u_\lambda \|_{\mathcal{L}^r_d\mathcal{FB}^s_{p,q}} \lesssim \| u \|_{\mathcal{L}^r_d\mathcal{FB}^s_{p,q}}.$$

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3. Results

Proceeding formally, we can apply the Fourier transform in the $n-1$ first variables in (1.1) in order to get

$$\begin{cases} -\partial^2_{x_n x_n}\widehat{u}(\xi', x_n) + 4\pi^2|\xi'|^2(\widehat{u})(\xi', x_n) = [K_1\widehat{(\partial^\beta u)^{\rho_1}}], & \text{for } \xi' \in \mathbb{R}^{n-1},\ x_n > 0, \\ \partial_{x_n}\widehat{u}(\xi', 0) = [\widehat{Vu}](\xi') + [\widehat{K_2 u^{\rho_2}}](\xi') + \widehat{f}(\xi'), & \text{for } \xi' \in \mathbb{R}^{n-1}. \end{cases} \tag{3.1}$$

Note that above we are identifying $\xi' = (\xi', 0) \in \mathbb{R}^{n-1}$. Solving (3.1) with respect to the x_n-variable, we arrive at the following integral equation in Fourier variables:

$$\begin{aligned} \widehat{u}(\xi', x_n) = \int_0^\infty & G(\xi', x_n, t)[K_1\widehat{(\partial^\beta u)^{\rho_1}}](\xi', t)\mathrm{d}t \\ & + G(\xi', x_n, 0)[(\widehat{Vu})(\xi') + (\widehat{K_2 u^{\rho_2}})(\xi') + \widehat{f}(\xi')], \end{aligned} \tag{3.2}$$

where

$$G(\xi', x, t) = \frac{\mathrm{e}^{-2\pi|\xi'||x+t|} + \mathrm{e}^{-2\pi|\xi'||x-t|}}{4\pi|\xi'|}, \quad \text{for } \xi' \in \mathbb{R}^{n-1}/\{0\},\ x \geqslant 0,\ \text{and } t \geqslant 0, \tag{3.3}$$

is the Green function associated with problem (3.1) (see [**43**] for more details).

Remark 3.1. Replacing the Neumann boundary condition in (3.1) with the Robin condition

$$\frac{\partial u}{\partial \eta} + \lambda u = V(x')u + K_2 u^{\rho_2} + f(x'),$$

and proceeding analogously to the above, we obtain the integral formulation (3.2) with the Green function

$$\tilde{G}(\xi', x, t) = \frac{(2\pi|\xi'| + \lambda)\mathrm{e}^{-2\pi|\xi'||x+t|} + (2\pi|\xi'| - \lambda)\mathrm{e}^{-2\pi|\xi'||x-t|}}{(8\pi^2|\xi'|^2 + \lambda|\xi'|)}, \tag{3.4}$$

instead of (3.4). For both cases, we have the pointwise estimate

$$|G(\xi', x, t)| \leqslant \frac{1}{2\pi|\xi'|}\mathrm{e}^{-2\pi|\xi'||x-t|} \quad \text{and} \quad |\tilde{G}(\xi', x, t)| \leqslant \frac{1}{2\pi|\xi'|}\mathrm{e}^{-2\pi|\xi'||x-t|}. \tag{3.5}$$

We can see from the integral equation (3.2) that it is necessary to evaluate the boundary values on $\partial\mathbb{R}^n_+$. However, since we are going to work with spaces of rough functions without a trace notion, we need to consider a functional setting that carries information on u both within $\mathbb{R}^n_+$ and on the boundary of the domain $\partial\mathbb{R}^n_+$. In this way, writing $u_1 = u|_{\mathbb{R}^n_+}$ and $u_2 = u|_{\partial\mathbb{R}^n_+}$, equation (3.2) can be equivalently

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rewritten as

$$\begin{cases} \widehat{u}_1(\xi', x_n) = \int_0^\infty G(\xi', x_n, t)(K_1\widehat{(\partial^\beta u_1})^{\rho_1})(\xi', t)\mathrm{d}t \\ \qquad\qquad + G(\xi', x_n, 0)[(\widehat{Vu_2})(\xi') + (\widehat{K_2 u_2^{\rho_2}})(\xi') + \widehat{f}(\xi')] \\ \widehat{u}_2(\xi') = \int_0^\infty G(\xi', 0, t)(K_1\widehat{(\partial^\beta u_1})^{\rho_1})(\xi', t)\mathrm{d}t \\ \qquad\qquad + G(\xi', 0, 0)[(\widehat{Vu_2})(\xi') + (\widehat{K_2 u_2^{\rho_2}})(\xi') + \widehat{f}(\xi')] \end{cases} \tag{3.6}$$

Naturally, in a setting with enough regularity for u, note that u_2 should be the trace of u_1 in $\partial\mathbb{R}^n_+$. This follows directly from the uniqueness of solution for the problem that will be obtained in theorem 3.2 (see more below).

To handle (3.6), we define the following operators in Fourier variables:
$I(u_1, u_2) = (I_1(u_1), I_2(u_1))$ with

$$\widehat{I_1}(u_1) = \int_0^\infty G(\xi', x_n, t)(K_1(\partial^\beta u_1)^{\rho_1})^\wedge(\xi', t)\mathrm{d}t$$
$$\text{and} \quad \widehat{I_2}(u_1) = \int_0^\infty G(\xi', 0, t)(K_1(\partial^\beta u_1)^{\rho_1})^\wedge(\xi', t)\mathrm{d}t; \tag{3.7}$$

$N(u_1, u_2) = (N_1(u_2), N_2(u_2))$ with

$$\widehat{N_1}(u_2) = G(\xi', x_n, 0)(Vu_2)^\wedge(\xi') \quad \text{and} \quad \widehat{N_2}(u_2) = G(\xi', 0, 0)(Vu_2)^\wedge(\xi'); \tag{3.8}$$

$T(u_1, u_2) = (T_1(u_2), T_2(u_2))$ with

$$\widehat{T_1}(u_2) = G(\xi', x_n, 0)(K_2 u_2^{\rho_2})^\wedge(\xi') \quad \text{and} \quad \widehat{T_2}(u_2) = G(\xi', 0, 0)(K_2 u_2^{\rho_2})^\wedge(\xi'); \tag{3.9}$$

$L(f) = (L_1(f), L_2(f))$ with

$$\widehat{L_1}(u_1) = G(\xi', x_n, 0)\widehat{f}(\xi') \quad \text{and} \quad \widehat{L_2}(u_2) = G(\xi', 0, 0)\widehat{f}(\xi'). \tag{3.10}$$

Then, we can express (3.6) through the formulation

$$u = I(u) + N(u) + T(u) + L(f), \tag{3.11}$$

where $u = (u_1, u_2)$. If u satisfies equation (3.11), we say that u is an integral solution for (1.1).

In what follows, we carry out a scaling analysis to find suitable indexes for the corresponding FB-spaces of u, V, f. For that purpose and just a moment, consider V and f homogeneous distributions of degree h_1 and h_2, respectively. Also, denote

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$u_\lambda(x) = \lambda^\gamma u(\lambda x)$ and assume that

$$\gamma = \rho_1(\gamma + \beta) - 2 = \gamma - h_1 - 1 = \rho_2\gamma - 1 = -h_2 - 1, \tag{3.12}$$

or equivalently

$$\gamma = \frac{2 - \rho_1\beta}{\rho_1 - 1}, \quad h_1 = -1, \quad h_2 = \frac{\rho_1(\beta - 1) - 1}{\rho_1 - 1}, \quad (\rho_2 - \rho_1)\gamma = \rho_1\beta - 1. \tag{3.13}$$

Making a scaling analysis, we have that u_λ verifies (3.11) if so does u. Thus, we have the scaling map

$$u \to u_\lambda. \tag{3.14}$$

A space of tempered distribution is said to be critical for (3.11) when it is invariant under (3.14).

Let us point out that the scaling map carries structural information about the BVP, showing the degree of homogeneity preserved by it. Thus, studying the BVP in spaces (with the correct indexes) that preserve such homogeneity (critical spaces), in principle, should provide a good environment for estimating the terms of its integral formulation via tools such as Fourier transform, product estimates, estimates for potential operators, among others. This way, a suitable balance is obtained between the two sides of the needed estimates for the operators of the integral formulation; in our case the operators $I(\cdot)$, $N(\cdot)$, $T(\cdot)$, and $L(\cdot)$ in (3.11). These aspects are even more prominent in the case of homogeneous versions of spaces such as the homogeneous Sobolev spaces, the homogeneous Besov spaces, the homogeneous Fourier–Besov spaces, as is our case in question. They are also relevant in the case where the space of original variables is invariant by homotheties $x \to \lambda x$ ($\lambda > 0$), such as $\mathbb{R}^n$ or the half-space $\mathbb{R}^n_+$, in which certain embeddings and estimates work well only for exact indexes or a precise relation between them.

Next, we define the functional setting where we are going to analyse (3.11). For $n \geqslant 3$, $1 \leqslant p \leqslant \infty$, s_1, $s_2 \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$, and $d > 0$ satisfying the relation

$$s_1 - d + \frac{(n-1)}{p} = s_2 + \frac{(n-1)}{p} = (n-1) - \gamma, \tag{3.15}$$

we consider

$$\mathcal{X} = \mathcal{X}^{s_1,s_2}_{p,d} = \mathcal{L}^\infty_d \mathcal{FB}^{s_1}_{p,\infty}(\mathbb{R}^n_+) \times \mathcal{FB}^{s_2}_{p,\infty}(\mathbb{R}^{n-1}), \tag{3.16}$$

endowed with the norm:

$$\|(u_1, u_2)\|_{\mathcal{X}} = \|u_1\|_{\mathcal{L}^\infty_d \mathcal{FB}^{s_1}_{p_1,\infty}} + \|u_2\|_{\mathcal{FB}^{s_2}_{p_2,\infty}}. \tag{3.17}$$

Note that in view of (3.13) and (3.15), we have that $\mathcal{X}$ is a critical space for (3.11), namely

$$\|(u_1, u_2)\|_{\mathcal{X}} \simeq \|(\lambda^\gamma u_1(\lambda x), \lambda^\gamma u_2(\lambda x))\|_{\mathcal{X}}.$$

Furthermore, for ρ_1, $\rho_2 \geqslant 2$ integers and $\beta \geqslant 0$, define the regularity indexes $\tilde{s}$ and $\overline{s}$ as

$$\tilde{s} = (n-1) - \frac{n-1}{p} - 1 \quad \text{and} \quad \overline{s} = (n-1) - \frac{(n-1)}{p} - 1 - \frac{2 - \rho_1\beta}{\rho_1 - 1}. \tag{3.18}$$

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THEOREM 3.2. *Let $n \geqslant 3$, $\rho_1, \rho_2 \geqslant 2$ integers with $\rho_2 \geqslant (\rho_1+1)/2$, $1 \leqslant p \leqslant \infty$, $s_1, s_2, \tilde{s}, \overline{s} \in \mathbb{R}$, $d > 0$, and $0 \leqslant \beta < 2/\rho_1$. Assume the scaling relations (3.13), (3.15), and (3.18). Suppose also the conditions*

$$d < \min\left\{\frac{1}{\rho_1 - 1}, \frac{2-\beta}{\rho_1 - 1}\right\}, \quad s_1 > 2 - (\rho_1 - 1)d, \quad \text{and} \quad s_2 > 2 - \rho_1 d. \tag{3.19}$$

Then, there are $\varepsilon > 0$ and $\delta_1, \delta_2 > 0$ such that equation (3.11) has a unique solution $u = (u_1, u_2)$ satisfying $\|u\|_{\mathcal{X}} \leqslant \varepsilon$ provided that $f \in \mathcal{FB}^{\overline{s}}_{p,\infty}$ and $V \in \mathcal{FB}^{\tilde{s}}_{p,\infty}$ with $\|f\|_{\mathcal{FB}^{\overline{s}}_{p,\infty}} \leqslant \delta_1$ and $\|V\|_{\mathcal{FB}^{\tilde{s}}_{p,\infty}} \leqslant \delta_2$. Furthermore, the solution u depends continuously on f and V.

REMARK 3.3.

