Can. J. Math., Vol. XXXVIII, No. 1, 1986, pp. 179-214

ON A FAMILY OF DISTRIBUTIONS OBTAINED FROM ORBITS

JAMES ARTHUR

Introduction. Suppose that G is a reductive algebraic group defined over a number field F. The trace formula is an identity

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(f) = \sum_{\chi\in\mathcal{X}} J_{\chi}(f), \quad f \in C_c^{\infty}(G(\mathbf{A})^1),$$

of distributions. The terms on the right are parametrized by "cuspidal automorphic data", and are defined in terms of Eisenstein series. They have been evaluated rather explicitly in [**3**]. The terms on the left are parametrized by semisimple conjugacy classes and are defined in terms of related $G(\mathbf{A})$ orbits. The object of this paper is to evaluate these terms.

In previous papers we have already evaluated $J_{\mathfrak{o}}(f)$ in two special cases. The easiest case occurs when $\mathfrak{o}$ corresponds to a regular semisimple conjugacy class $\{\sigma\}$ in $G(F)$. We showed in Section 8 of [**1**] that for such an $\mathfrak{o}$, $J_{\mathfrak{o}}(f)$ could be expressed as a weighted orbital integral over the conjugacy class of σ. (We actually assumed that $\mathfrak{o}$ was "unramified", which is slightly more general.) The most difficult case is the opposite extreme, in which $\mathfrak{o}$ corresponds to $\{1\}$. This was the topic of [**5**]. We were able to express the distribution, which we denoted by J_{unip}, as a finite linear combination of weighted orbital integrals over unipotent conjugacy classes. The general case is a mixture of these two. If $\mathfrak{o}$ corresponds to an arbitrary semisimple conjugacy class $\{\sigma\}$, let G_{σ} be the connected component of the centralizer of σ in G. In this paper we shall reduce the study of $J_{\mathfrak{o}}(f)$ to the unipotent case on subgroups of G_{σ}. We will then be able to appeal to the results of [**5**].

Suppose that S is a finite set of valuations of F which contains the Archimedean places. We can embed $C_c^{\infty}(G(F_S)^1)$ into $C_c^{\infty}(G(\mathbf{A})^1)$ by multiplying any function $f \in C_c^{\infty}(G(F_S)^1)$ by the characteristic function of a maximal compact subgroup of $\prod_{v\notin S} G(F_v)$. Suppose that M is a Levi component of a parabolic subgroup of G which is defined over F. If γ is any point in $M(F_S)$, the weighted orbital integral $J_M(\gamma, f)$ is the integral of f over the $G(F_S)$-conjugacy class of γ, with respect to a certain noninvariant measure. The measure is easily defined in terms of the

Received November 19, 1984. This research was supported by an E.W.R. Steacie Memorial Fellowship, NSERC Operating Grant A3483, and also in part by the Institute for Advanced Study.

volume of a certain convex hull if $M_\gamma = G_\gamma$, but in general is more delicate. In any case it is defined in [**4**]. The main result of [**5**] (Theorem 8.1) is a formula

$$J_{\text{unip}}(f) = \sum_M \sum_u |W_0^M| \, |W_0^G|^{-1} a^M(S, u) J_M(u, f),$$

where u ranges over the unipotent conjugacy classes in $M(F_S)$ which are the images of unipotent classes in $M(F)$, and $\{a^M(S, \gamma)\}$ are certain constants. In Corollary 8.5 of [**5**] we showed that

$$a^M(S, 1) = \text{vol}(M(F)\backslash M(\mathbf{A})^1),$$

but the other constants remain undetermined. The main result of this paper (Theorem 8.1) is a similar formula

$$J_{\mathfrak{o}}(f) = \sum_M \sum_\gamma |W_0^M| \, |W_0^G|^{-1} a^M(S, \gamma) J_M(\gamma, f)$$

for arbitrary $\mathfrak{o}$, in which γ ranges over classes in $M(F_S)$ with semisimple Jordan component $\{\sigma\}$, and $\{a^M(S, \gamma)\}$ are given in terms of the constants $\{a^{M_\sigma}(S, u)\}$.

The distribution $J_{\mathfrak{o}}(f)$ is defined as the value at $T = T_0$ of a certain polynomial $J_{\mathfrak{o}}^T(f)$. Our starting point will be an earlier formula for $J_{\mathfrak{o}}^T(f)$ (Theorem 8.1 of [**1**]). In Section 3 we change this formula into an expression which contains a certain alternating sum (3.4) of characteristic functions of chambers. Sections 4 and 5 are a combinatorial analysis of this alternating sum. We introduce some functions $\Gamma_R^G(X, \mathcal{Y}_R)$ which generalize the functions $\Gamma'_P(X, Y)$ used in [**2**] to prove $J_{\mathfrak{o}}^T(f)$ a polynomial. The main fact we require is Lemma 4.1. It asserts that $\Gamma_R^G(X, \mathcal{Y}_R)$ is compactly supported in X, and that its integral is a sum of integrals of functions $\Gamma'_P(\cdot, Y)$. The proof of Lemma 4.1 requires a combinatorial property (Lemma 4.2) which we establish in Section 5. Having proved Lemma 4.1, we return in Section 6 to our formula for $J_{\mathfrak{o}}(f)$. We make various changes of variable which reduce $J_{\mathfrak{o}}$ to a linear combination of distributions J_{unip} on subgroups of G_σ (Lemma 6.2). This allows us to apply the results of [**5**] in Section 7. Combined with a descent formula for weighted orbital integrals, they eventually lead to Theorem 8.1.

Theorem 8.1 and the related Theorem 9.2 will be important for future applications of the trace formula. They can be used to prove a general formula for the traces of Hecke operators. They will also play a role in the comparison of $GL(n)$ with its inner twistings, and in base change for $GL(n)$. Details will appear in a future paper with Clozel.

We have actually written this paper in the context of the twisted trace formula, which of course is a generalization of the ordinary trace formula. The twisted trace formula was proved by Clozel, Labesse and Langlands in a seminar at The Institute for Advanced Study during the academic year 1983-84. Their results are to appear in a future volume on the subject. In the meantime, we refer the reader to the lecture notes **[6]** from the seminar.

I thank Langlands, and also Stephen Halperin, for enlightening conversations related to this paper.

1. Assumptions on G. We would like our discussion to apply to the twisted trace formula proved in **[6]**, so we shall work with algebraic groups which are not connected. In Section 1 of the paper **[4]** we introduce some notions for such groups. For convenience, we shall describe the ones we will use here.

Suppose that G is a connected component of an algebraic group $\tilde{G}$ (not necessarily connected) which is defined over a number field F. We shall write G^+ for the subgroup of $\tilde{G}$ generated by G, and G^0 for the connected component of 1 in G^+. We shall assume that $G(F)$ is nonempty.

Assume that $\tilde{G}$ is reductive. A *parabolic subset* of G is a set $P = \tilde{P} \cap G$, where $\tilde{P}$ is the normalizer in $\tilde{G}$ of a parabolic subgroup of G^0 which is defined over F. Notice that

$$P^0 = \tilde{P} \cap G^0 = P^+ \cap G^0.$$

We shall let N_P denote the unipotent radical of P^0. A *Levi component* of P is a set $M = \tilde{M} \cap P$, where $\tilde{M}$ is the normalizer in $\tilde{G}$ of a Levi component of P^0 which is defined over F. Clearly $P = MN_P$. We call any such M a *Levi subset* of G. Let A_M denote the split component of the centralizer of M in M^0. It is a split torus over F. Let $X(M)_F$ be the group of characters of M^+ which are defined over F, and set

$$\mathfrak{a}_M = \mathrm{Hom}(X(M)_F, \mathbf{R}).$$

Then $\mathfrak{a}_M$ is a real vector space whose dimension equals that of the torus A_M. Observe that

$$A_M \subset A_{M^0} \quad \text{and} \quad \mathfrak{a}_M \subset \mathfrak{a}_{M^0}.$$

We fix, for once and for all, a minimal Levi subset M_0 of G^0. (Of course M_0 is actually a subgroup of G^0.) Set $A_0 = A_{M_0}$ and $\mathfrak{a}_0 = \mathfrak{a}_{M_0}$. Write $\mathscr{F}$ for the parabolic subsets P of G such that P^0 contains M_0. Similarly, write $\mathscr{L}$ for the Levi subsets M of G such that M^0 contains M_0. Both $\mathscr{F}$ and $\mathscr{L}$ are finite sets. Any $P \in \mathscr{F}$ has a unique Levi component M_P in $\mathscr{L}$, so we can write $P = M_PN_P$. Suppose that $M \in \mathscr{L}$. Write $\mathscr{F}(M)$ (respectively $\mathscr{L}(M)$) for the set of elements in $\mathscr{F}$ (respectively $\mathscr{L}$) which contain M. We also write $\mathscr{P}(M)$ for the set of $P \in \mathscr{F}$ such that $M_P = M$.

