TWO CONJECTURES OF G.D. BIRKHOFF

CHRISTOPHER K. MCCORD and KENNETH R. MEYER*
Department of Mathematical Sciences, University of Cincinnati, Cincinnati, Ohio 45221-0025,
E-mail:CHRIS.MCCORD@UC.EDU and KEN.MEYER@UC.EDU

The spatial (planar) three-body problem admits the ten (six) integrals of energy, center of mass, linear momentum and angular momentum. Fixing these integrals defines an eight (six) dimensional algebraic set called the integral manifold, $\mathfrak{M}(c, h)$ ($\mathfrak{m}(c, h)$), which depends on the energy level h and the magnitude c of the angular momentum vector. The seven (five) dimensional reduced integral manifold, $\mathfrak{M}_R(c, h)$ ($\mathfrak{m}_R(c, h)$), is the quotient space $\mathfrak{M}(c, h)/SO_2$ ($\mathfrak{m}(c, h)/SO_2$) where the SO_2 action is rotation about the angular momentum vector. We want to determine how the geometry or topology of these sets depends on c and h. It turns out that there is one bifurcation parameter, $\nu = -c^2h$, and nine (six) special values of this parameter, $\nu_i, i = 1, \ldots, 9$.

At each of the special values the geometric restrictions imposed by the integrals change, but one of these values, ν_5, does not give rise to a change in the topology of the integral manifolds $\mathfrak{M}(c, h)$ and $\mathfrak{M}_R(c, h)$. The other *eight special values give rise to nine different topologically distinct cases.* We give a complete description of the geometry of these sets along with their homology. These results confirm some conjectures and refutes several others.

Birkhoff in his classic book *Dynamical Systems*, (1927) states as a theorem that these sets change their topological type only at values of ν corresponding to relative equilibrium solutions. This is the first "Birkhoff Conjecture". In the seventies Easton, Iacob and Smale (Easton 1971; Iacob, 1973; Smale, 1970a, 1970b) prove this theorem for the planar problem, but in (McCord *et al.*, 1998) we prove that there are other bifurcation values due to "critical points at infinity". Thus the first Birkhoff conjecture is true for the planar problem, but false for the spatial problem.

In the same book Birkhoff notes that $\mathfrak{m}_R(c, h)$ is a codimension two invariant set in $\mathfrak{M}_R(c, h)$ and wonders if $\mathfrak{m}_R(c, h)$ is the boundary of a general cross section for the flow on $\mathfrak{M}_R(c, h)$. This is the second "Birkhoff conjecture". If this were to be the case the Euler-Poincaré characteristics of $\mathfrak{M}_R(c, h)$ and $\mathfrak{m}_R(c, h)$ would have to agree (McCord and Meyer, 1998). Our computations of the homologies show this not to be the case and so the second Birkhoff conjecture is false also.

We not only study the integral manifolds, but also the six dimensional Hill's regions, $\mathfrak{H}(c, h)$, and the five dimensional reduced Hill's region

$$\mathfrak{H}_R(c, h) = \mathfrak{H}(c, h)/SO_2.$$

* This research partially supported by grants from the National Science Foundation and the Taft Foundation.

J. Henrard and S. Ferraz-Mello (eds.), Impact of Modern Dynamics in Astronomy, 439–440.

The Hill's region is the projection of the integral manifold onto position space. We determine the homotopy type of the reduced Hill's region, and the homology groups of $\mathfrak{H}(c, h)$ and $\mathfrak{H}_R(c, h)$. This does not detect all the changes in the homeomorphism type of $\mathfrak{H}(c, h)$, so we investigate the topology of the boundary points of these Hill's regions. From this finer analysis, we conclude that these Hill's regions undergo bifurcations at eight of the special values of the bifurcation parameter, but none of the topological invariants that we calculate detect a bifurcation at the parameter value ν_5.

References

A. Albouy: 1993, Integral manifold of the N-body problem, *Invent. Math.*, **14**, 463-488.

G.D. Birkhoff: 1927, *Dynamical Systems* , Amer. Math. Soc., Providence, RI.

R. Easton: 1971, Some topology of the three-body problem, *J. Dif. Eqs.*, **10**, 371-377.

A. Iacob: 1973, *Metode Topologice î Mecanica Clasică*, Editura Academiei Socialiste România, Bucure sti,.

C. McCord, K. Meyer and Q. Wang: 1998, The integral manifolds of thre three-body problem, *Mem. Amer. Math. Soc*, **628**, 1-91.

C. K. McCord and K. R. Meyer: 1998, Cross sections in the three-body problem, to appear.

S. Smale: 1970a, Topology and mechanics,*Invent. Math.*, **10**, 305-331.

S. Smale: 1970b, Topology and mechanics II,*Invent. Math.*, **11**, 45-64.