(i) (Lipschitz dependence on f, V) In fact, the proof of theorem 3.2 gives that the data-solution map $(f, V) \to u = (u_1, u_2)$ is Lipschitz continuous. More precisely, if $u = (u_1, u_2)$ and $w = (w_1, w_2) \in B_{\mathcal{X}}$ are solutions of (3.11) corresponding to the pairs (V, f) and $(\widetilde{V}, \widetilde{f})$, respectively, then we have positive constants η, ζ independent of $V, \widetilde{V}, f, \widetilde{f}, u, w$ such that

$$\|u - w\|_{\mathcal{X}} \leqslant \eta \parallel V - \widetilde{V} \parallel_{\mathcal{FB}^{\tilde{s}}_{p,\infty}} + \zeta \parallel f - \widetilde{f} \parallel_{\mathcal{FB}^{\overline{s}}_{p,\infty}} \tag{3.20}$$

(ii) Note that the conditions in theorem 3.2 are non-empty. As a matter of fact, it follows from (3.13) that $\beta = (2\rho_2 - \rho_1 - 1)/(\rho_2 - 1)$. For $\rho_2 > (\rho_1+1)/2$, it follows that $0 < \beta < 2/\rho_1$. Then, we can take $d > 0$ satisfying $(\rho_1+1)d < 1$ and $d < (2-\beta)/(\rho_1 - 1)$. Also, consider n, p such that

$$(n-1)\left(1 - \frac{1}{p}\right) > \rho_1\left(\frac{2-\beta}{\rho_1 - 1} - d\right).$$

Now, for γ as in (3.13), we can choose s_1 and s_2 such that (3.15) and (3.19) hold true. The case $\beta = 0$ is similar but we need to consider an odd integer $\rho_1 \geqslant 3$, because $\rho_2 = (\rho_1+1)/2$.

(iii) With suitable adaptations in theorem 3.2, we could treat problem (1.1) with the first equation being $-\Delta u = K_1(\partial^\beta u)^{\rho_1} + h$, that is, with an additional forcing term h. For example, we need to assume $h \in \mathcal{FB}^{s_3}_{p,\infty}$ with $s_3 = n - 2 - \gamma - n/p$ and $\|h\|_{\mathcal{FB}^{s_3}_{p,\infty}} \leqslant \delta_3$, for some small $\delta_3 > 0$.

(iv) From a more general viewpoint, formulation (3.11) can be interpreted within the perspective of nonlinearly perturbed linear problems. This broad class of problems has attracted the attention of several authors; see, for example [**16**, **24**, **32**] and references therein.

REMARK 3.4.

(i) (Singular potentials) Theorem 3.2 covers potentials V as in (1.3) which are homogeneous of degree -1. In fact, for potentials homogeneous of degree $-\sigma$,

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by proposition 2.1(i) we have that $V \in \mathcal{FB}^{\tilde{s}}_{p,\infty}$ with $\tilde{s} = n - 1 - ((n-1)/p) - \sigma$ that corresponds to the index $\tilde{s}$ in (3.18) when $\sigma = 1$.

(ii) (Measures as forcing terms) We can consider f as a Radon measure by taking $\overline{s} = -(n-1)/p$. In this case, using (3.18) we have $(n-1) - \gamma = 1$ and then it follows from (3.15) that the indexes s_1, s_2 should satisfy

$$s_1 - d = s_2 = 1 - \frac{(n-1)}{p}. \tag{3.21}$$

Therefore, in order to have the conditions on s_1 and s_2 in (3.19), we need to assume

$$d > \frac{1}{\rho_1} + \frac{n-1}{p\rho_1} \quad \text{and} \quad \frac{n-1}{p} < \min\left\{\frac{1}{\rho_1 - 1}, \frac{1 + \rho_1 - \beta\rho_1}{\rho_1 - 1}\right\} \tag{3.22}$$

which is compatible with the other conditions in theorem 3.2. For example, in the case $p = \infty$, condition (3.22) reduces to the simple one $d > 1/\rho_1$.

To analyse the regularity of solutions for (3.11), we need to consider another functional setting which is a suitable half-space version of the Fourier–Sobolev space $H^{1,s}(\mathbb{R}^n)$ defined in (2.2). Let $H^{1,s}_d = H^{1,s}_d(\mathbb{R}^n_+)$ be the Banach space of all Bochner measurable functions $u : (0, \infty) \longrightarrow H^{1,s}(\mathbb{R}^{n-1})$ such that the norm $\|\cdot\|_{H^{1,s}_d}$ is finite, where

$$\|f\|_{H^{1,s}_d} = \operatorname*{ess\,sup}_{x_n > 0} x_n^d \left\| (1 + |\cdot|^s)\widehat{f}(x_n, \cdot) \right\|_1. \tag{3.23}$$

Consider the Banach space $\mathcal{H}^{1,s}_d = H^{1,s}_d \times H^{1,s}$ with the norm

$$\|(u_1, u_2)\|_{\mathcal{H}^{1,s}_d} = \|u_1\|_{H^{1,s}_d} + \|u_2\|_{H^{1,s}}. \tag{3.24}$$

We have the following result.

Theorem 3.5. *Under the same hypotheses of theorem 3.2, let $s \in \mathbb{R}$ and suppose further that $\rho_1 \geqslant 4$, $\rho_2 < \rho_1$, $d < \min\{1/(\rho_1 + 1), (2 - \rho_1\beta)/(\rho_1 - 1), (1 - \beta)/\rho_1\}$, and $s \geqslant \beta$.*

There exist $\delta_1, \delta_2 > 0$ such that, if $f \in \mathcal{FB}^{\overline{s}}_{p,\infty} \cap H^{1,s}$ and $V \in \mathcal{FB}^{\tilde{s}}_{p,\infty} \cap H^{1,s}$ satisfy

$$\|f\|_{\mathcal{FB}^{\overline{s}}_{p,\infty} \cap H^{1,s}} \leqslant \delta_1 \quad \text{and} \quad \|V\|_{\mathcal{FB}^{\tilde{s}}_{p,\infty} \cap H^{1,s}} \leqslant \delta_2,$$

then the solution $u = (u_1, u_2)$ of (3.11) obtained in theorem 3.2 belongs to $\mathcal{X} \cap \mathcal{H}^{1,s}_d$. Moreover, we have that $u_1(\cdot, x_n) \in C_0^{\lfloor s \rfloor}(\mathbb{R}^{n-1})$, for each $x_n > 0$, and $u_2 \in C_0^{\lfloor s \rfloor}(\mathbb{R}^{n-1})$, where $\lfloor \cdot \rfloor$ stands for the greatest integer function.

Remark 3.6. For index s large enough in theorem 3.5, we obtain a solution u for equation (3.11) smooth w.r.t. the variables $x' = (x_1, \ldots, x_{n-1})$.

In the next result, we present a result on axial symmetry of solutions.

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THEOREM 3.7. *Under the hypotheses of theorem 3.2. Assume further that V and f are radially symmetric in $\mathbb{R}^{n-1}$. Let $u = (u_1, u_2)$ be the solution of (3.11) obtained in theorem 3.2 corresponding to V and f. Then, u is invariant under rotations around the axis $\overrightarrow{Ox_n}$, that is, u_1 is invariant under rotations around $\overrightarrow{Ox_n}$ and the trace component u_2 is radially symmetric in $\mathbb{R}^{n-1}$.*

4. Estimates for the terms of formulation (3.11)

The purpose of this section is to develop the key estimates for the operators $L(f)$, $I(u)$, $T(u)$, and $N(u)$ defined in (3.7)–(3.10).

4.1. Estimates in spaces of FB-type

Consider the Banach spaces

$$\mathcal{Y} = \mathcal{L}^\infty_d \mathcal{FB}^{s_1}_{p,\infty}(\mathbb{R}^n_+) \quad \text{and} \quad \mathcal{Z} = \mathcal{FB}^{s_2}_{p,\infty}(\mathbb{R}^{n-1}), \tag{4.1}$$

with the respective norms $\|u_1\|_{\mathcal{Y}} = \|u_1\|_{\mathcal{L}^\infty_d \mathcal{FB}^{s_1}_{p,\infty}}$ and $\|u_2\|_{\mathcal{Z}} = \|u_2\|_{\mathcal{FB}^{s_2}_{p,\infty}}$. Note that $\mathcal{Z}$ is a trace space and the space $\mathcal{X}$ in theorem 3.2 can be expressed as $\mathcal{X} = \mathcal{Y} \times \mathcal{Z}$.

To deal with the product operator and nonlinearities in $\mathcal{Y}$ and $\mathcal{Z}$, we need to work with some decompositions in frequency variables. For that, let $w, v \in \mathcal{S}'/\mathcal{P}$ and $1 \leqslant p \leqslant \infty$. Recalling Bony's paraproduct formula, we have that

$$\begin{aligned} wv &= \sum_{k\in\mathbb{Z}} S_{k-3} v \Delta_k w + \sum_{k\in\mathbb{Z}} S_{k-3} w \Delta_k v + \sum_{l,k\in\mathbb{Z}} \sum_{|l-k|\leqslant 2} \Delta_l w \Delta_k v \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

Then, for each $j \in \mathbb{Z}$, it follows that

$$\|\widehat{\phi}_j (\widehat{wv})\|_p \leqslant \|\widehat{\phi}_j \widehat{A_1}\|_p + \|\widehat{\phi}_j \widehat{A_2}\|_p + \|\widehat{\phi}_j \widehat{A_3}\|_p .$$

Using that $\operatorname{supp}(\widehat{S_{k-3} v \Delta_k w}) \subset \{\xi' \in \mathbb{R}^{n-1};\ 2^{k-2} \leqslant |\xi'| \leqslant 2^{k+2}\}$ (similarly for the parcels of $\widehat{A}_2$) and

$$\operatorname{supp}\left(\sum_{|l-k|\leqslant 2} \widehat{\phi}_l \widehat{w} * \widehat{\phi}_k \widehat{v} \right) \subset \{\xi' \in \mathbb{R}^{n-1}; \quad 0 < |\xi'| < 2^{l+4}\},$$

we can decompose

$$\begin{aligned} \| \widehat{\phi}_j (\widehat{wv}) \|_p \leqslant & \sum_{|k-j|\leqslant 3} \sum_{l=-\infty}^{k-3} \| \widehat{\phi}_l \widehat{v} * \widehat{\phi}_k \widehat{w} \|_p + \sum_{|k-j|\leqslant 3} \sum_{l=-\infty}^{k-3} \| \widehat{\phi}_l \widehat{w} * \widehat{\phi}_k \widehat{v} \|_p \\ & + \sum_{|j-l|<5} \sum_{|k-l|\leqslant 2} \| \widehat{\phi}_l \widehat{w} * \widehat{\phi}_k \widehat{v} \|_p + \sum_{l=j-3}^{\infty} \sum_{|k-l|\leqslant 2} \| \widehat{\phi}_l \widehat{w} * \widehat{\phi}_k \widehat{v} \|_p \\ =: & B_1 + B_2 + B_3 + B_4. \end{aligned} \tag{4.2}$$

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Proceeding similarly to the above for the product zvw, we arrive at

$$\begin{aligned}\| \widehat{\phi}_j(\widehat{zvw}) \|_p &\leqslant \sum_{|k-j|\leqslant 3} \sum_{l=-\infty}^{k-3} \| \widehat{\phi}_l \widehat{z} * \widehat{\phi}_k \widehat{vw} \|_p + \sum_{|k-j|\leqslant 3} \sum_{l=-\infty}^{k-3} \| \widehat{\phi}_l \widehat{vw} * \widehat{\phi}_k \widehat{z} \|_p \\ &\quad + \sum_{|j-l|<5} \sum_{|k-l|\leqslant 2} \| \widehat{\phi}_l \widehat{z} * \widehat{\phi}_k \widehat{vw} \|_p + \sum_{l=j-3}^{\infty} \sum_{|k-l|\leqslant 2} \| \widehat{\phi}_l \widehat{z} * \widehat{\phi}_k \widehat{vw} \|_p \\ &=: D_1 + D_2 + D_3 + D_4. \end{aligned} \tag{4.3}$$

Moreover, we can estimate the parcel D_1 as follows:

$$\begin{aligned} D_1 &= \sum_{|k-j|\leqslant 3} \sum_{l=-\infty}^{k-3} \| \widehat{\phi}_l \widehat{z} * \widehat{\phi}_k \widehat{vw} \|_p \\ &\leqslant \sum_{|k-j|\leqslant 3} \sum_{l=-\infty}^{k-3} \| \widehat{\phi}_l \widehat{z} \|_1 \left(\sum_{|m-k|\leqslant 3} \sum_{\eta=-\infty}^{m-3} \| \widehat{\phi}_\eta \widehat{w} * \widehat{\phi}_m \widehat{v} \|_p \right. \\ &\quad + \sum_{|m-k|\leqslant 3} \sum_{\eta=-\infty}^{m-3} \| \widehat{\phi}_\eta \widehat{v} * \widehat{\phi}_m \widehat{w} \|_p + \sum_{|k-m|<5} \sum_{|m-\eta|\leqslant 2} \| \widehat{\phi}_\eta \widehat{w} * \widehat{\phi}_m \widehat{v} \|_p \\ &\quad \left. + \sum_{m=k-3}^{\infty} \sum_{|\eta-m|\leqslant 2} \| \widehat{\phi}_\eta \widehat{w} * \widehat{\phi}_m \widehat{v} \|_p \right) \\ &=: D_1^1 + D_1^2 + D_1^3 + D_1^4. \end{aligned} \tag{4.4}$$

In the same way, for D_2, D_3, D_4 we obtain the estimates

$$D_i \leqslant D_i^1 + D_i^2 + D_i^3 + D_i^4, \quad \text{for } i = 2, 3, 4,$$

where the parcels D_i^j are as in (4.4) with the natural small modifications.