Suppose that $P \in \mathscr{F}$. Set $\mathfrak{a}_P = \mathfrak{a}_{M_P}$ and $A_P = A_{M_P}$. The roots of (P, A_P) are defined by taking the adjoint action of A_P on the Lie algebra of N_P. We will regard them either as characters on A_P or as elements in the dual space $\mathfrak{a}_P^*$ of $\mathfrak{a}_P$. The usual properties in the connected case carry over to the

https://doi.org/10.4153/CJM-1986-009-4 Published online by Cambridge University Press

present setting. In particular, we can define the simple roots Δ_P of (P, A_P), the associated "co-roots" $\{\alpha^\vee : \alpha \in \Delta_P\}$ in $\mathfrak{a}_P$, the weights $\hat{\Delta}_P$ and the associated "co-weights" $\{\varpi^\vee : \varpi \in \hat{\Delta}_P\}$ as in Section 1 of [**1**]. The roots of (P, A_P) divide $\mathfrak{a}_P$ into chambers. We shall write $\mathfrak{a}_P^+$ for the chamber on which the roots Δ_P are positive. Suppose that Q is another element in $\mathscr{F}$ such that $P \subset Q$. Then there are canonical embeddings $\mathfrak{a}_Q \subset \mathfrak{a}_P$ and $\mathfrak{a}_Q^* \subset \mathfrak{a}_P^*$, and canonical complementary subspaces $\mathfrak{a}_P^Q \subset \mathfrak{a}_P$ and $(\mathfrak{a}_P^Q)^* \subset \mathfrak{a}_P$. Let Δ_P^Q denote the set of roots in Δ_P which vanish on $\mathfrak{a}_Q$. They can be identified with the simple roots of the parabolic subset $P \cap M_Q$ of M_Q.

Let W_0^G denote the set of linear isomorphisms of $\mathfrak{a}_0$ induced by elements of G which normalize A_0. It is actually the set

$$W_0 = W_0^{G^0}$$

that is our main concern. This, of course, is just the Weyl group of (G^0, A_0). It acts simply transitively on W_0^G (on either the left or the right). In general, for any two subspaces $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathfrak{a}_0$, we shall let $W(\mathfrak{a}, \mathfrak{b})$ denote the set (possibly empty) of isomorphisms from $\mathfrak{a}$ onto $\mathfrak{b}$ which can be obtained by restricting elements in W_0 to $\mathfrak{a}$. Suppose that P_0 is a minimal parabolic subset in $\mathscr{F}$. Then $\mathfrak{a}_{P_0}$ is a subspace of $\mathfrak{a}_0$. The centralizer of $\mathfrak{a}_{P_0}$ in G^0 is

$$M_{P_0}^0 = M_{P_0^0} = M_0.$$

It follows that every element in $W(\mathfrak{a}_{P_0}, \mathfrak{a}_{P_0})$ is the restriction to $\mathfrak{a}_{P_0}$ of a unique element in W_0. We therefore identify $W(\mathfrak{a}_{P_0}, \mathfrak{a}_{P_0})$ with a subgroup of W_0. If P is any element in $\mathscr{F}$ which contains P_0, the set of chambers of $\mathfrak{a}_P$ can be recovered as the disjoint union

$$\bigcup_{\{Q \in \mathscr{F} : Q \supset P_0\}} \ \bigcup_{s \in W(\mathfrak{a}_P, \mathfrak{a}_Q)} s^{-1}\mathfrak{a}_Q^+.$$

(See Lemma 9.2.2 of [**6**].) This generalizes a well known result for connected groups. In those parts of the paper in which P_0 is fixed, the sets

$$W(\mathfrak{a}_P, \mathfrak{a}_Q), \quad P, Q \supset P_0,$$

will be regarded as subsets of $W(\mathfrak{a}_{P_0}, \mathfrak{a}_{P_0})$ and hence also of W_0. For if s belongs to $W(\mathfrak{a}_P, \mathfrak{a}_Q)$, we extend s to the unique element in $W(\mathfrak{a}_{P_0}, \mathfrak{a}_{P_0})$ such that $s(\alpha)$ is a root of (P_0, A_{P_0}) for every root $\alpha \in \Delta_{P_0}^P$.

Let $\mathbf{A}$ be the adèle ring of F. We can form the adèlized variety $G(\mathbf{A})$, as well as the adèle group $G^0(\mathbf{A})$. If

$$x = \prod_v x_v$$

is any point in either $G(\mathbf{A})$ or $G^0(\mathbf{A})$, define a vector $H_G(x)$ in $\mathfrak{a}_G$ by

$$e^{\langle H_G(x), \chi \rangle} = |\chi(x)| = \prod_v |\chi(x_v)|_v, \quad \chi \in X(G^+)_F.$$

(Here and in the future the products over v stand for products over the valuations of F.) Let $G(\mathbf{A})^1$ and $G^0(\mathbf{A})^1$ be the subsets of $G(\mathbf{A})$ and $G^0(\mathbf{A})$ respectively of elements x such that $H_G(x) = 0$. (Our notation is somewhat ambiguous in that $G^0(\mathbf{A})^1$ depends on G and not just G^0.) If $G_{\mathbf{Q}}$ is the variety over $\mathbf{Q}$ obtained from G by restriction of scalars, let A_G^{∞} denote the identity component of the Lie group $A_{G_{\mathbf{Q}}}(\mathbf{R})$. Then $G^0(\mathbf{A})$ is the direct product of $G^0(\mathbf{A})^1$ and A_G^{∞}, and $G(\mathbf{A})$ equals $G(\mathbf{A})^1 A_G^{\infty}$.

For each valuation v, let K_v^+ be a maximal compact subgroup of $G^+(F_v)$, and set

$$K_v = K_v^+ \cap G(F_v).$$

Then $K = \prod_v K_v$ is a maximal compact subgroup of $G^0(\mathbf{A})$. We assume that it is admissible relative to M_0 in the sense of Section 1 of [**2**]. Then if P is any element in $\mathscr{F}$,

$$G^0(\mathbf{A}) = P^0(\mathbf{A})K = N_P(\mathbf{A})M_P^0(\mathbf{A})K.$$

For any point

$$x = n_P m_P k_P, \quad n_P \in N_P(\mathbf{A}),\ m_P \in M_P(\mathbf{A}),\ k_P \in K,$$

in $G^0(\mathbf{A})$, define

$$H_P(x) = H_{M_P}(m_P).$$

If $x \in G(\mathbf{A})$, we can define $H_P(x)$ in a similar fashion.

2. The distributions $J_{\mathfrak{o}}^T$. The distributions we propose to study are parametrized by $G^0(F)$-orbits of semisimple elements in $G(F)$. They were defined for connected groups in [**1**]. Our references will henceforth be mostly to papers that apply only to connected groups. The analogous results for arbitrary G have been proved by Clozel, Labesse and Langlands. They can all be found in the lecture notes [**6**]. The references for the trace formula are actually for groups defined over $\mathbf{Q}$. However, the results can all be carried over to arbitrary F, by restricting scalars, or by directly transcribing the proofs.

Suppose that σ is a semisimple element in $G(F)$. We shall write G_σ for the identity component of the centralizer of σ in G^0. It is a connected reductive group defined over F. For any subgroup H of G^0 which is defined over F, we shall write $H(F, \sigma)$ for the centralizer of σ in $H(F)$. Then $G_\sigma(F)$ is a subgroup of finite index in $G^0(F, \sigma)$. We will let

$$\iota^0(\sigma) = G_\sigma(F)\backslash G^0(F, \sigma)$$

denote the quotient group.

There is a Jordan decomposition for elements in $G(F)$. Any element $\gamma \in G(F)$ can be decomposed uniquely as $\gamma = \sigma u$, where σ is a semisimple element in $G(F)$ and u is a unipotent element in $G_\sigma(F)$. Let γ_s denote the semisimple component σ of γ. As in [**1**], define two elements γ and γ' to be equivalent if γ_s and γ'_s are in the same $G^0(F)$ orbit. Let $\mathcal{O}$ be the set of equivalence classes in $G(F)$. It is clearly in bijective correspondence with the set of semisimple $G^0(F)$ orbits in $G(F)$.

The space $C_c^\infty(G(\mathbf{A})^1)$ of smooth, compactly supported functions on $G(\mathbf{A})^1$ can be defined in the usual fashion. Our objects of study are distributions on $G(\mathbf{A})^1$ which are indexed by the classes in $\mathcal{O}$. As originally defined, they depend on a minimal parabolic subset P_0 in $\mathcal{F}$ and also a point $T \in \mathfrak{a}_0$ which is suitably regular with respect to P_0, in the sense that $\alpha(T)$ is large for every root α in Δ_{P_0}. Given P_0, a *standard* parabolic subset will be, naturally, an element $P \in \mathcal{F}$ which contains P_0. Let $\hat{\tau}_P$ be the characteristic function of

$$\{H \in \mathfrak{a}_0 : \varpi(H) > 0, \ \varpi \in \hat{\Delta}_P\}.$$

The distribution

$$J_{\mathfrak{o}}^T(f), \quad \mathfrak{o} \in \mathcal{O}, f \in C_c^\infty(G(\mathbf{A})^1),$$

is then defined by the formula

$$(2.1) \qquad \int_{G^0(F)\backslash G^0(\mathbf{A})^1} \sum_{P \supset P_0} (-1)^{\dim(A_P/A_G)} \times \sum_{\delta \in P^0(F)\backslash G^0(F)} K_{P,\mathfrak{o}}(\delta x, \delta x)\hat{\tau}_P(H_P(\delta x) - T)dx,$$

where

$$K_{P,\mathfrak{o}}(y, y) = \sum_{\gamma \in M_P(F) \cap \mathfrak{o}} \int_{N_P(\mathbf{A})} f(y^{-1}\gamma n y)dn.$$

(See [**1**], p. 947 and Theorem 7.1)

We would like to find a formula for $J_{\mathfrak{o}}^T(f)$ in terms of locally defined objects. A first step in this direction is the formula given by Theorem 8.1 of [**1**]. If $Q \supset P_0$ is a standard parabolic, set

$$J_{Q,\mathfrak{o}}(y, y) = \sum_{\gamma \in M_Q(F) \cap \mathfrak{o}} \ \sum_{\nu \in N_Q(F,\gamma_s)\backslash N_Q(F)} \times \int_{N_Q(\mathbf{A},\gamma_s)} f(y^{-1}\nu^{-1}\gamma n \nu y)dn,$$

where $N_Q(\mathbf{A}, \gamma_s)$ is the centralizer of γ_s in $N_Q(\mathbf{A})$. Then for T sufficiently regular, $J_{\mathfrak{o}}^T(f)$ equals

$$(2.2) \qquad \int_{G^0(F)\backslash G^0(A)^1} \sum_{Q \supset P_0} (-1)^{\dim(A_Q/A_G)} \times \sum_{\delta \in Q^0(F)\backslash G^0(F)} J_{Q,\mathfrak{o}}(\delta x, \delta x)\hat{\tau}_Q(H_Q(\delta x) - T)dx.$$

This will be our starting point for the next section.

It was shown in Proposition 2.3 of [**2**] that $J_{\mathfrak{o}}^T(f)$ was a polynomial in T, and could therefore be defined for all T. We are interested in its value at a particular point T_0, whose definition (Lemma 1.1 of [**2**]) we recall. For each element $s \in W_0$, write w_s and $\widetilde{w}_s$ for representatives of s in $G^0(F)$ and K respectively. These cannot in general be chosen to be the same. The obstruction is T_0, which is the point in $\mathfrak{a}_0$, uniquely determined modulo $\mathfrak{a}_G$, such that

$$H_{M_0}(w_s^{-1}) = T_0 - s^{-1}T_0$$

for each $s \in W_0$. Set

$$J_{\mathfrak{o}}(f) = J_{\mathfrak{o}}^{T_0}(f).$$

This distribution is independent of P_0, (see the discussion in [**2**] following Proposition 2.3), and will be our main object of study.

In [**5**] we considered a special case. For any connected reductive subgroup H of G^0 which is defined over F, let $\mathscr{U}_H$ denote the Zariski closure in H of the set of unipotent elements in $H(F)$. It is an algebraic variety, defined over F. If $H = G = G^0$, then $\mathscr{U}_G(F)$ belongs to $\mathscr{O}$. As in [**5**], we denote the corresponding distributions by J_{unip}^T and J_{unip} respectively (or by $J_{\text{unip}}^{G,T}$ and J_{unip}^G when we wish to emphasize the role of G). Observe that in this case the formulas (2.1) and (2.2) are the same.

3. A preliminary formula. We choose a class $\mathfrak{o} \in \mathscr{O}$ and a function $f \in C_c^\infty(G(\mathbf{A})^1)$. We propose to keep these objects fixed until Section 9. We shall begin by examining the formula for $J_{\mathfrak{o}}^T(f)$, so for the time being we want also to fix the minimal parabolic P_0 in $\mathscr{F}$. For $\mathfrak{o}$ unramified in the sense of [**1**] we showed (p. 950 of [**1**]) that (2.2) could be written as a weighted orbital integral of f. In this section we shall perform similar manipulations for our arbitrary $\mathfrak{o}$ to obtain at least a reduction of (2.2).

Fix a semisimple element σ in $\mathfrak{o}$. We can choose σ so that it belongs to $M_{P_1}(F)$ for a fixed standard parabolic subset P_1 of G, but so that it belongs to no proper parabolic subset of M_{P_1}. We shall write $M_1 = M_{P_1}$, $A_1 = A_{M_1}$ and $\mathfrak{a}_1 = \mathfrak{a}_{M_1}$. Then

$$G_\sigma = \text{Cent}(\sigma, G^0)^0$$

is a connected reductive group with minimal Levi subgroup

$$M_{1\sigma} = \text{Cent}(\sigma, M_1^0)^0 = M_1^0 \cap G_\sigma$$

and standard minimal parabolic subgroup

$$P_{1\sigma} = \text{Cent}(\sigma, P_1^0)^0 = P_1^0 \cap G_\sigma,$$

both defined over F. Notice that

$$A_1 = A_{1\sigma}$$

is the split component of both P_1 and $P_{1\sigma}$. In general, the centralizer of σ in G^0 is not connected. Its group of rational points is $G^0(F, \sigma)$. We shall have to be careful to distinguish between this group and $G_\sigma(F)$.