First, we treat the operators $L_1(\cdot)$ and $L_2(\cdot)$.

Lemma 4.1. *Let s_1, $s_2 \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$, $\rho_1 \geqslant 2$, $\beta \geqslant 0$, $d > 0$ satisfy (3.15) with $\gamma = (2-\rho_1\beta)/(\rho_1 - 1)$. Consider $\overline{s} \in \mathbb{R}$ as in (3.18). Then, there exists a constant $C > 0$ such that*

$$\|L_1(f)\|_{\mathcal{Y}} \leqslant C \|f\|_{\mathcal{FB}^{\overline{s}}_{p,\infty}} \quad \text{and} \quad \|L_2(f)\|_{\mathcal{Z}} \leqslant C \|f\|_{\mathcal{FB}^{\overline{s}}_{p,\infty}}, \tag{4.5}$$

for all $f \in \mathcal{FB}^{\overline{s}}_{p,\infty}(\mathbb{R}^{n-1})$.

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Proof. For each $j \in \mathbb{Z}$, using that $s_1 - 1 - d = s_2 - 1 = \overline{s}$, we can estimate

$$\begin{aligned}
2^{s_1 j} x_n^d \parallel \widehat{\phi_j} G(\xi', x_n, 0)\widehat{f} \parallel_p &\leqslant C 2^{s_1 j} x_n^d \parallel \widehat{\phi_j} \frac{1}{2\pi|\xi'|} \mathrm{e}^{-2\pi|\xi'||x_n|} \widehat{f} \parallel_p \\
&\leqslant C 2^{(s_1-1)j} x_n^d \mathrm{e}^{-2\pi|2^j x_n|} \parallel \widehat{\phi_j}\widehat{f} \parallel_p \\
&= C 2^{(s_1-1-d)j} (2^j x_n)^d \mathrm{e}^{-2\pi|2^j x_n|} \parallel \widehat{\phi_j}\widehat{f} \parallel_p \\
&\leqslant C 2^{\overline{s}j} \parallel \widehat{\phi_j}\widehat{f} \parallel_p
\end{aligned} \tag{4.6}$$

and

$$2^{s_2 j} \parallel \widehat{\phi_j} G(\xi', 0, 0)\widehat{f} \parallel_p \leqslant C 2^{(s_2-1)j} \parallel \widehat{\phi_j}\widehat{f} \parallel_p = C 2^{\overline{s}j} \parallel \widehat{\phi_j}\widehat{f} \parallel_p . \tag{4.7}$$

The estimates in (4.5) follows by taking the supremum over $x_n > 0$ and $j \in \mathbb{Z}$ in (4.6) and the supremum over $j \in \mathbb{Z}$ in (4.7). □

The lemma below contains estimates for the operator $I_1(u)$ defined in (3.7).

Lemma 4.2. *Let $s_1 \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$, $\rho_1 \geqslant 2$ integer, $\beta \geqslant 0$, and let $d > 0$ be such that*

$$d < \min\left\{\frac{1}{\rho_1 - 1}, \frac{2-\beta}{\rho_1 - 1}\right\}, \quad s_1 > 2 - (\rho_1 - 1)d$$

and

$$s_1 - d + \frac{n-1}{p} = (n-1) - \frac{2 - \rho_1\beta}{\rho_1 - 1}. \tag{4.8}$$

Then, there exists a constant $C > 0$ such that

$$\|I_1(u_1) - I_1(w_1)\|_{\mathcal{Y}} \leqslant C \|u_1 - w_1\|_{\mathcal{Y}} \sum_{i=0}^{\rho_1 - 1} \|u_1\|_{\mathcal{Y}}^{\rho_1 - 1 - i} \|w_1\|_{\mathcal{Y}}^{i}, \tag{4.9}$$

for all u_1, $w_1 \in \mathcal{Y}$.

Proof. Assume initially that $\rho_1 = 2$. For each $j \in \mathbb{Z}$, in view of (3.5), we have that

$$\begin{aligned}
&2^{s_1 j} \parallel \widehat{\phi_j}(\xi') \int_0^\infty G(\xi', x_n, t)[K_1((\partial^\beta u_1)^2)^\wedge(\xi', t) - K_1((\partial^\beta w_1)^2)^\wedge(\xi', t)]\mathrm{d}t \parallel_p \\
&\quad \leqslant C 2^{(s_1-1)j} \int_0^\infty \mathrm{e}^{-2\pi 2^j |x_n - t|} \parallel \widehat{\phi_j}(\xi')[\partial^\beta(u_1 - w_1)\partial^\beta(u_1 + w_1)]^\wedge(\xi', t) \parallel_p \mathrm{d}t.
\end{aligned} \tag{4.10}$$

Taking $w = \partial^\beta(u_1 - w_1)(\cdot, t)$ and $v = \partial^\beta(u_1 + w_1)(\cdot, t)$, and using (4.2), we obtain that

$$\text{R.H.S. of (4.10)} \leqslant C 2^{(s_1-1)j} \int_0^\infty \mathrm{e}^{-2\pi 2^j |x_n - t|}(B_1 + B_2 + B_3 + B_4)\mathrm{d}t. \tag{4.11}$$

In what follows, we separately treat the parcels in (4.11). For the parcel with B_1, in view of (4.8) with $\rho_1 = 2$, employing Young inequality in L^p and Bernstein-type

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inequality (2.10), we have that

$$
\begin{aligned}
&2^{(s_1-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j|x_n-t|}B_1\mathrm{d}t\\
&\quad\leqslant C2^{(s_1-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j|x_n-t|}\sum_{|k-j|\leqslant 3}\sum_{l=-\infty}^{k-3}2^{[(n-1)-((n-1)/p)+\beta-s_1]l}\\
&\qquad\times 2^{s_1 l}\parallel \widehat{\phi}_l(u_1-w_1)^\wedge(\xi',t)\parallel_p 2^{k\beta}\parallel \widehat{\phi}_k(u_1+w_1)^\wedge(\xi',t)\parallel_p \mathrm{d}t\\
&\quad\leqslant C\parallel u_1-w_1\parallel_{\mathcal{L}^\infty_d\mathcal{F}\mathcal{B}^{s_1}_{p,\infty}}\int_0^\infty t^{-2d}\mathrm{e}^{-2\pi 2^j|x_n-t|}\mathrm{d}t\sum_{|k-j|\leqslant 3}2^{(s_1-1)j}2^{k[2-d]}\\
&\qquad\times\sup_{t>0}t^d\parallel \widehat{\phi}_k(u_1+w_1)^\wedge(\cdot,t)\parallel_p,
\end{aligned}
$$

because $(n-1)-((n-1)/p)+\beta-s_1=(2-2\beta)-d+\beta$. Using now that $\mathrm{e}^{-2\pi 2^j|x_n-t|}(2\pi 2^j||x_n-t|)^M<1$ with $M<1$, it follows that

$$
\begin{aligned}
&\int_0^\infty t^{-2d}\mathrm{e}^{-2\pi 2^j|x_n-t|}\mathrm{d}t\\
&\quad\leqslant C2^{-jM}\left(\int_0^{x_n}t^{-2d}(|x_n-t|)^{-M}\mathrm{d}t+\int_{x_n}^\infty t^{-2d}(|x_n-t|)^{-M}\mathrm{d}t\right)\\
&\quad=2^{-jM}x_n^{1-M-2d}(\mathcal{B}(1-2d,1-M)+\mathcal{B}(2d+M-1,1-M))\\
&\quad\leqslant C2^{-jM}x_n^{1-M-2d},
\end{aligned}
$$

where $\mathcal{B}(\cdot,\cdot)$ is the beta function (see (2.4)) and we use the change of variables $t=x_ns$ and $t=x_n/w$. Here, we need $1-2d>0$, $1-M>0$, and $2d+M-1>0$ for the convergence of $\mathcal{B}$. So, taking $M+2d-1=d$, we arrive at

$$
\begin{aligned}
2^{(s_1-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j|x_n-t|}B_1\mathrm{d}t\leqslant C\,\|u_1-w_1\|_{\mathcal{L}^\infty_d\mathcal{F}\mathcal{B}^{s_1}_{p,\infty}}\,x_n^{-d}\sum_{|k-j|\leqslant 3}2^{(s_1-1-M)(j-k)}2^{s_1k}\\
\times\sup_{t>0}t^d\parallel \widehat{\phi}_k(u_1+w_1)^\wedge(\cdot,t)\parallel_p,
\end{aligned}
$$

since $s_1-1-M+2-d=s_1$. For B_2, B_3, and B_4, proceeding analogously to above, we have respectively that

$$
\begin{aligned}
&2^{(s_1-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j|x_n-t|}B_2\mathrm{d}t\\
&\quad\leqslant C2^{(s_1-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j|x_n-t|}\sum_{|k-j|\leqslant 3}\sum_{l=-\infty}^{k-3}2^{[(n-1)-((n-1)/p)+\beta-s_1]l}
\end{aligned}
$$

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$$\times\, 2^{s_1 l} \parallel \widehat{\phi}_l(u_1+w_1)^\wedge(\xi',t) \parallel_p 2^{k\beta} \parallel \widehat{\phi}_k(u_1-w_1)^\wedge(\xi',t) \parallel_p \mathrm{d}t$$
$$\leqslant C \, \|u_1+w_1\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}} \, x_n^{-d} \sum_{|k-j|\leqslant 3} 2^{(s_1-1-M)(j-k)} 2^{s_1 k}$$
$$\times \sup_{t>0} t^d \parallel \widehat{\phi}_k(u_1-w_1)^\wedge(\cdot,t) \parallel_p,$$

$$2^{(s_1-1)j} \int_0^\infty \mathrm{e}^{-2\pi 2^j |x_n-t|} B_3 \mathrm{d}t$$
$$\leqslant C 2^{(s_1-1)j} \int_0^\infty \mathrm{e}^{-2\pi 2^j |x_n-t|} \sum_{|l-j|<5} \sum_{|k-l|\leqslant 2} 2^{[(n-1)-((n-1)/p)+\beta-s_1]k}$$
$$\times\, 2^{s_1 k} \parallel \widehat{\phi}_k(u_1-w_1)^\wedge(\xi',t) \parallel_p 2^{l\beta} \parallel \widehat{\phi}_l(u_1+w_1)^\wedge(\xi',t) \parallel_p \mathrm{d}t$$
$$\leqslant C \parallel u_1-w_1 \parallel_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}} x_n^{-d} \sum_{|l-j|<5} 2^{(s_1-1-M)(j-l)} 2^{s_1 l}$$
$$\times \sup_{t>0} t^d \parallel \widehat{\phi}_l(u_1+w_1)^\wedge(\cdot,t) \parallel_p,$$

and

$$2^{(s_1-1)j} \int_0^\infty \mathrm{e}^{-2\pi 2^j |x_n-t|} B_4 \mathrm{d}t$$
$$\leqslant C 2^{(s_1-1)j} \int_0^\infty \mathrm{e}^{-2\pi 2^j |x_n-t|} \sum_{l=j-3}^{\infty} \sum_{|k-l|\leqslant 2} 2^{[(n-1)-((n-1)/p)+\beta-s_1]k}$$
$$\times\, 2^{s_1 k} \parallel \widehat{\phi}_k(u_1-w_1)^\wedge(\xi',t) \parallel_p 2^{l\beta} \parallel \widehat{\phi}_l(u_1+w_1)^\wedge(\xi',t) \parallel_p \mathrm{d}t$$
$$\leqslant C \, \|u_1-w_1\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}} x_n^{-d} \sum_{l=j-3}^{\infty} 2^{(s_1-1-M)(j-l)} 2^{s_1 l}$$
$$\times \sup_{t>0} t^d \parallel \widehat{\phi}_l(u_1+w_1)^\wedge(\cdot,t) \parallel_p .$$

Next, bearing in mind (4.10)–(4.11), multiplying both sides of the above estimates by x_n^d, taking the supremum over $x_n > 0$, afterwards the supremum over $j \in \mathbb{Z}$, and applying Young inequality for discrete convolutions in $\mathbb{Z}$, we arrive at

$$\|I_1(u_1) - I_1(w_1) \parallel_{\mathcal{Y}} \leqslant C \parallel u_1 - w_1 \parallel_{\mathcal{Y}} (\|u_1 \parallel_{\mathcal{Y}} + \|w_1 \parallel_{\mathcal{Y}}),$$

where we have used that $2-d-\beta > 0$ and $s_1 > 2-d$ in order to ensure convergence of the series in the estimates involving B_i's.