We shall take a standard parabolic subset Q of G and study its contribution to the integrand of (2.2). We must then look at the formula for $J_{Q,\mathfrak{o}}(y, y)$. Suppose that γ belongs to $M_Q(F) \cap \mathfrak{o}$. We know that γ is $G^0(F)$-conjugate to an element σu, $u \in \mathcal{U}_{G_\sigma}(F)$, but we will have to be more precise. The semisimple constituent γ_s commutes with a torus in G^0 which is a $G^0(F)$-conjugate of A_1. This torus is in turn $M_Q^0(F)$-conjugate to a torus A_{Q_1}, where $Q_1 \subset Q$ is a standard parabolic subset of G which is associated to P_1. Thus we can write

$$\gamma = \mu^{-1} w_s \sigma u w_s^{-1} \mu,$$

for

$$s \in W(\mathfrak{a}_{P_1}, \mathfrak{a}_{Q_1}),$$
$$\mu \in M_Q^0(F) \quad \text{and}$$
$$u \in w_s^{-1} M_Q^0(F) w_s \cap \mathcal{U}_{G_\sigma}(F).$$

The Weyl element s is uniquely determined up to multiplication on the left by Weyl elements of M_Q and multiplication on the right by the Weyl group of $(G^0(F, \sigma), A_1)$. Once s is fixed, μ is uniquely determined modulo

$$M_Q^0(F) \cap w_s G^0(F, \sigma) w_s^{-1}.$$

The element u is clearly uniquely determined by μ and w_s. Let $W(\mathfrak{a}_1; Q, G_\sigma)$ be the union over all standard $Q_1 \subset Q$ of those elements

$$s \in W(\mathfrak{a}_{P_1}, \mathfrak{a}_{Q_1})$$

such that $s^{-1}\alpha$ is positive for every root α in $\Delta_{Q_1}^Q$ and such that $s\beta$ is positive for every positive root β of (G_σ, A_1). This set would uniquely represent all elements s arising above were it not for the fact that the Weyl group of $(G^0(F, \sigma), A_1)$ could be larger than that of $(G_\sigma(F), A_1)$. However, in the formula for $J_{Q,\mathfrak{o}}(y, y)$ we will be able to take a sum over $W(\mathfrak{a}_1; Q, G_\sigma)$ if at the same time we sum μ modulo

$$M_Q^0(F) \cap w_s G_\sigma(F) w_s^{-1},$$

and then divide by $|\iota^G(\sigma)|$, the index of $G_\sigma(F)$ in $G^0(F, \sigma)$.

It follows from this discussion that $J_{Q,\mathfrak{o}}(y, y)$ equals the sum over s in $W(\mathfrak{a}_1; Q, G_\sigma)$ of

$$|\iota^G(\sigma)|^{-1} \sum_\mu \sum_\nu \sum_u \int f(y^{-1}\nu^{-1}\mu^{-1} w_s \sigma u w_s^{-1} \mu n \nu y) dn,$$

in which μ, ν and u are summed over

$$M_Q^0(F) \cap w_s G_\sigma(F) w_s^{-1} \backslash M_Q^0(F),$$
$$N_Q(F, \mu^{-1} w_s \sigma w_s^{-1} \mu) \backslash N_Q(F),$$

and

$$w_s^{-1}M_Q^0(F)w_s \cap \mathcal{U}_{G_\sigma}(F)$$

respectively, while the integral is over n in

$$N_Q(\mathbf{A}, \mu^{-1}w_s\sigma w_s^{-1}\mu).$$

In this expression we replace $w_s^{-1}\mu n$ by $nw_s^{-1}\mu$, changing the integral to one over

$$w_s^{-1}N_Q(\mathbf{A}, \sigma)w_s = w_s^{-1}N_Q(\mathbf{A})w_s \cap G_\sigma(\mathbf{A}).$$

Since

$$N_Q(F, \mu^{-1}w_s\sigma w_s^{-1}\mu) = \mu^{-1}(N_Q(F) \cap w_sG_\sigma(F)w_s^{-1})\mu,$$

we can change the sum over μ and ν to a sum over π in

$$Q^0(F) \cap w_sG_\sigma(F)w_s^{-1}\backslash Q^0(F).$$

We obtain

$$J_{Q,o}(y, y) = |\iota^G(\sigma)|^{-1} \sum_s \sum_\pi \sum_u \int f(y^{-1}\pi^{-1}w_s\sigma unw_s^{-1}\pi y)dn.$$

We substitute this into the expression

$$(3.1) \qquad \sum_{\delta \in Q^0(F)\backslash G^0(F)} J_{Q,o}(\delta x, \delta x)\hat{\tau}_Q(H_Q(\delta x) - T)$$

which occurs in the formula (2.2). Take the sum over δ inside the sum over s, and then combine it with the sum over π. For a given s, this produces a sum over ξ in

$$Q^0(F) \cap w_sG_\sigma(F)w_s^{-1}\backslash G^0(F).$$

The expression (3.1) becomes

$$|\iota^G(\sigma)|^{-1} \sum_s \sum_\xi \sum_u \int f(x^{-1}\xi^{-1}w_s\sigma unw_s^{-1}\xi x)$$
$$\times \hat{\tau}_Q(H_Q(\xi x) - T)dn.$$

Finally, replace ξ by $w_s\xi$, changing the corresponding sum to one over $R(F)\backslash G^0(F)$, where

$$R = w_s^{-1}Q^0w_s \cap G_\sigma.$$

Clearly R is a standard parabolic subgroup of G_σ with Levi decomposition

$$R = M_RN_R = (w_s^{-1}M_Q^0w_s \cap G_\sigma)(w_s^{-1}N_Qw_s \cap G_\sigma).$$

It follows that (3.1) equals the sum over

$$s \in W(\mathfrak{a}_1; Q, G_\sigma)$$

of

$$|\iota^G(\sigma)|^{-1} \sum_{\xi \in R(F)\backslash G^0(F)} \sum_{u \in \mathcal{U}_{M_R}(F)} \int_{N_R(\mathbf{A})} f(x^{-1}\xi^{-1}\sigma un\xi x)$$

$$\times \hat{\tau}_Q(H_Q(w_s\xi x) - T)dn.$$

We can now rewrite (2.2). We see that $J_0^T(f)$ equals the integral over x in $G^0(F)\backslash G^0(\mathbf{A})^1$ and the sum over standard parabolic subgroups R of G_σ and elements ξ in $R(F)\backslash G^0(F)$ of the product of

$$|\iota^G(\sigma)|^{-1} \sum_{u \in \mathcal{U}_{M_R}(F)} \int_{N_R(\mathbf{A})} f(x^{-1}\xi^{-1}\sigma un\xi x)dn$$

with

(3.2) $$\sum_Q \sum_s (-1)^{\dim(A_Q/A_G)}\hat{\tau}_Q(H_Q(w_s\xi x) - T).$$

In (3.2), Q and s are to be summed over the set

$$\{Q \supset P_0, s \in W(\mathfrak{a}_1; Q, G_\sigma): w_s^{-1}Q^0w_s \cap G_\sigma = R\}.$$

We shall free this expression from its dependence on the standard parabolic Q.

We shall write $\mathcal{F}_R(M_1)$ for the set of parabolic subsets $P \in \mathcal{F}(M_1)$ such that $P_\sigma = R$. Suppose that Q and s are as in (3.2). Then

$$P = w_s^{-1}Qw_s$$

is a parabolic subset in $\mathcal{F}_R(M_1)$. The corresponding summand in (3.2) is easily expressed in terms of P. For

$$\hat{\tau}_Q(H_Q(w_s\xi x) - T) = \hat{\tau}_Q(H_Q(\tilde{w}_s\xi x) + H_Q(w_s) - T),$$

where $\tilde{w}_s$ is a representative of s in K. By Lemma 1.1 of [**2**] this equals

$$\hat{\tau}_Q(H_Q(\tilde{w}_s\xi x) + T_0 - sT_0 - T).$$

Since

$$H_Q(\tilde{w}_s\xi x) = sH_P(\xi x),$$

this is the same as

$$\hat{\tau}_P(H_P(\xi x) - Z_P(T - T_0) - T_0),$$

where

(3.3) $$Z_P(T - T_0) = s^{-1}(T - T_0).$$

Conversely suppose that P is any parabolic subset in $\mathcal{F}_R(M_1)$. Then there is a unique standard parabolic Q and an element $w \in W_0$ such that

$$P = w_s^{-1} Q w_s.$$

We demand that $s^{-1}\alpha$ be positive for each root α in $\Delta_{P_0}^Q$, so that s will also be uniquely determined. Since the space $s\mathfrak{a}_{P_1}$ contains $\mathfrak{a}_Q$, it must be of the form $\mathfrak{a}_{Q_1}$ for a standard parabolic Q_1. Otherwise s^{-1} would map some positive, nonsimple linear combination of roots in $\Delta_{P_0}^Q$ to a simple root in Δ_{P_0}, a contradiction. Moreover, combining this property with the fact that Q^0 contains $w_s R w_s^{-1}$, we see that $s\beta$ is positive for every positive root β of (G_σ, A_1). It follows that the restriction of s to $\mathfrak{a}_{P_1}$ defines a unique element in $W(\mathfrak{a}_1; Q, G_\sigma)$. Therefore, the double sum in (3.2) can be replaced by the sum over $P \in \mathscr{F}_R(M_1)$.

We have established

LEMMA 3.1. *For sufficiently regular T, $J_0^T(f)$ equals the integral over x in $G^0(F)\backslash G^0(\mathbf{A})^1$, and the sum over standard parabolic subgroups R of G_σ and elements ξ in $R(F)\backslash G^0(F)$ of the product of*

$$|\iota^G(\sigma)|^{-1} \sum_{u \in \mathscr{U}_{M_R}(F)} \int_{N_R(\mathbf{A})} f(x^{-1}\xi^{-1}\sigma u n \xi x)dn$$

with

$$(3.4) \qquad \sum_{P \in \mathscr{F}_R(M_1)} (-1)^{\dim(A_P/A_G)} \hat{\tau}_P(H_P(\xi x) - Z_P(T - T_0) - T_0).$$

4. A construction. The next two sections represent a combinatorial digression. The expression (3.4) is a sum over parabolic subsets of G. We shall introduce a construction designed to transform it into a sum over parabolic subgroups of G_σ.

The discussion will center around the group G_σ, where σ is a fixed semisimple element in $G(F)$. In the next two sections we will not single out standard parabolic subsets of either G or G_σ. We do assume, however, that there is a fixed Levi subset M_1 in $\mathscr{L}$ which contains σ, and such that $M_{1\sigma}$ is a minimal Levi subgroup of G_σ. Then, as in Section 3, we set

$$A_1 = A_{M_1} = A_{M_{1\sigma}}$$

and

$$\mathfrak{a}_1 = \mathfrak{a}_{M_1} = \mathfrak{a}_{M_{1\sigma}}.$$

Set

$$\mathscr{F}^\sigma = \mathscr{F}^{G_\sigma}(M_{1\sigma}),$$

the set of parabolic subgroups of G_σ which contain $M_{1\sigma}$. We have the map

$$P \to P_\sigma = \mathrm{Cent}(\sigma, P^0)^0, \quad P \in \mathscr{F}(M_1),$$

from $\mathscr{F}(M_1)$ onto $\mathscr{F}^\sigma$. Suppose that R is a group in $\mathscr{F}^\sigma$. We shall be concerned with the three successively embedded subsets

$$\mathscr{F}_R^0(M_1) = \{P \in \mathscr{F}(M_1): P_\sigma = R, \mathfrak{a}_P = \mathfrak{a}_R\},$$

$$\mathscr{F}_R(M_1) = \{P \in \mathscr{F}(M_1): P_\sigma = R\},$$

and

$$\bar{\mathscr{F}}_R(M_1) = \{P \in \mathscr{F}(M_1): P_\sigma \supset R\}$$

of $\mathscr{F}(M_1)$. It is helpful to think of these sets geometrically. Associated to R is the positive chamber $\mathfrak{a}_R^+$ of $\mathfrak{a}_R$. Its closure is the disjoint union over $P \in \bar{\mathscr{F}}_R(M_1)$ of the chambers $\mathfrak{a}_P^+$. The set $\mathscr{F}_R(M_1)$ corresponds to those chambers which are actually contained in $\mathfrak{a}_R^+$, while the first set $\mathscr{F}_R^0(M_1)$ corresponds to those chambers which are open in $\mathfrak{a}_R^+$. Observe that $\mathscr{F}_R^0(M_1)$ consists of the minimal parabolic subsets from $\bar{\mathscr{F}}_R(M_1)$.

Our construction will actually be a generalization of a definition from Section 2 of [**2**]. The earlier definition will be the special case here that $G_\sigma = G = G^0$. We shall begin by recalling the earlier definition, or rather its extension to our connected component G. Take a point $Y \in \mathfrak{a}_0$ and for any $P \in \mathscr{F}$ let Y_P be the projection of Y onto $\mathfrak{a}_P$. Then there is a function

$$\Gamma_P^G(X, Y_P), \quad X \in \mathfrak{a}_0,$$

for any $P \in \mathscr{F}$, such that

$$(4.1) \qquad \Gamma_P^G(X, Y_P) = \sum_{\{Q \in \mathscr{F}: Q \supset P\}} (-1)^{\dim(A_Q/A_G)} \tau_P^Q(X) \hat{\tau}_Q(X - Y_Q),$$

and

$$(4.2) \qquad \hat{\tau}_P(X - Y_P) = \sum_{\{Q \in \mathscr{F}: Q \supset P\}} (-1)^{\dim(A_Q/A_G)} \hat{\tau}_P^Q(X) \Gamma_Q^G(X, Y_Q).$$

The function $\Gamma_P^G(X, Y_P)$ depends only on the projection of X onto $\mathfrak{a}_P^G$, and it is compactly supported as a function of X in $\mathfrak{a}_P^G$. (See Lemma 2.1 of [**2**]. In [**2**] we wrote $\Gamma'_P(X, Y)$ instead of $\Gamma_P^G(X, Y_P)$ and we treated only the case that $G = G^0$. However, the proof applies equally well to arbitrary G.)

Fix a group R in $\mathscr{F}^\sigma$. Suppose that

$$\mathscr{Y} = \{Y_P : P \in \mathscr{F}_R^0(M_1)\}$$

is a set of points in $\mathfrak{a}_0$ with the usual compatibility condition. Namely, if $P, P' \in \mathscr{F}_R^0(M_1)$ are adjacent (that is, their chambers share a common wall), then $Y_{P'} - Y_P$ is a multiple of the co-root $\alpha^\vee$, where α is the unique root in $\Delta_{P'} \cap (-\Delta_P)$. Let Q be any parabolic element in $\bar{\mathscr{F}}_R(M_1)$. Then Q contains a P in $\mathscr{F}_R^0(M_1)$. Define Y_Q to be the projection of Y_P onto $\mathfrak{a}_Q$.

Because of our condition on $\mathcal{Y}$, Y_Q will be independent of which group P is chosen. For any $S \in \mathcal{F}^\sigma$ with $S \supset R$, set

$$\mathcal{Y}_S = \{Y_Q : Q \in \mathcal{F}_S(M_1)\}.$$

For each such R and $\mathcal{Y}$ we define a function

$$\Gamma_R^G(X, \mathcal{Y}_R), \quad X \in \mathfrak{a}_1,$$

such that

$$(4.1)^* \quad \Gamma_R^G(X, \mathcal{Y}_R) = \sum_{\{S \in \mathcal{F}^\sigma : S \supset R\}} \tau_R^S(X) \Big(\sum_{Q \in \mathcal{F}_S(M_1)} (-1)^{\dim(A_Q/A_G)} \hat{\tau}_Q(X - Y_Q) \Big)$$

and

$$(4.2)^* \quad \Big(\sum_{P \in \mathcal{F}_R(M_1)} (-1)^{\dim(A_P/A_G)} \hat{\tau}_P(X - Y_P) \Big) = \sum_{\{S \in \mathcal{F}^\sigma : S \supset R\}} (-1)^{\dim(A_R/A_S)} \hat{\tau}_R^S(X) \Gamma_S^G(X, \mathcal{Y}_S).$$

Either formula serves to define this function while the other follows from the fact that

$$(4.3) \quad \sum_{\{S \in \mathcal{F}^\sigma : R \subset S \subset R'\}} (-1)^{\dim(A_R/A_S)} \tau_R^S(X) \hat{\tau}_S^{R'}(X) = 0,$$

for any groups $R \subsetneq R'$ in $\mathcal{F}^\sigma$. (See the remark following Corollary 6.2 of [**1**].) The second formula is evidently the one which pertains to (3.4). It is clear that $\Gamma_R^G(X, \mathcal{Y}_R)$ depends only on the projection of X onto $\mathfrak{a}_R$. We shall show that it is compactly supported as a function of X in the orthogonal complement, $\mathfrak{a}_R^G$, of $\mathfrak{a}_G$ in $\mathfrak{a}_R$, and we shall find its Fourier transform.