Now, we turn to the case $\rho_1 = 3$. First, note that

$$(\partial^\beta u_1)^3 - (\partial^\beta w_1)^3 = \partial^\beta(u_1-w_1)(\partial^\beta u_1)^2 + \partial^\beta(u_1-w_1)(\partial^\beta u_1)(\partial^\beta w_1)$$
$$+\, \partial^\beta(u_1-w_1)(\partial^\beta w_1)^2.$$

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Thus,

$$\begin{aligned}
&2^{s_1 j} \parallel \widehat{\phi_j}(\xi') \int_0^\infty G(\xi', x_n, t)[K_1((\partial^\beta u_1)^3)^\wedge - K_1((\partial^\beta w_1)^3)^\wedge](\xi', t)\mathrm{d}t \parallel_p \\
&\leqslant C 2^{(s_1-1)j} \int_0^\infty \mathrm{e}^{-2\pi 2^j |x_n - t|} \Big(\|\widehat{\phi_j}(\xi')(\partial^\beta (u_1 - w_1)(\partial^\beta u_1)^2)^\wedge(\xi', t) \parallel_p \\
&\qquad + \|\widehat{\phi_j}(\xi')(\partial^\beta (u_1 - w_1)(\partial^\beta u_1)(\partial^\beta w_1))^\wedge(\xi', t) \parallel_p \\
&\qquad + \|\widehat{\phi_j}(\xi')(\partial^\beta (u_1 - w_1)(\partial^\beta w_1)^2)^\wedge(\xi', t) \parallel_p \Big) \mathrm{d}t \\
&=: J_1 + J_2 + J_3. \qquad (4.12)
\end{aligned}$$

Let us provide an estimate for J_2. Considering $z = \partial^\beta(u_1 - w_1)$, $v = \partial^\beta w_1$, and $w = \partial^\beta u_1$ in (4.3), we have the corresponding parcels D_i's. We are going to show how to handle D_1. The others can be treated similarly, being left to the reader. We have that

$$2^{(s_1-1)j} \int_0^\infty \mathrm{e}^{-2\pi 2^j |x_n - t|} D_1 \mathrm{d}t \leqslant C 2^{(s_1-1)j} \int_0^\infty \mathrm{e}^{-2\pi 2^j |x_n - t|} (D_1^1 + D_1^2 + D_1^3 + D_1^4) \mathrm{d}t.$$

For the terms D_1^i's, due to the triple product in (4.3), we use Young inequality in L^p and (2.10) twice (see e.g. (4.4)), as well as (4.8) with $\rho_1 = 3$, in order to estimate

$$\begin{aligned}
&2^{(s_1-1)j} \int_0^\infty \mathrm{e}^{-2\pi 2^j |x_n - t|} D_1^1 \mathrm{d}t \\
&\leqslant C 2^{(s_1-1)j} \|u_1 - w_1\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}} \|u_1\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}} \int_0^\infty t^{-3d} \mathrm{e}^{-2\pi 2^j |x_n - t|} \mathrm{d}t \\
&\quad \times \sum_{|k-j| \leqslant 3} 2^{k[1-d-(1/2)\beta]} \sum_{|m-k| \leqslant 3} \sum_{\eta=-\infty}^{m-3} 2^{\eta[1-d-(1/2)\beta]} 2^{m\beta} \sup_{t>0} t^d \parallel (\widehat{\phi}_m \widehat{w_1})(\cdot, t) \parallel_p \\
&\leqslant C x_n^{1-3d-M} \|u_1 - w_1\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}} \|u_1\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}} \sum_{|k-j| \leqslant 3} 2^{(j-k)(s_1-1-M)} \\
&\quad \times 2^{k[2-2d-1-M]} \sum_{|m-k| \leqslant 3} 2^{(k-m)s_1} 2^{m s_1} \sup_{t>0} t^d \parallel (\widehat{\phi}_m \widehat{w_1})(\cdot, t) \parallel_p,
\end{aligned}$$

$$\begin{aligned}
&2^{(s_1-1)j} \int_0^\infty \mathrm{e}^{-2\pi 2^j |x_n - t|} D_1^2 \mathrm{d}t \\
&\leqslant C x_n^{1-3d-M} \|u_1 - w_1\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}} \|w_1\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}} \sum_{|k-j| \leqslant 3} 2^{(j-k)(s_1-1-M)} \\
&\quad \times 2^{k[2-2d-1-M]} \sum_{|k-m| \leqslant 3} 2^{(k-m)s_1} 2^{m s_1} \sup_{t>0} t^d \parallel \widehat{\phi}_m \widehat{u_1}(\cdot, t) \parallel_p,
\end{aligned}$$

$$
\begin{aligned}
&2^{(s_1-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j|x_n-t|}D_1^3\mathrm{d}t\\
&\quad\leqslant Cx_n^{1-3d-M}\left\|u_1-w_1\right\|_{\mathcal{L}_d^\infty\mathcal{FB}^{s_1}_{p,\infty}}\left\|u_1\right\|_{\mathcal{L}_d^\infty\mathcal{FB}^{s_1}_{p,\infty}}\sum_{|k-j|\leqslant 3}2^{(j-k)(s_1-1-M)}\\
&\quad\times 2^{k[2-2d-1-M]}\sum_{|k-m|<5}2^{(k-m)s_1}2^{ms_1}\sup_{t>0}t^d\parallel \widehat{\phi}_m\widehat{w_1}(\cdot,t)\parallel_p,
\end{aligned}
$$

and

$$
\begin{aligned}
&2^{(s_1-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j|x_n-t|}D_1^4\mathrm{d}t\\
&\quad\leqslant Cx_n^{1-3d-M}\left\|u_1-w_1\right\|_{\mathcal{L}_d^\infty\mathcal{FB}^{s_1}_{p,\infty}}\left\|u_1\right\|_{\mathcal{L}_d^\infty\mathcal{FB}^{s_1}_{p,\infty}}\sum_{|k-j|\leqslant 3}2^{(j-k)(s_1-1-M)}\\
&\quad\times 2^{k[2-2d-1-M]}\sum_{m=k-3}^{\infty}2^{(k-m)s_1}2^{ms_1}\sup_{t>0}t^d\parallel (\widehat{\phi}_m\widehat{w_1})(\cdot,t)\parallel_p .
\end{aligned}
$$

Thus, considering the similar estimates for D_2, D_3, D_4 and taking $M=1-2d$ yield

$$
\begin{aligned}
\sup_{j\in\mathbb{Z}}\sup_{x_n>0}x_n^d\ J_2 &= C\sup_{j\in\mathbb{Z}}2^{(s_1-1)j}\sup_{x_n>0}x_n^d\int_0^\infty \mathrm{e}^{-2\pi 2^j|x_n-t|}\\
&\qquad\times\|\widehat{\phi_j}(\xi')(\partial^\beta(u_1-w_1)(\partial^\beta u_1)(\partial^\beta w_1))^\wedge(\xi',t)\parallel_p \mathrm{d}t\\
&\leqslant C\left\|u_1-w_1\right\|_{\mathcal{Y}}\left\|u_1\right\|_{\mathcal{Y}}\left\|w_1\right\|_{\mathcal{Y}},
\end{aligned}\tag{4.13}
$$

where we use $1-d-\beta/2>0$ and $s_1>2-2d$. Following the same reasoning, we can also show that

$$
\begin{aligned}
\sup_{j\in\mathbb{Z}}\sup_{x_n>0}x_n^d\ J_1 &= C\sup_{j\in\mathbb{Z}}2^{(s_1-1)j}\sup_{x_n>0}x_n^d\int_0^\infty \mathrm{e}^{-2\pi 2^j|x_n-t|}\\
&\qquad\times\|\widehat{\phi_j}(\xi')(\partial^\beta(u_1-w_1)(\partial^\beta u_1)^2)^\wedge(\xi',t)\parallel_p \mathrm{d}t\\
&\leqslant C\left\|u_1-w_1\right\|_{\mathcal{Y}}\left\|u_1\right\|_{\mathcal{Y}}^2
\end{aligned}\tag{4.14}
$$

and

$$
\begin{aligned}
\sup_{j\in\mathbb{Z}}\sup_{x_n>0}x_n^dJ_3 &= C\sup_{j\in\mathbb{Z}}2^{(s_1-1)j}\sup_{x_n>0}x_n^d\int_0^\infty \mathrm{e}^{-2\pi 2^j|x_n-t|}\\
&\qquad\times\|\widehat{\phi_j}(\xi')(\partial^\beta(u_1-w_1)(\partial^\beta w_1)^2)^\wedge(\xi',t)\parallel_p \mathrm{d}t\\
&\leqslant C\left\|u_1-w_1\right\|_{\mathcal{Y}}\left\|w_1\right\|_{\mathcal{Y}}^2 .
\end{aligned}\tag{4.15}
$$

Considering (4.13), (4.14), (4.15) in (4.12), we obtain (4.9) with $\rho_1=3$. The general case follows by proceeding as above and employing an induction argument for $\rho_1\geqslant 2$ even and $\rho_1\geqslant 3$ odd. □

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In the next lemma, we develop estimates for the trace-type operator I_2 from $\mathcal{Y}$ to $\mathcal{Z}$.

LEMMA 4.3. *Let s_1, $s_2 \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$, $\rho_1 \geqslant 2$ integer, $\beta \geqslant 0$ and let $d > 0$ be such that*

$$d < \min\left\{\frac{1}{\rho_1 - 1}, \frac{2-\beta}{\rho_1 - 1}\right\}, \quad s_2 > 2 - \rho_1 d$$

and

$$s_1 - d + \frac{n-1}{p} = s_2 + \frac{(n-1)}{p} = (n-1) - \frac{2-\rho_1\beta}{\rho_1 - 1}. \tag{4.16}$$

Then, there exists a constant $C > 0$ such that

$$\|I_2(u_1) - I_2(w_1)\|_{\mathcal{Z}} \leqslant C\,\|u_1 - w_1\|_{\mathcal{Y}} \sum_{i=0}^{\rho_1 - 1} \|u_1\|_{\mathcal{Y}}^{\rho_1 - 1 - i}\,\|w_1\|_{\mathcal{Y}}^{i}, \tag{4.17}$$

for all u_1, $w_1 \in \mathcal{Y}$.

Proof. Again we show (4.17) in the cases $\rho_1 = 2$ and $\rho_1 = 3$. The general case follows by induction for $\rho_1 \geqslant 2$ even and $\rho_1 \geqslant 3$ odd.