Given R and $\mathcal{Y}$, and also $P \in \mathcal{F}_R(M_1)$, set

$$c_P'(\lambda) = \int_{\mathfrak{a}_P^G} \Gamma_P^G(X, Y_P) e^{\lambda(X)} dX, \quad \lambda \in i\mathfrak{a}_P^*.$$

As the Fourier transform of a compactly supported function, $c_P'(\lambda)$ extends to an entire function of λ. It has a simple formula in terms of the function

$$c_P(\lambda) = e^{\lambda(Y_P)}$$

(Lemma 2.2 of [**2**]) and agrees with the general definitions of Section 6 of [**2**]. Define

$$c_R'(\lambda) = \sum_{P \in \mathcal{F}_R^0(M_1)} c_P'(\lambda).$$

LEMMA 4.1. *For any R and $\mathscr{Y}$, the support of the function*

$$X \to \Gamma_R^G(X, \mathscr{Y}_R), \quad X \in \mathfrak{a}_R^G,$$

is compact and depends continuously on $\mathscr{Y}_R$. Moreover,

$$\int_{\mathfrak{a}_R^G} \Gamma_R^G(X, \mathscr{Y}_R)e^{\lambda(X)}dX = c_R'(\lambda), \quad \lambda \in i\mathfrak{a}_R^*.$$

Proof. For any X in $\mathfrak{a}_R$, we have

$$\begin{aligned}
&\Gamma_R^G(X, \mathscr{Y}_R) \\
&= \sum_{\{S \in \mathscr{F}^\sigma: S \supset R\}} \tau_R^S(X)\Big(\sum_{Q \in \mathscr{F}_S(M_1)} (-1)^{\dim(A_Q/A_G)}\hat{\tau}_Q(X - Y_Q)\Big) \\
&= \sum_S \tau_R^S(X) \sum_{Q \in \mathscr{F}_S(M_1)} \sum_{\{P \in \mathscr{F}: P \supset Q\}} (-1)^{\dim(A_Q/A_P)}\hat{\tau}_Q^P(X)\Gamma_P^G(X, Y_P),
\end{aligned}$$

by (4.1)* and (4.2). This equals

$$\sum_{P \in \bar{\mathscr{F}}_R(M_1)} \Gamma_P^G(X, Y_P) \times \Big(\sum_{\{Q \in \bar{\mathscr{F}}_R(M_1): Q \subset P\}} (-1)^{\dim(A_Q/A_P)}\tau_R^{Q_\sigma}(X)\hat{\tau}_Q^P(X)\Big).$$

In the next section we shall prove

LEMMA 4.2. *For $R \in \mathscr{F}^\sigma$ and $P \in \bar{\mathscr{F}}_R(M_1)$, the expression*

$$\sum_{\{Q \in \bar{\mathscr{F}}_R(M_1): Q \subset P\}} (-1)^{\dim(A_Q/A_P)}\tau_R^{Q_\sigma}(X)\hat{\tau}_Q^P(X), \quad X \in \mathfrak{a}_R,$$

equals 1 if P belongs to $\mathscr{F}_R(M_1)$ and X belongs to $\mathfrak{a}_P$, and equals 0 otherwise.

Assuming Lemma 4.2, we obtain

$$(4.4) \qquad \Gamma_R^G(X, \mathscr{Y}_R) = \sum_{P \in \mathscr{F}_R(M_1)} \Gamma_P^G(X, Y_P)\epsilon_P(X), \quad X \in \mathfrak{a}_R,$$

where $\epsilon_P(X)$ equals 1 if X belongs to $\mathfrak{a}_P$, and equals 0 otherwise. This equals

$$\sum_{P \in \mathscr{F}_R^0(M_1)} \Gamma_P^G(X, Y_P)$$

almost everywhere on $\mathfrak{a}_R^G$. Lemma 4.1 follows from this fact and Lemma 2.1 of [**2**].

Continuing a convention from [**2**], we shall often denote the values of $c_P'(\lambda)$ and $c_R'(\lambda)$ at $\lambda = 0$ by c_P' and c_R' respectively.

https://doi.org/10.4153/CJM-1986-009-4 Published online by Cambridge University Press

5. Combinatorial lemmas. The purpose of this section is to prove Lemma 4.2. We continue with the notation of Section 4. Since the center of G_σ could be larger than that of G, the space $\mathfrak{a}_G$ could be properly contained in the analogous space $\mathfrak{a}_{G_\sigma}$ for G_σ. We are denoting the complementary subspace by $\mathfrak{a}_{G_\sigma}^G$.

Let R be a fixed group in $\mathscr{F}^\sigma$. Then

$$\mathfrak{a}_G \subset \mathfrak{a}_{G_\sigma} \subset \mathfrak{a}_R,$$

and

$$\mathfrak{a}_R = \mathfrak{a}_G \oplus \mathfrak{a}_{G_\sigma}^G \oplus \mathfrak{a}_R^{G_\sigma}.$$

LEMMA 5.1. *The function*

$$\sum_{P \in \bar{\mathscr{F}}_R(M_1)} (-1)^{\dim(A_P/A_G)} \hat{\tau}_P(X), \quad X \in \mathfrak{a}_R,$$

equals the characteristic function of the set

$$\{X \in \mathfrak{a}_G + \mathfrak{a}_R^{G_\sigma} : \varpi(X) \leqq 0,\ \varpi \in \hat{\Delta}_R\}.$$

Proof. We shall define a simplicial complex

$$C = \bigcup_P c_P,$$

whose simplices are indexed by the parabolic subsets $P \in \bar{\mathscr{F}}_R(M_1)$ with $P \neq G$. For any such P set

$$\hat{\Delta}_P = \{\varpi_1, \ldots, \varpi_n\},$$

and then define c_P to be the simplex

$$\{t_1\varpi_1 + \ldots + t_n\varpi_n : t_i \geqq 0,\ t_1 + \ldots + t_n = 1\}.$$

It lies in $(\mathfrak{a}_R^G)^*$, the complement of $\mathfrak{a}_G^*$ in $\mathfrak{a}_R^*$. Let C be the union of the simplices c_P. If $R = G_\sigma$, C is homeomorphic to the unit sphere in $(\mathfrak{a}_R^G)^*$. If $R \neq G_\sigma$, C is homeomorphic to the intersection of the unit sphere with a closed convex cone.

If $X \in \mathfrak{a}_G$ the required formula is immediate, both sides being equal to 1. Suppose then that $X \notin \mathfrak{a}_G$. Fix a small positive number ϵ and set

$$\mathscr{H}^+ = \{\lambda \in (\mathfrak{a}_R^G)^* : \lambda(X) \geqq \epsilon\}.$$

Let $\mathscr{P}^+$ be the set of elements P whose simplex c_P belongs to the interior of $\mathscr{H}^+$, and let $\mathscr{P}^0$ be the set of P whose simplex meets the boundary of $\mathscr{H}^+$. We can choose ϵ so that the boundary of $\mathscr{H}^+$ contains no zero simplex of C. Moreover, we can assume that $\mathscr{P}^+$ consists of those P for which $\lambda(X)$ is strictly positive for every λ in c_P. Now $\hat{\tau}_P$ is the characteristic function of the open cone in $\mathfrak{a}_P$ which is dual to the positive chamber in $(\mathfrak{a}_P^G)^*$. It follows that $\hat{\tau}_P(X)$ equals 1 if P belongs to $\mathscr{P}^+$, and equals 0 for any other group $P \in \bar{\bar{\mathscr{F}}}_R(M_1)$ with $P \neq G$. Consequently

$$(5.1) \qquad \sum_{P \in \bar{\bar{\mathscr{F}}}_R(M_1)} (-1)^{\dim(A_P/A_G)} \hat{\tau}_P(X)$$

$$= 1 + \sum_{\{P \in \bar{\bar{\mathscr{F}}}_R(M_1): P \neq G\}} (-1)^{\dim(A_P/A_G)} \hat{\tau}_P(X)$$

$$= 1 + \sum_{P \in \mathscr{P}^+} (-1)^{\dim(A_P/A_G)}.$$

This equals

$$1 - \chi(C^+),$$

where C^+ is the simplicial complex consisting of those simplices in $\mathscr{P}^+$ and $\chi(C^+)$ is its Euler characteristic.