Starting with $\rho_1 = 2$, for each $j \in \mathbb{Z}$ we can estimate

$$\begin{aligned} &2^{s_2 j}\,\|\,\widehat{\phi_j}(\xi')\int_0^\infty G(\xi',0,t)[K_1((\partial^\beta u_1)^2)^\wedge(\xi',t) - K_1((\partial^\beta w_1)^2)^\wedge(\xi',t)]\mathrm{d}t\,\|_p \\ &\quad \leqslant C2^{(s_2-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j t}\,\|\,\widehat{\phi_j}(\xi')[\partial^\beta(u_1 - w_1)\partial^\beta(u_1 + w_1)]^\wedge(\xi',t)\,\|_p\,\mathrm{d}t. \end{aligned} \tag{4.18}$$

Considering $w = \partial^\beta(u_1 - w_1)(\cdot, t)$ and $v = \partial^\beta(u_1 + w_1)(\cdot, t)$, decomposition (4.2) leads us to

$$\text{R.H.S. of (4.18)} \leqslant C2^{(s_2-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j t}(B_1 + B_2 + B_3 + B_4)\mathrm{d}t. \tag{4.19}$$

For the parcel with B_1, we proceed as follows:

$$\begin{aligned} &2^{(s_2-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j t}B_1\mathrm{d}t \\ &\leqslant C2^{(s_2-1)j}\,\|u_1 - w_1\|_{\mathcal{L}_d^\infty \mathcal{FB}_{p,\infty}^{s_1}}\int_0^\infty t^{-2d}\mathrm{e}^{-2\pi 2^j t}\mathrm{d}t \\ &\qquad \times \sum_{|k-j|\leqslant 3} 2^{k(2-d-\beta)}2^{k\beta}\sup_{t>0} t^d\,\|\,\widehat{\phi}_k(\cdot)(u_1 + w_1)^\wedge(\cdot,t)\,\|_p \\ &\leqslant C\,\|\,u_1 - w_1\,\|_{\mathcal{L}_d^\infty \mathcal{FB}_{p,\infty}^{s_1}} \\ &\quad \times \sum_{|k-j|\leqslant 3} 2^{(j-k)(s_2+2d-2)}2^{ks_1}\cdot\sup_{t>0} t^d\,\|\,\widehat{\phi}_k(\cdot)(u_1 + w_1)^\wedge(\cdot,t)\,\|_p, \end{aligned} \tag{4.20}$$

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where above we use that $\int_0^\infty t^{-2d}\mathrm{e}^{-2\pi 2^j t}\mathrm{d}t \leqslant C2^{j(2d-1)}\Gamma(1-2d) \leqslant C2^{j(2d-1)}$ and (4.16) with $\rho_1 = 2$. For the terms with B_2, B_3, and B_4, we have the estimates

$$2^{(s_2-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j t}B_2\mathrm{d}t \leqslant C \parallel u_1 + w_1 \parallel_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}^{s_1}_{p,\infty}} \sum_{|k-j|\leqslant 3} 2^{(j-k)(s_2+2d-2)}2^{ks_1} \times \sup_{t>0} t^d \parallel \widehat{\phi}_k(\cdot)(u_1 - w_1)^\wedge(\cdot,t) \parallel_p, \tag{4.21}$$

$$2^{(s_2-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j t}B_3\mathrm{d}t \leqslant C \parallel u_1 - w_1 \parallel_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}^{s_1}_{p,\infty}} \sum_{|l-j|<5} 2^{(j-l)(s_2+2d-2)}2^{ls_1} \times \sup_{t>0} t^d \parallel \widehat{\phi}_l(\cdot)(u_1 + w_1)^\wedge(\cdot,t) \parallel_p, \tag{4.22}$$

and

$$2^{(s_2-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j t}B_4\mathrm{d}t \leqslant C \parallel u_1 - w_1 \parallel_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}^{s_1}_{p,\infty}} \sum_{l=j-3}^{\infty} 2^{(s_2-2+2d)(j-l)}2^{ls_1} \times \sup_{t>0} t^d \parallel \widehat{\phi}_l(\cdot)(u_1 + w_1)^\wedge(\cdot,t) \parallel_p . \tag{4.23}$$

Inserting (4.20)–(4.23) into (4.19), taking the supremum over $j \in \mathbb{Z}$, and then applying Young inequality for discrete convolutions, the resulting estimate is

$$\|I_2(u_1) - I_2(w_1)\|_{\mathcal{F}\mathcal{B}^{s_2}_{p,\infty}(\mathbb{R}^{n-1})} \leqslant C\,\|u_1 - w_1\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}^{s_1}_{p,\infty}(\mathbb{R}^n_+)}\,\|u_1 + w_1\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}^{s_1}_{p,\infty}(\mathbb{R}^n_+)},$$

which implies (4.17) with $\rho_1 = 2$. Note that above we have used that $2 - d - \beta > 0$ and $s_2 > 2 - 2d$ for the convergence of the corresponding series.

We conclude by performing the proof for $\rho_1 = 3$. In this case, we can split

$$\begin{aligned} &2^{s_2 j} \parallel \widehat{\phi_j}(\xi')\int_0^\infty G(\xi',0,t)[K_1(\partial^\beta u_1)^3)^\wedge - K_1(\partial^\beta w_1)^3)^\wedge(\xi',t)\mathrm{d}t \parallel_p \\ &\leqslant C2^{(s_2-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j t}\Big(\|\widehat{\phi_j}(\xi')(\partial^\beta(u_1-w_1)(\partial^\beta u_1)^2)^\wedge(\xi',t)\parallel_p \\ &\qquad + \|\widehat{\phi_j}(\xi')(\partial^\beta(u_1-w_1)(\partial^\beta u_1)(\partial^\beta w_1))^\wedge(\xi',t)\parallel_p \\ &\qquad + \|\widehat{\phi_j}(\xi')(\partial^\beta(u_1-w_1)(\partial^\beta w_1)^2)^\wedge(\xi',t)\parallel_p\Big)\,\mathrm{d}t \\ &=: J_1 + J_2 + J_3. \end{aligned} \tag{4.24}$$

In what follows, we explain how to estimate J_1. For $z = \partial^\beta(u_1 - w_1)$, $v = \partial^\beta u_1$, and $w = \partial^\beta u_1$ in (4.3), we have the decomposition of $\parallel \widehat{\phi_j}(\xi')\widehat{zvw} \parallel_p$ in terms of D_i's. Below, we treat the term D_1. The other ones D_2, D_3, D_4 can be estimated

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similarly. In this direction, bearing in mind (4.16) with $\rho_1 = 3$, we have that

$$2^{s_2 j}\int_0^\infty \mathrm{e}^{-2\pi 2^j t} D_1 \mathrm{d}t \leqslant C 2^{(s_2-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j t}(D_1^1 + D_1^2 + D_1^3 + D_1^4)\mathrm{d}t,$$

with the respective estimates for the parcels with D_1^1, D_1^2, D_1^3, and D_1^4:

$$\begin{aligned}
&2^{(s_2-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j t} D_1^1 \mathrm{d}t \\
&\quad\leqslant C\left\|u_1 - w_1\right\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}}\left\|u_1\right\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}}\sum_{|k-j|\leqslant 3} 2^{(j-k)(s_2-2+3d)} \\
&\qquad\times \sum_{|m-k|\leqslant 3} 2^{(k-m)(s_2+2d-1-(1/2)\beta)} 2^{ms_1}\sup_{t>0} t^d \left\| \widehat{\phi}_m(\cdot)\widehat{u_1}(\cdot,t)\right\|_p, \\
&2^{(s_2-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j t} D_1^2 \mathrm{d}t \\
&\quad\leqslant C\left\|u_1 - w_1\right\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}}\left\|u_1\right\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}}\sum_{|k-j|\leqslant 3} 2^{(j-k)(s_2-2+3d)} \\
&\qquad\times \sum_{|k-m|\leqslant 3} 2^{(k-m)(s_2+2d-1-(1/2)\beta)} 2^{ms_1}\sup_{t>0} t^d \left\| \widehat{\phi}_m(\cdot)\widehat{u_1}(\cdot,t)\right\|_p, \\
&2^{(s_2-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j t} D_1^3 \mathrm{d}t \\
&\quad\leqslant C\left\|u_1 - w_1\right\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}}\left\|u_1\right\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}}\sum_{|k-j|\leqslant 3} 2^{(j-k)(s_2-2+3d)} \\
&\qquad\times \sum_{|k-m|<5} 2^{(k-m)(s_2+2d-1-(1/2)\beta)} 2^{ms_1}\sup_{t>0} t^d \left\| \widehat{\phi}_m(\cdot)\widehat{u_1}(\cdot,t)\right\|_p,
\end{aligned}$$

and

$$\begin{aligned}
&2^{(s_2-1)j}\int_0^\infty \mathrm{e}^{-2\pi 2^j t} D_1^4 \mathrm{d}t \\
&\quad\leqslant C\left\|u_1 - w_1\right\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}}\left\|u_1\right\|_{\mathcal{L}_d^\infty \mathcal{F}\mathcal{B}_{p,\infty}^{s_1}}\sum_{|k-j|\leqslant 3} 2^{(j-k)(s_2-2+3d)} \\
&\qquad\times \sum_{m=k-3}^{\infty} 2^{(k-m)(s_2+2d-1-(1/2)\beta)} 2^{ms_1}\sup_{t>0} t^d \left\| \widehat{\phi}_m(\cdot)\widehat{u_1}(\cdot,t)\right\|_p .
\end{aligned}$$

Now, considering the similar estimates for D_2, D_3, D_4 and recalling the conditions $1 - d - \beta/2 > 0$ and $s_2 > 2 - 3d$, we arrive at

$$\sup_{j\in\mathbb{Z}} J_1 \leqslant C\left\|u_1 - w_1\right\|_{\mathcal{Y}}\left\|u_1\right\|_{\mathcal{Y}}^2 . \tag{4.25}$$

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For J_2 and J_3, proceeding as in the proof of (4.25), but with $v = \partial^\beta u_1$, $w = \partial^\beta w_1$ and $v = \partial^\beta w_1$, $w = \partial^\beta w_1$ instead of $v = \partial^\beta u_1$, $w = \partial^\beta u_1$, we obtain that

$$\sup_{j\in\mathbb{Z}} J_2 \leqslant C\,\|u_1 - w_1\|_{\mathcal{Y}}\,\|u_1\|_{\mathcal{Y}}\,\|w_1\|_{\mathcal{Y}} \quad \text{and} \quad \sup_{j\in\mathbb{Z}} J_3 \leqslant C\,\|u_1 - w_1\|_{\mathcal{Y}}\,\|w_1\|_{\mathcal{Y}}^2 . \tag{4.26}$$

Estimate (4.17) with $\rho_1 = 3$ follows by taking the supremum over $j \in \mathbb{Z}$ in both sides of (4.24) and then considering (4.25) and (4.26). □

The subject of the following lemma are estimates for the operators $T_1 : \mathcal{Z} \to \mathcal{Y}$ and $T_2 : \mathcal{Z} \to \mathcal{Z}$.

Lemma 4.4. *Let s_1, $s_2 \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$, $\rho_2 \geqslant 2$ integer, $\beta \geqslant 0$ and $d > 0$.*

(i) *Assuming that*

$$s_1 > 1 + d \quad \text{and} \quad s_1 - d + \frac{n-1}{p} = s_2 + \frac{(n-1)}{p} = (n-1) - \frac{1}{\rho_2 - 1}, \tag{4.27}$$

we have the estimate

$$\|T_1(u_2) - T_1(w_2)\|_{\mathcal{Y}} \leqslant C\,\|u_2 - w_2\|_{\mathcal{Z}} \sum_{i=0}^{\rho_2 - 1} \|u_2\|_{\mathcal{Z}}^{\rho_2 - 1 - i}\,\|w_2\|_{\mathcal{Z}}^{i}, \tag{4.28}$$

where $C > 0$ is a constant independent of u_2, $w_2 \in \mathcal{Z}$.

(ii) *Supposing that*

$$s_2 > 1 \quad \text{and} \quad s_2 + \frac{(n-1)}{p} = (n-1) - \frac{1}{\rho_2 - 1}, \tag{4.29}$$

we have the estimate

$$\| T_2(u_2) - T_2(w_2) \|_{\mathcal{Z}} \leqslant C \| u_2 - w_2 \|_{\mathcal{Z}} \sum_{i=0}^{\rho_2 - 1} \| u_2 \|_{\mathcal{Z}}^{\rho_2 - 1 - i} \| w_2 \|_{\mathcal{Z}}^{i}, \tag{4.30}$$

where $C > 0$ is a constant independent of u_2, $w_2 \in \mathcal{Z}$.

Proof. For (4.28), considering the basic cases $\rho_2 = 2$ and $\rho_2 = 3$ and proceeding by induction, this time we need to handle the expressions

$$\begin{aligned} &2^{s_1 j} \| \widehat{\phi_j}(\xi') G(\xi', x_n, 0)[(K_2(u_2)^2)^\wedge - (K_2(w_2)^2)^\wedge](\xi') \|_p \\ &\quad \leqslant C 2^{(s_1 - 1)j} \mathrm{e}^{-2\pi 2^j x_n} \| \widehat{\phi_j}[(u_2 - w_2)(u_2 + w_2)]^\wedge \|_p \end{aligned}$$

and

$$\begin{aligned} &2^{s_1 j} \| \widehat{\phi_j}(\xi') G(\xi', x_n, 0)[(K_2(u_2)^3)^\wedge - (K_1(w_2)^3)^\wedge](\xi') \|_p \\ &\quad \leqslant C 2^{(s_1 - 1 - d)j} x_n^{-d} \Big(\| \widehat{\phi_j}(u_2 - w_2)(u_2)^2 \|_p \\ &\qquad + \|\widehat{\phi_j}(u_2 - w_2) u_2 w_2 \|_p + \|\widehat{\phi_j}(u_2 - w_2)(w_2)^2 \|_p \Big), \end{aligned}$$

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instead of (4.10) and (4.12), respectively. For that, we can employ decompositions (4.2) and (4.3) and proceed as in the proof of lemma 4.2 with a slight adaptation of the arguments. The same follows for (4.30) but proceeding as in lemma 4.3. We leave the details to the reader. □

We finish this subsection by treating the operators N_1 and N_2 that depend on the boundary potential V.

LEMMA 4.5. *Let s_1, s_2, $\tilde{s} \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$ and $d > 0$ satisfy (3.18) and (3.15) with $\gamma = (2-\rho_1\beta)/(\rho_1 - 1)$. Let $V \in \mathcal{FB}^{\tilde{s}}_{p,\infty}$ and suppose further that $s_1 > 1 + d$. Then, there exists a constant $C > 0$ such that*

$$\| N_1(u_2) - N_1(w_2) \|_{\mathcal{Y}} \leqslant C \|V\|_{\mathcal{FB}^{\tilde{s}}_{p,\infty}} \|u_2 - w_2\|_{\mathcal{Z}}\,, \tag{4.31}$$

$$\| N_2(u_2) - N_2(w_2) \|_{\mathcal{Z}} \leqslant C \| V \|_{\mathcal{FB}^{\tilde{s}}_{p,\infty}} \| u_2 - w_2 \|_{\mathcal{Z}}, \tag{4.32}$$

for all u_2, $w_2 \in \mathcal{Z}$.

Proof. For each $j \in \mathbb{Z}$, we have the estimate:

$$2^{s_1 j} \| \widehat{\phi_j} G(\xi', x_n, 0)[(\widehat{Vu_2}) - (\widehat{Vw_2})] \|_p \leqslant C 2^{(s_1-1-d)j} x_n^{-d} \| \widehat{\phi_j}[V(u_2-w_2)]^\wedge \|_p\,. \tag{4.33}$$

Considering $w = V$ and $v = u_2 - w_2$ in (4.2) yields the decomposition

$$2^{(s_1-1-d)j} x_n^{-d} \| \widehat{\phi_j}[V(u_2-w_2)]^\wedge \|_p \leqslant C 2^{(s_1-d-1)j} x_n^{-d}(B_1+B_2+B_3+B_4). \tag{4.34}$$

Moreover, recalling (3.18) and using Young inequality and (2.10), we can handle the parcels B_i's as follows:

$$\begin{aligned}
&2^{(s_1-1-d)j} B_1 \\
&\leqslant C \sum_{|k-j|\leqslant 3} 2^{(s_1-1-d)j} \sum_{l=-\infty}^{k-3} \left\| \widehat{\phi}_l (u_2-w_2)^\wedge \right\|_1 \| \widehat{\phi}_k \widehat{V} \|_p \\
&\leqslant C \sum_{|k-j|\leqslant 3} 2^{(s_1-1-d)j} \\
&\quad \times \sum_{l=-\infty}^{k-3} 2^{[(n-1)-((n-1)/p)-s_2]l} 2^{ls_2} \left\| \widehat{\phi}_l (u_2-w_2)^\wedge \right\|_p \| \widehat{\phi}_k \widehat{V} \|_p \\
&\leqslant C \|u_2 - w_2\|_{\mathcal{FB}^{s_2}_{p,\infty}} \sum_{|k-j|\leqslant 3} 2^{(s_1-1-d)(j-k)} 2^{[s_1-1-d+(n-1)-((n-1)/p)-s_2]k} \| \widehat{\phi}_k \widehat{V} \|_p,
\end{aligned} \tag{4.35}$$

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$$
\begin{aligned}
&2^{(s_1-1-d)j}B_2\\
&\leqslant C\sum_{|k-j|\leqslant 3}2^{(s_1-1-d)j}\sum_{l=-\infty}^{k-3}\parallel \widehat{\phi}_l\widehat{V}\parallel_1\left\|\widehat{\phi}_k(u_2-w_2)^\wedge\right\|_p\\
&\leqslant C\parallel V\parallel_{\mathcal{FB}^{\tilde{s}}_{p,\infty}}\\
&\quad\times\sum_{|k-j|\leqslant 3}2^{(s_1-1-d)(j-k)}2^{[s_1-1-d+(n-1)-((n-1)/p)-\tilde{s}]k}\left\|\widehat{\phi}_k(u_2-w_2)^\wedge\right\|_p, \qquad (4.36)\\
&2^{(s_1-1-d)j}B_3\\
&\leqslant C\parallel V\parallel_{\mathcal{FB}^{\tilde{s}}_{p,\infty}}\\
&\quad\times\sum_{|l-j|<5}2^{(s_1-1-d)(j-l)}2^{[s_1-1-d+(n-1)-((n-1)/p)-\tilde{s}]l}\left\|\widehat{\phi}_l(u_2-w_2)^\wedge\right\|_p, \qquad (4.37)
\end{aligned}
$$

and

$$
\begin{aligned}
&2^{(s_1-1-d)j}B_4\\
&\leqslant C\parallel V\parallel_{\mathcal{FB}^{\tilde{s}}_{p,\infty}}\sum_{l=j-3}^{\infty}2^{(s_1-1-d)(j-l)}2^{[s_1-1-d+(n-1)-((n-1)/p)-\tilde{s}]l}\left\|\widehat{\phi}_l(u_2-w_2)^\wedge\right\|_p. \qquad (4.38)
\end{aligned}
$$

Now, in view of the condition $s_1-d>1$, estimate (4.31) follows by multiplying (4.33) by x_n^d, using (4.35)–(4.38) to estimate the R.H.S. of (4.33), and taking the supremum over $x_n>0$, and then over $j\in\mathbb{Z}$.

Finally, for estimate (4.32), we have that $s_2=s_1-d>1$ and

$$
2^{s_2j}\parallel\widehat{\phi_j}G(\xi',0,0)[(\widehat{Vu_2})-(\widehat{Vw_2})]\parallel_p\leqslant C2^{(s_2-1)j}\parallel\widehat{\phi_j}[V(u_2-w_2)]^\wedge\parallel_p,
$$

whose R.H.S. can be handled similar to that of (4.33) with some slight modifications of the arguments. We omit the details, leaving them to the reader. □

4.2. Regularity estimates in Fourier–Sobolev spaces

This subsection is devoted to presenting some regularity estimates for the terms in (3.11). With this in mind, in addition to the spaces $\mathcal{Y}$ and $\mathcal{Z}$, here we shall employ the spaces $H^{1,s}=H^{1,s}(\mathbb{R}^{n-1})$ and $H^{1,s}_d=H^{1,s}_d(\mathbb{R}^n_+)$ defined in (2.2) and (3.23).

Let $R>0$ be fixed but arbitrary. Assume the same hypotheses of theorem 3.2. Suppose also that $\rho_1\geqslant 4$, $\rho_2<\rho_1$, $d<\min\{1/(\rho_1+1),(2-\rho_1\beta)/(\rho_1-1),(1-\beta)/(\rho_1)\}$, and $s\geqslant\beta$. Let $V\in\mathcal{FB}^{\tilde{s}}_{p,\infty}\cap H^{1,s}$.

Then, there exists a universal constant $C>0$ (independent of R and V) such that the following estimates hold true for the components of the operators

$L(f)$, $I(u)$, $T(u)$, and $N(u)$, respectively:

$$\| L_1(f) \|_{H^{1,s}_d} \leqslant C\frac{1}{R^{d+1}} \| f \|_{H^{1,s}} + C(1+R^s)R^{(2-\rho_1\beta)/(\rho_1-1)} \| f \|_{\mathcal{FB}^{\bar{s}}_{p,\infty}}, \quad \forall f \in \mathcal{FB}^{\bar{s}}_{p,\infty} \cap H^{1,s}; \tag{4.39}$$

$$\| L_2(f) \|_{H^{1,s}} \leqslant C\frac{1}{R} \| f \|_{H^{1,s}} + C(1+R^s)R^{(2-\rho_1\beta)/(\rho_1-1)+d} \| f \|_{\mathcal{FB}^{\bar{s}}_{p,\infty}}, \quad \forall f \in \mathcal{FB}^{\bar{s}}_{p,\infty} \cap H^{1,s}; \tag{4.40}$$

$$\begin{aligned} &\| I_1(u_1) - I_1(w_1) \|_{H^{1,s}_d} \\ &\leqslant C\frac{1}{R^{2-\beta-(\rho_1-1)d}} \| u_1 - w_1 \|_{H^{1,s}_d} \sum_{i=0}^{\rho_1-1} \| u_1 \|^{\rho_1-1-i}_{H^{1,s}_d} \| w_1 \|^{i}_{H^{1,s}_d} \\ &\quad + C(1+R^s)R^{(2-\rho_1\beta)/(\rho_1-1)} \| u_1 - w_1 \|_{\mathcal{Y}} \\ &\quad \times \sum_{i=0}^{\rho_1-1} \| u_1 \|^{\rho_1-1-i}_{\mathcal{Y}} \| w_1 \|^{i}_{\mathcal{Y}}, \quad \forall u_1, w_1 \in \mathcal{Y} \cap H^{1,s}_d; \end{aligned} \tag{4.41}$$

$$\begin{aligned} &\| I_2(u_1) - I_2(w_1) \|_{H^{1,s}} \\ &\leqslant C\frac{1}{R^{2-\beta-ad}} \| u_1 - w_1 \|_{H^{1,s}_d} \sum_{i=0}^{\rho_1-1} \| u_1 \|^{\rho_1-1-i}_{H^{1,s}_d} \| w_1 \|^{i}_{H^{1,s}_d} \\ &\quad + C(1+R^s)R^{(2-\rho_1\beta)/(\rho_1-1)+d} \| u_1 - w_1 \|_{\mathcal{Y}} \\ &\quad \times \sum_{i=0}^{\rho_1-1} \| u_1 \|^{\rho_1-1-i}_{\mathcal{Y}} \| w_1 \|^{i}_{\mathcal{Y}}, \quad \forall u_1, w_1 \in \mathcal{Y} \cap H^{1,s}_d; \end{aligned} \tag{4.42}$$

$$\begin{aligned} &\| T_1(u_2) - T_1(w_2) \|_{H^{1,s}_d} \\ &\leqslant C\frac{1}{R^{l+1}} \| u_2 - w_2 \|_{H^{1,s}} \sum_{i=0}^{\rho_2-1} \| u_2 \|^{\rho_2-1-i}_{H^{1,s}} \| w_2 \|^{i}_{H^{1,s}} \\ &\quad + (1+R^s)R^{(2-\rho_1\beta)/(\rho_1-1)} \| u_2 - w_2 \|_{\mathcal{Z}} \\ &\quad \times \sum_{i=0}^{\rho_2-1} \| u_2 \|^{\rho_2-1-i}_{\mathcal{Z}} \| w_2 \|^{i}_{\mathcal{Z}}, \quad \forall u_2, w_2 \in \mathcal{Z} \cap H^{1,s}; \end{aligned} \tag{4.43}$$

$$\begin{aligned} &\| T_2(u_2) - T_2(w_2) \|_{H^{1,s}} \\ &\leqslant C\frac{1}{R} \| u_2 - w_2 \|_{H^{1,s}} \sum_{i=0}^{\rho_2-1} \| u_2 \|^{\rho_2-1-i}_{H^{1,s}} \| w_2 \|^{i}_{H^{1,s}} \end{aligned}$$