Let D^+ be the intersection of C with $\mathscr{H}^+$. If $R = G_\sigma$, D^+ is a hemisphere and is contractible. If $R \neq G_\sigma$, D^+ is either contractible or the empty set. Therefore, the Euler characteristic $\chi(D^+)$ equals 1 unless D^+ is empty, in which case it equals 0 by definition. Now D^+ is the union of C^+ with

$$\bigcup_{P \in \mathscr{P}^0} (c_P \cap \mathscr{H}^+).$$

We shall exhibit a retraction from D^+ onto C^+. Suppose that $P \in \mathscr{P}^0$. Since ϵ is small, we can take

$$\hat{\Delta}_P = \{\tilde{\varpi}_1, \ldots, \tilde{\varpi}_n\},$$

with

$$\tilde{\varpi}_i(X) \leqq 0, \quad 1 \leqq i \leqq r,$$

and

$$\tilde{\varpi}_j(X) > \epsilon, \quad r + 1 \leqq j \leqq n.$$

An arbitrary element in $c_P \cap \mathscr{H}^+$ is of the form

$$\lambda = t_1\tilde{\varpi}_1 + \ldots + t_n\tilde{\varpi}_n,$$

with $t_i \geqq 0$, $t_1 + \ldots + t_n = 1$, and $\lambda(X) \geqq \epsilon$. The point

$$\lambda_1 = (t_{r+1} + \ldots + t_n)^{-1}(t_{r+1}\tilde{\varpi}_{r+1} + \ldots + t_n\tilde{\varpi}_n)$$

belongs to the intersection of $(c_P \cap \mathscr{H}^+)$ with C^+. If λ belongs to two spaces $c_P \cap \mathscr{H}^+$, λ_1 is still uniquely defined. In fact, it is clear that $\lambda \to \lambda_1$ extends to a continuous retraction of D^+ onto C^+. Therefore,

$$\chi(C^+) = \chi(D^+).$$

We have shown that (5.1) equals 1 if D^+ is empty and equals 0

https://doi.org/10.4153/CJM-1986-009-4 Published online by Cambridge University Press

otherwise. Since ϵ is small, D^+ is empty precisely when $\lambda(X) \leqq 0$ for every $\lambda \in C$. However C and $\hat{\Delta}_R \cup (\mathfrak{a}_{G_\sigma}^G)^*$ both generate the same positive cones in $(\mathfrak{a}_R^G)^*$. (This is just a restatement of the fact that the closure of $\mathfrak{a}_R^+$ is the union over $P \in \bar{\mathscr{F}}_R(M_1)$ of the chambers $\mathfrak{a}_P^+$.) Therefore D^+ is empty precisely when the projection of X onto $\mathfrak{a}_{G_\sigma}^G$ vanishes, and when $\varpi(X) \leqq 0$ for every $\varpi \in \hat{\Delta}_R$. The lemma follows.

LEMMA 5.2. *The expression*

$$\sum_{P \in \mathscr{F}_R(M_1)} (-1)^{\dim(A_P/A_G)} \hat{\tau}_P(X), \quad X \in \mathfrak{a}_R,$$

equals

$$(-1)^{\dim(A_R/A_{G_\sigma})} \hat{\tau}_R(X)$$

if X belongs to $\mathfrak{a}_G + \mathfrak{a}_R^{G_\sigma}$, and vanishes otherwise.

Proof. The given expression equals

$$\sum_{\{S \in \mathscr{F}^\sigma : S \supset R\}} (-1)^{\dim(A_R/A_S)} \times \sum_{P \in \bar{\mathscr{F}}_S(M_1)} (-1)^{\dim(A_P/A_G)} \hat{\tau}_P(X).$$

This follows from the fact that for a given $P \in \bar{\mathscr{F}}_R(M)$, the groups $S \in \mathscr{F}^\sigma$ such that $R \subset S \subset P_\sigma$ are parametrized by the subsets of $\hat{\Delta}_R \setminus \hat{\Delta}_{P_\sigma}$. Apply Lemma 5.1. If X does not belong to $\mathfrak{a}_G + \mathfrak{a}_R^{G_\sigma}$, it projects onto none of the spaces $\mathfrak{a}_G + \mathfrak{a}_S^{G_\sigma}$, and the expression is zero. Suppose then that X does belong to $\mathfrak{a}_G + \mathfrak{a}_R^{G_\sigma}$. Let R' be the group in $\mathscr{F}^\sigma$, with $R \subset R'$, such that

$$\hat{\Delta}_{R'} = \{\varpi \in \hat{\Delta}_R : \varpi(X) \leqq 0\}.$$

Our expression becomes

$$\sum_{\{S \in \mathscr{F}^\sigma : R' \subset S\}} (-1)^{\dim(A_R/A_S)}.$$

It equals 0 if $R' \neq G_\sigma$ and it equals

$$(-1)^{\dim(A_R/A_{G_\sigma})}$$

if $R' = G_\sigma$. However, $R' = G_\sigma$ precisely when $\varpi(X) > 0$ for every $\varpi \in \hat{\Delta}_R$. Therefore the expression equals

$$(-1)^{\dim(A_R/A_{G_\sigma})} \hat{\tau}_R(X),$$

as required.

LEMMA 5.3. *The expression*

$$\sum_{P \in \bar{\mathscr{F}}_R(M_1)} (-1)^{\dim(A_P/A_G)} \tau_R^{P_\sigma}(X) \hat{\tau}_P(X), \quad X \in \mathfrak{a}_R.$$

equals 1 *if* $R = G_\sigma$ *and* X *belongs to* $\mathfrak{a}_G$, *and vanishes otherwise.*

Proof. The given expression equals

$$\sum_{\{S \in \mathscr{F}^\sigma : S \supset R\}} \tau_R^S(X) \sum_{P \in \mathscr{F}_S(M_1)} (-1)^{\dim(A_P/A_G)} \hat{\tau}_P(X).$$

Apply Lemma 5.2. If X does not belong to $\mathfrak{a}_G + \mathfrak{a}_R^{G_\sigma}$, it projects onto none of the spaces $\mathfrak{a}_G + \mathfrak{a}_S^{G_\sigma}$, and the expression is 0. Suppose then that X does belong to $\mathfrak{a}_G + \mathfrak{a}_R^{G_\sigma}$. Our expression becomes

$$\sum_{\{S \in \mathscr{F}^\sigma : S \supset R\}} (-1)^{\dim(A_S/A_{G_\sigma})} \tau_R^S(X) \hat{\tau}_S(X).$$

As we noted in (4.3), this vanishes if $R \neq G_\sigma$. If $R = G_\sigma$, so that $\mathfrak{a}_R^{G_\sigma} = \{0\}$, it clearly equals 1. The lemma follows.

We can now prove Lemma 4.2. We must evaluate the expression

$$(5.2) \qquad \sum_{\{Q \in \bar{\mathscr{F}}_R(M_1) : Q \subset P\}} (-1)^{\dim(A_Q/A_P)} \tau_R^{Q_\sigma}(X) \hat{\tau}_Q^P(X), \quad X \in \mathfrak{a}_R,$$

where now P is a fixed element in $\bar{\mathscr{F}}_R(M_1)$. Let $M = M_P$. The sum in (5.2) may be replaced by a sum over $\bar{\mathscr{F}}^M_{M_\sigma \cap R}(M_1)$, the set of parabolic subsets Q' of M which contain M_1 and such that Q'_σ contains $M_\sigma \cap R$. The resulting expression is just that of Lemma 5.3, but with (G, G_σ, R) replaced by $(M, M_\sigma, M_\sigma \cap R)$. It is easy to see that $M_\sigma = M_\sigma \cap R$ if and only if $P_\sigma = R$. It follows that (5.2) equals 1 if $P_\sigma = R$ and X belongs to $\mathfrak{a}_M = \mathfrak{a}_P$, and equals 0 otherwise. This was the assertion of Lemma 4.2.