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$$+ C(1+R^s)R^{((2-\rho_1\beta)/(\rho_1-1))+d} \parallel u_2 - w_2 \parallel_Z \times \sum_{i=0}^{\rho_2-1} \parallel u_2 \parallel_Z^{\rho_2-1-i} \parallel w_2 \parallel_Z^i, \quad \forall u_2, w_2 \in \mathcal{Z} \cap H^{1,s}; \tag{4.44}$$

$$\begin{aligned} &\parallel N_1(u_2) - N_1(w_2) \parallel_{H_d^{1,s}} \\ &\quad \leqslant C\frac{1}{R^{d+1}} \parallel V \parallel_{H^{1,s}} \parallel u_2 - w_2 \parallel_{H^{1,s}} \\ &\qquad + C(1+R^s)R^{(2-\rho_1\beta)/(\rho_1-1)} \parallel V \parallel_{\mathcal{FB}^{\tilde{s}}_{p,\infty}} \parallel u_2 - w_2 \parallel_{\mathcal{Z}}, \quad \forall u_2, w_2 \in \mathcal{Z} \cap H^{1,s}; \end{aligned} \tag{4.45}$$

$$\begin{aligned} &||N_2(u_2) - N_2(w_2) \parallel_{H^{1,s}} \\ &\quad \leqslant C\frac{1}{R} \parallel V \parallel_{H^{1,s}} \parallel u_2 - w_2 \parallel_{H^{1,s}} \\ &\qquad + C(1+R^s)R^{(2-\rho_1\beta)/(\rho_1-1)+d} \parallel V \parallel_{\mathcal{FB}^{\tilde{s}}_{p,\infty}} \parallel u_2 - w_2 \parallel_{\mathcal{Z}}, \quad \forall u_2, w_2 \in \mathcal{Z} \cap H^{1,s}. \end{aligned} \tag{4.46}$$

For reasons of length of the paper, we prove two of the eight estimates above. More precisely, we show estimates (4.39) and (4.45). The others can be proved by adapting the proof developed for (4.39)–(4.45), as well as employing some of the arguments presented in § 4.1. The details are left to the reader.

4.2.1. Proof of estimate (4.39) Using $R > 0$ to split the integral within the $H_d^{1,s}$-norm in low and high frequencies, we obtain that

$$\begin{aligned} x_n^d \parallel (1+|\xi'|^s)\widehat{L(f)} \parallel_1 &= x_n^d \parallel (1+|\xi'|^s)G(\xi', x_n, 0)\widehat{f} \parallel_1 \\ &\leqslant x_n^d \int_{|\xi'|\leqslant R} |(1+|\xi'|^s)G(\xi', x_n, 0)\widehat{f}(\xi')|\mathrm{d}\xi' \\ &\quad + x_n^d \int_{|\xi'|>R} |(1+|\xi'|^s)G(\xi', x_n, 0)\widehat{f}(\xi')|\mathrm{d}\xi' \\ &=: P_1 + P_2. \end{aligned} \tag{4.47}$$

For the parcel P_1, using (2.10) and recalling (3.18), we have that

$$\begin{aligned} P_1 &\leqslant C(1+R^s) \sum_{j\leqslant 1+\lfloor \log_2 R \rfloor} 2^{-j} x_n^d \mathrm{e}^{-2\pi 2^j x_n} \parallel \widehat{\phi_j}\widehat{f}(\xi') \parallel_1 \\ &\leqslant C(1+R^s) \\ &\quad \times \sum_{j\leqslant 1+\lfloor \log_2 R \rfloor} (2^j x_n)^d \mathrm{e}^{-2\pi 2^j x_n} 2^{\bar{s}j} \parallel \widehat{\phi_j}\widehat{f}(\xi') \parallel_p 2^{[(n-1)-((n-1)/p)-1-\bar{s}-d]j} \\ &\leqslant C(1+R^s)R^{(2-\rho_1\beta)/(\rho_1-1)} \parallel f \parallel_{\mathcal{FB}^{\bar{s}}_{p,\infty}}, \end{aligned} \tag{4.48}$$

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because $(n-1)-((n-1)/p)-1-\overline{s}-d=((2-\rho_1\beta)/(\rho_1-1))-d>0$. For the high frequency part, we can estimate

$$P_2 \leqslant CR^{-1}x_n^d \int_{|\xi'|>R} \mathrm{e}^{-2\pi R x_n}(1+|\xi'|^s)|\widehat{f}|\mathrm{d}\xi' \leqslant CR^{-1-d} \parallel f \parallel_{H^{1,s}} . \tag{4.49}$$

Now, we obtain (4.47) by taking the supremum over $x_n>0$ in both sides of (4.47) and using (4.48) and (4.49).

4.2.2. Proof of estimate (4.45) For $R>0$, we can estimate

$$\begin{aligned} & x_n^d \,\|(1+|\xi'|^s)(N_1(u_2)-N_1(w_2))\|_1 \\ & \quad \leqslant x_n^d \parallel (1+|\xi'|^s)G(\xi',x_n,0)[V(u_2-w_2)]^{\wedge}(\xi') \parallel_1 \\ & \quad \leqslant x_n^d \int_{|\xi'|\leqslant R}(1+|\xi'|^s)\,|G(\xi',x_n,0)[V(u_2-w_2)]^{\wedge}(\xi')|\,\mathrm{d}\xi' \\ & \qquad + x_n^d \int_{|\xi'|>R}(1+|\xi'|^s)\,|G(\xi',x_n,0)[V(u_2-w_2)]^{\wedge}(\xi')|\,\mathrm{d}\xi' \\ & \quad =: P_1+P_2. \end{aligned} \tag{4.50}$$

The integral P_1 can be handled as follows:

$$P_1 \leqslant C(1+R^s) \sum_{j\leqslant 1+\lfloor \log_2 R\rfloor} (2^j x_n)^d \mathrm{e}^{-2\pi 2^j x_n} \parallel \widehat{\phi_j}[V(u_2-w_2)]^{\wedge}(\xi') \parallel_1 \times 2^{[(n-1)-((n-1)/p)-1-d]j}.$$

Taking $w=V$ and $v=u_2-w_2$ in (4.2), proceeding as in (4.34) and recalling $\tilde{s}=(n-1)-((n-1)/p)-1$, we get

$$2^{(\tilde{s}-d)j} \parallel \widehat{\phi_j}[V(u_2-w_2)]^{\wedge} \parallel_p \leqslant B_1+B_2+B_3+B_4,$$

where (see also (4.35))

$$\begin{aligned} B_1 & \leqslant C \sum_{|k-j|\leqslant 3} 2^{(\tilde{s}-d)j} \sum_{l=-\infty}^{k-3} \left\| \widehat{\phi}_l (u_2-w_2)^{\wedge}\right\|_1 \parallel \widehat{\phi}_k \widehat{V} \parallel_p \\ & \leqslant C \sum_{|k-j|\leqslant 3} 2^{(\tilde{s}-d)j} \sum_{l=-\infty}^{k-3} 2^{[(n-1)-((n-1)/p)-s_2]l} 2^{ls_2} \left\| \widehat{\phi}_l (u_2-w_2)^{\wedge}\right\|_p \parallel \widehat{\phi}_k \widehat{V} \parallel_p \\ & \leqslant C\,\|u_2-w_2\|_{\mathcal{Z}} \parallel V \parallel_{\mathcal{FB}^{\tilde{s}}_{p,\infty}} \sum_{|k-j|\leqslant 3} 2^{(\tilde{s}-d)(j-k)} 2^{[\tilde{s}-d+(n-1)-((n-1)/p)-s_2-\tilde{s}]k}. \end{aligned}$$

In the same way, we can estimate the parcels B_2, B_3, and B_4 by proceeding similarly to (4.36), (4.37), and (4.38), respectively.

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Now, with the corresponding estimates for the B_i's in hand, using that $(n-1) - ((n-1)/p) - s_2 - d = \gamma - d > 0$ and $\gamma = (2-\rho_1\beta)/(\rho_1 - 1)$, it follows that

$$\begin{aligned} P_1 &\leqslant C(1+R^s)\,\|u_2 - w_2\|_{\mathcal{Z}}\,\|\,V\,\|_{\mathcal{FB}^{\tilde{s}}_{p,\infty}} \\ &\quad\times \sum_{j\leqslant 1+\lfloor \log_2 R\rfloor}\ \sum_{|k-j|\leqslant 3} 2^{(\tilde{s}-d)(j-k)} 2^{[(n-1)-((n-1)/p)-s_2-d]k} \\ &\leqslant C(1+R^s)\,\|u_2 - w_2\|_{\mathcal{Z}}\,\|\,V\,\|_{\mathcal{FB}^{\tilde{s}}_{p,\infty}} \sum_{j\leqslant 1+\lfloor \log_2 R\rfloor} 2^{[\gamma-d]j} \\ &\leqslant C(1+R^s)R^{(2-\rho_1\beta)/(\rho_1-1)}\,\|u_2 - w_2\|_{\mathcal{Z}}\,\|\,V\,\|_{\mathcal{FB}^{\tilde{s}}_{p,\infty}}. \end{aligned} \tag{4.51}$$

For the parcel P_2, we have that

$$\begin{aligned} P_2 &\leqslant CR^{-1}x_n^d \mathrm{e}^{-2\pi|R|x_n} \int_{|\xi'|>R} (1+|\xi'|^s)|[V(u_2-w_2)]^\wedge(\xi')|\mathrm{d}\xi' \\ &\leqslant CR^{-1-d}(Rx_n)^d \mathrm{e}^{-2\pi|R|x_n}\,\|V(u_2-w_2)\|_{H^{1,s}(\mathbb{R}^{n-1})} \\ &\leqslant CR^{-1-d}\,\|\,V\,\|_{H^{1,s}}\|\,u_2 - w_2\,\|_{H^{1,s}}, \end{aligned} \tag{4.52}$$

where the last pass was obtained via (2.3) and $(Rx_n)^d \mathrm{e}^{-2\pi|R|x_n} \leqslant C$ for all R, $x_n > 0$. Considering now (4.51) and (4.52) in (4.50) and then taking $\sup_{x_n>0}$, we are done.

5. Proofs

This section is devoted to the proofs of results stated in § 3.

5.1. Proof of theorem 3.2

With the estimates developed in § 4.1 in hand, we are able to employ a contraction argument and show the solvability of BVP (1.1). For that, recall the spaces $\mathcal{Y} = \mathcal{L}^\infty_d \mathcal{FB}^{s_1}_{p,\infty}(\mathbb{R}^n_+)$, $\mathcal{Z} = \mathcal{FB}^{s_2}_{p,\infty}(\mathbb{R}^{n-1})$, and $\mathcal{X} = \mathcal{Y}\times\mathcal{Z}$ with the norm $\|\cdot\|_{\mathcal{X}} = \|\cdot\|_{\mathcal{Y}} + \|\cdot\|_{\mathcal{Z}}$ (see (3.16)–(3.17)). Consider the operators

$$\Psi_1(u) = I_1(u_1) + N_1(u_2) + T_1(u_2) + L_1(f), \quad \text{for } u = (u_1, u_2) \in \mathcal{X}, \tag{5.1}$$

and

$$\Psi_2(u) = I_2(u_1) + N_2(u_2) + T_2(u_2) + L_2(f), \quad \text{for } u = (u_1, u_2) \in \mathcal{X}. \tag{5.2}$$

So, we can define $\Psi(u) = (\Psi_1(u), \Psi_2(u))$ in $\mathcal{X} = \mathcal{Y}\times\mathcal{Z}$ with the norm $\|\cdot\|_{\mathcal{X}} = \|\cdot\|_{\mathcal{Y}} + \|\cdot\|_{\mathcal{Z}}$ (see (3.16)–(3.17)).

Let ε, δ_1, $\delta_2 > 0$ be such that

$$\begin{aligned} &\rho_1\varepsilon^{\rho_1-1} + \rho_2\varepsilon^{\rho_2-1} < \frac{1}{4C}, \quad \delta_2 \leqslant \frac{1}{4C} - \rho_1\varepsilon^{\rho_1-1} - \rho_2\varepsilon^{\rho_2-1}, \\ &\text{and} \quad \delta_1 + \delta_2\varepsilon \leqslant \frac{\varepsilon}{2C} - \rho_1\varepsilon^{\rho_1} - \rho_2\varepsilon^{\rho_2}, \end{aligned} \tag{5.3}$$

where $C > 0$ is the largest constant obtained among those in lemmas 4.1 to 4.5.

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We are going to show that Ψ is a contraction in the closed ball $B_{\mathcal{X}} = \{u = (u_1, u_2) \in \mathcal{X}; \|(u_1, u_2)\|_{\mathcal{X}} \leqslant \varepsilon\}$. In view of the estimates in the aforementioned lemmas, we can handle Ψ as follows:

$$\begin{aligned} \| \Psi(u) \|_{\mathcal{X}} &\leqslant C(\rho_1 \| (u_1, u_2) \|_{\mathcal{X}}^{\rho_1} \\ &\quad + \rho_2 \| (u_1, u_2) \|_{\mathcal{X}}^{\rho_2} + \| V \|_{\mathcal{F}\mathcal{B}^{\tilde{s}}_{p,\infty}} \| (u_1, u_2) \|_{\mathcal{X}} + \| f \|_{\mathcal{F}\mathcal{B}^{\overline{s}}_{p,\infty}}) \\ &\leqslant C(\rho_1 \varepsilon^{\rho_1} + \rho_2 \varepsilon^{\rho_2} + \delta_2 \varepsilon + \delta_1) \leqslant \frac{\varepsilon}{2}, \end{aligned} \tag{5.4}$$

provided that $(u_1, u_2) \in B_{\mathcal{X}}$. It follows that Ψ maps from $B_{\mathcal{X}}$ to $B_{\mathcal{X}}$. Moreover, for $u = (u_1, u_2)$ and $w = (w_1, w_2) \in \mathcal{X}$, we have that

$$\begin{aligned} \| \Psi(u) - \Psi(w) \|_{\mathcal{X}} &\leqslant C(\delta_2 + \rho_1 \varepsilon^{\rho_1 - 1} + \rho_2 \varepsilon^{\rho_2 - 1}) \| (u_1 - w_1, u_2 - w_2) \|_{\mathcal{X}} \\ &\leqslant \frac{1}{4} \| u - w \|_{\mathcal{X}}, \end{aligned} \tag{5.5}$$

which gives the contraction property for Ψ. Then, by the contraction mapping principle, there exists a unique solution $u \in \mathcal{X}$ for (3.11) $(u = \Psi(u))$ satisfying $\|u\|_{\mathcal{X}} \leqslant \varepsilon$.

In the sequel, we show the continuity of the data-solution map. Let $u = (u_1, u_2)$ and $w = (w_1, w_2)$ be solutions in $B_{\mathcal{X}}$ for (3.11) corresponding to V, f and $\widetilde{V}$, $\widetilde{f}$, respectively. Proceeding as in (5.5) and using that $\Psi(u) = u$ and $\Psi(w) = w$, we can estimate

$$\begin{aligned} \| u - w \|_{\mathcal{X}} &= \| \Psi(u) - \Psi(w) \|_{\mathcal{X}} \\ &\leqslant C(\delta_2 + \rho_1 \varepsilon^{\rho_1 - 1} + \rho_2 \varepsilon^{\rho_2 - 1}) \| u - w \|_{\mathcal{X}} \\ &\quad + C\varepsilon \| V - \widetilde{V} \|_{\mathcal{F}\mathcal{B}^{\tilde{s}}_{p,\infty}} + C \| f - \widetilde{f} \|_{\mathcal{F}\mathcal{B}^{\overline{s}}_{p,\infty}}, \end{aligned}$$

which yields the desired continuity, since $C(\delta_2 + \rho_1 \varepsilon^{\rho_1 - 1} + \rho_2 \varepsilon^{\rho_2 - 1}) < 1$.

5.2. Proof of theorem 3.5

Let ε, δ_1, $\delta_2 > 0$, and $R > 1$ be such that

$$\begin{aligned} &\left[\frac{1}{R^{2-\beta-ad}} + 2(1 + R^s) R^{((2-\rho_1\beta)/(\rho_1 - 1)) + d} + \frac{1}{R} \right] (\rho_1 \varepsilon^{\rho_1 - 1} + \rho_2 \varepsilon^{\rho_2 - 1}) \leqslant \frac{1}{8C} \quad \text{and} \\ &\left[\frac{1}{R} + \frac{1}{R^{d+1}} + 2(1 + R^s) R^{\frac{2-\rho_1\beta}{\rho_1 - 1} + d} \right] (\delta_1 + \delta_2) \leqslant \frac{1}{8C}, \end{aligned} \tag{5.6}$$

with ε and δ_1, δ_2 satisfying also the relations in (5.6).

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Recall the space $\mathcal{H}_d^{1,s} = H_d^{1,s} \times H^{1,s}$ (3.24)) and consider the closed ball

$$B_{\mathcal{X}\cap\mathcal{H}_d^{1,s}} = \{u = (u_1, u_2) \in \mathcal{X} \cap \mathcal{H}_d^{1,s}; \|(u_1, u_2)\|_{\mathcal{X}} + \|(u_1, u_2)\|_{\mathcal{H}_d^{1,s}} \leqslant \varepsilon\}.$$

Employing estimates (4.39)–(4.43), (4.45), and (4.46), and proceeding as in the proof of theorem 3.2, we can show that

$$\| \Psi(u_1, u_2) \|_{\mathcal{H}_d^{1,s}} \leqslant \frac{\varepsilon}{2}, \tag{5.7}$$

$$\| \Psi(u_1, u_2) - \Psi(w_1, w_2) \|_{\mathcal{H}_d^{1,s}} \leqslant \frac{1}{4} \| (u_1, u_2) - (w_1, w_2) \|_{\mathcal{H}_d^{1,s}}, \tag{5.8}$$

for all $(u_1,\, u_2),\, (w_1,\, w_2) \in \mathcal{H}_d^{1,s}$, where $\Psi = (\Psi_1,\, \Psi_2)$ is defined via (5.4)–(5.7).

Putting together estimates (5.4)–(5.5) and (5.7)–(5.8) yields that Ψ is also a contraction in $B_{\mathcal{X}\cap\mathcal{H}_d^{1,s}}$. So, by uniqueness, it follows that the solution $u \in \mathcal{X}$ obtained through theorem 3.2 also belongs to $\mathcal{H}_d^{1,s}$.

Finally, recalling (2.2) and (3.23), and using that $u \in \mathcal{H}_d^{1,s}$, we have that

$$\|(1 + |\xi'|^s)\widehat{u_1} \|_1 \leqslant x_n^{-d} \|u\|_{\mathcal{H}_d^{1,s}} \quad \text{and} \quad \| (1 + |\xi'|^s)\widehat{u_2} \|_1 \leqslant \|u\|_{\mathcal{H}_d^{1,s}} .$$

Thus, for each $x_n > 0$, it follows that

$$(\partial_{x'}^{\alpha} u_1(\cdot, x_n))^{\wedge} \in L^1(\mathbb{R}^{n-1}) \quad \text{and} \quad (\partial_{x'}^{\alpha} u_2)^{\wedge} \in L^1(\mathbb{R}^{n-1}), \tag{5.9}$$

and then $\partial_{x'}^{\alpha} u_1(\cdot,\, x_n)$ and $\partial_{x'}^{\alpha} u_2$ belong to the space $C_0(\mathbb{R}^{n-1})$ of continuous functions vanishing at infinity, for all multi-index $|\alpha| \leqslant s$. Therefore, they belong to $C_0^{\lfloor s \rfloor}(\mathbb{R}^{n-1})$, as requested.

5.3. Proof of theorem 3.7 (axial symmetry)

First, we observe that the function ϕ of the Littlewood–Paley decomposition can be considered radially symmetric. This can be made without loss of generality since different functions ϕ's generate equivalent Fourier–Besov norms.

Due to the contraction argument, the solution $u = (u_1,\, u_2)$ obtained in theorem 3.2 is the limit in $\mathcal{X}$ of the following Picard sequence

$$u^{(1)} = (u_1^{(1)}, u_2^{(1)}) = (L_1(f), L_2(f))$$

and

$$u^{(m)} = (u_1^{(m)}, u_2^{(m)}) = (\Psi_1(u^{(m-1)}), \Psi_2(u^{(m-1)})), \quad \text{for } m = 2, 3, \ldots$$

Since f is radial, so is $\widehat{f}$. Also, for each rotation τ around the axis $\overrightarrow{Ox_n}$, note that

$$G(\tau(\xi', x), t) = G(\xi', x, t), \quad \text{for all } \xi' \in \mathbb{R}^{n-1},\ x \geqslant 0,\ t \in \mathbb{R}. \tag{5.10}$$

Then,

$$\widehat{L_1(f)}(\tau(\xi', x_n)) = \widehat{L_1(f)}(\xi', x_n) \quad \text{and} \quad \widehat{L_2(f)}(\xi') = \widehat{L_2(f)}(\tau(\xi', 0)) = \widehat{L_2(f)}(\xi'),$$

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for all $x_n > 0$ and $\xi' \in \mathbb{R}^{n-1}$. It follows that $\widehat{u^{(1)} \circ \tau} = \widehat{u^{(1)}} \circ \tau = \widehat{u^{(1)}}$ and

$$u^{(1)} \circ \tau = u^{(1)}. \tag{5.11}$$

Note also that $(\partial^\beta g)^\theta \circ \tau = (\partial^\beta g)^\theta$ (see (1.2)), $(g)^\theta \circ \tau = (g)^\theta$ and $(Vg) \circ \tau = Vg$, provided that $V \circ \tau = V$ and $g \circ \tau = g$. Using these properties, (5.11) and (5.10), we can show that $u^{(2)} = (\Psi_1(u^{(1)}), \Psi_2(u^{(1)}))$ is invariant under rotations τ around $\overrightarrow{Ox_n}$. In fact, by induction, it follows that

$$u^{(m)} \circ \tau = u^{(m)}, \quad \forall m \in \mathbb{N}, \tag{5.12}$$

for each rotation τ around $\overrightarrow{Ox_n}$. Moreover, if ϕ is radially symmetric then $\|g \circ \tau\|_{\mathcal{FB}^s_{p,q}} = \|g\|_{\mathcal{FB}^s_{p,q}}$, for all $g \in \mathcal{FB}^s_{p,q}$. Employing this invariance property of the $\mathcal{FB}^s_{p,q}$-norm, we get

$$\begin{aligned} \|w \circ \tau\|_{\mathcal{X}} &= \|w_1 \circ \tau\|_{\mathcal{L}^\infty_d \mathcal{FB}^{s_1}_{p,\infty}} + \|w_2 \circ \tau\|_{\mathcal{FB}^{s_2}_{p,\infty}} \\ &= \|w_1\|_{\mathcal{L}^\infty_d \mathcal{FB}^{s_1}_{p,\infty}} + \|w_2\|_{\mathcal{FB}^{s_2}_{p,\infty}} \\ &= \|w\|_{\mathcal{X}}, \quad \text{for all } w \in \mathcal{X}. \end{aligned} \tag{5.13}$$

Finally, in view of $u^{(m)} \to u$ in $\mathcal{X}$, (5.12) and (5.13), we can conclude that $u \circ \tau = u$, for each rotation τ around $\overrightarrow{Ox_n}$, as desired.

REMARK 5.1. In the proof of theorem 3.7 we have used the Picard sequence coming from the fixed point argument. Alternatively, we could show the same result by using the uniqueness property in theorem 3.2 together with the axial-invariance of the integral formulation (3.11). Note that such invariance has been proved in the above proof. Anyway, we prefer the use of the recurrent sequence because it illustrates a general procedure for obtaining qualitative properties that could be useful in other situations.

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