6. Reduction to the unipotent case. We shall resume the discussion of Section 3, with the symbols σ, G_σ, M_1 and $M_{1\sigma}$ having the same meaning as there. We will need to fix a maximal compact subgroup

$$K_\sigma = \prod_v K_{\sigma v}$$

of $G_\sigma(\mathbf{A})$ which is admissible relative to $M_{1\sigma}$ in the sense of Section 1 of [**2**]. Then for each group $R \in \mathscr{F}^\sigma$ we have the function

$$H_R : G_\sigma(\mathbf{A}) \to \mathfrak{a}_R$$

defined in the usual way from the decomposition

$$G_\sigma(\mathbf{A}) = N_R(\mathbf{A}) M_R(\mathbf{A}) K_\sigma.$$

For any $x \in G_\sigma(\mathbf{A})$, let $K_R(x)$ be the component of x in K_σ relative to this decomposition. It is determined up to multiplication on the left by elements in $M_R(\mathbf{A}) \cap K_\sigma$.

It is clear that any objects that have been associated to the triple (G, M_0, K) also exist for $(G_\sigma, M_{1\sigma}, K_\sigma)$. In particular, there is the analogue of the point T_0 described in Section 1, a point in $\mathfrak{a}_1$, which we shall denote by $T_{0\sigma}$. Let T_σ be the projection of the point $T - T_0 + T_{0\sigma}$ onto $\mathfrak{a}_1$. Then $T_{0\sigma}$ is the value of T_σ at $T = T_0$. For most of this section we will retain the standard minimal parabolic subset P_0 of G, and we will assume that there is a standard parabolic subset P_1 in $\mathscr{P}(M_1)$. If T is suitably regular with respect to P_0 then T_σ will be suitably regular with respect to $P_{1\sigma}$.

We start with the formula for $J_{\mathfrak{o}}^T(f)$ given by Lemma 3.1. We shall make some changes of variables, at first formally, leaving the justification until later. Change the sum and integral over (ξ, x) in

$$(R(F)\backslash G^0(F)) \times (G^0(F)\backslash G^0(\mathbf{A})^1)$$

to a sum and double integral over (δ, x, y) in

$$(R(F)\backslash G_\sigma(F)) \times (G_\sigma(F)\backslash G_\sigma(\mathbf{A}) \cap G^0(\mathbf{A})^1) \times (G_\sigma(\mathbf{A})\backslash G^0(\mathbf{A})).$$

(Here we should note that

$$G_\sigma(\mathbf{A})\backslash G^0(\mathbf{A}) \cong G_\sigma(\mathbf{A}) \cap G^0(\mathbf{A})^1\backslash G^0(\mathbf{A})^1.)$$

The expression (3.4) becomes

$$\sum_{P \in \mathscr{F}_R(M_1)} (-1)^{\dim(A_P/A_G)}\hat{\tau}_P(H_P(\delta xy) - Z_P(T - T_0) - T_0).$$

Since

$$\begin{aligned} H_P(\delta xy) &= H_R(\delta x) + H_P(K_R(\delta x)y) \\ &= H_R(\delta x) + H_P(K_{P_\sigma}(\delta x)y), \end{aligned}$$

this equals

$$\sum_{P \in \mathscr{F}_R(M_1)} (-1)^{\dim(A_P/A_G)}\hat{\tau}_P((H_R(\delta x) - T_\sigma) - Y_P^T(\delta x, y)),$$

with

$$(6.1) \qquad Y_P^T(\delta x, y) = -H_P(K_{P_\sigma}(\delta x)y) + Z_P(T - T_0) - T_\sigma + T_0.$$

The set

$$\mathscr{Y}_R^T(\delta x, y) = \{Y_P^T(\delta x, y): P \in \mathscr{F}_R(M_1)\}$$

satisfies the compatibility condition of Section 4, so by (4.2)* we can write this last expression as the sum over $\{S \in \mathscr{F}^\sigma: S \supset R\}$ of

$$(6.2) \qquad (-1)^{\dim(A_R/A_S)}\hat{\tau}_R^S(H_R(\delta x) - T_\sigma)$$

$$\times\ \Gamma_S^G(H_S(\delta x) - T_\sigma, \mathscr{Y}_S^T(\delta x, y)).$$

The formula for $J_o^T(f)$ becomes the integral and sum over

$$y \in G_\sigma(\mathbf{A})\backslash G^0(\mathbf{A}),$$
$$\{S \in \mathscr{F}^\sigma : S \supset P_{1\sigma}\},$$
$$x \in G_\sigma(F)\backslash G_\sigma(\mathbf{A}) \cap G^0(\mathbf{A})^1,$$
$$\{R \in \mathscr{F}^\sigma : P_{1\sigma} \subset R \subset S\}, \quad \text{and}$$
$$\delta \in R(F)\backslash G_\sigma(F)$$

of the product of

$$(6.3) \qquad |\iota^G(\sigma)|^{-1} \sum_{u \in \mathscr{U}_{M_R}(F)} \int_{N_R(\mathbf{A})} f(y^{-1}\sigma x^{-1}\delta^{-1}un\delta xy)dn$$

with (6.2).

Next, decompose the sum over $R(F)\backslash G_\sigma(F)$ into a double sum over (μ, ξ) in

$$(R(F) \cap M_S(F)\backslash M_S(F)) \times (S(F)\backslash G_\sigma(F)).$$

Take the resulting sum over ξ outside the sum over R, and combine it with the integral over

$$G_\sigma(F)\backslash G_\sigma(\mathbf{A}) \cap G^0(\mathbf{A})^1.$$

Then decompose the resulting integral over

$$S(F)\backslash G_\sigma(\mathbf{A}) \cap G^0(\mathbf{A})^1$$

into a multiple integral over (v, a, m, k) in

$$(N_S(F)\backslash N_S(\mathbf{A})) \times (A_S^\infty \cap G^0(\mathbf{A})^1) \times (M_S(F)\backslash M_S(\mathbf{A})^1) \times K_\sigma.$$

The variables of integration v and a drop out of (6.3), both being absorbed in the integral over $N_R(\mathbf{A})$. A Jacobian is introduced, but it cancels that of the last change of variables. v also drops out of (6.2), so the integral over $N_S(F)\backslash N_S(\mathbf{A})$ disappears. The variable a remains in (6.2) but occurs only in the second function Γ_S^G. As our final change of variables, we rewrite the integral over $N_R(\mathbf{A})$ in (6.3) as a double integral over (n_1, n_2) in

$$(N_R(\mathbf{A}) \cap M_S(\mathbf{A})) \times N_S(\mathbf{A}).$$

Our formula for $J_o^T(f)$ is now given by the integral and sum over y, S, k, a and m of the product of $|\iota^G(\sigma)|^{-1}$ with

$$(6.4) \qquad \sum_R \sum_\mu \sum_u \int (-1)^{\dim(A_R/A_S)} \Phi_{S,a,k,y}^T(m^{-1}\mu^{-1}un_1\mu m)$$
$$\times\ \hat{\tau}_R^S(H_R(\mu m) - T_\sigma)dn_1,$$

https://doi.org/10.4153/CJM-1986-009-4 Published online by Cambridge University Press

where

$$\Phi^T_{S,a,k,y}(m_2), \quad m_2 \in M_S(\mathbf{A})^1,$$

is defined as the product of

$$\delta_S(m_2)^{1/2} \int_{N_S(\mathbf{A})} f(y^{-1}\sigma k^{-1} m_2 n_2 k y) dn_2,$$

with

$$(6.5) \qquad \Gamma^G_S(H_S(a) - T_\sigma, \mathscr{Y}^T_S(k, y)).$$

(We can obviously insert δ_S, the modular function of $S(\mathbf{A})$. We have included it so that at a later stage our definitions will agree with [**4**].) It is clear that $\Phi^T_{S,a,k,y}$ is a function in $C^\infty_c(M_S(\mathbf{A})^1)$ which depends smoothly on y, k and a.

We must justify the changes of variables. It would be enough to show that (6.4) is absolutely integrable over y, S, k, a and m. For we would then be able to work backwards, verifying each successive change of variables by Fubini's theorem.

LEMMA 6.1. *Given a compact subset Δ of $G(\mathbf{A})^1$ we can choose a compact subset Σ of $G_\sigma(\mathbf{A})\backslash G^0(\mathbf{A})$ such that*

$$y^{-1}\sigma \mathscr{U}_{G_\sigma}(\mathbf{A}) y \cap \Delta = \emptyset, \quad y \in G_\sigma(\mathbf{A})\backslash G^0(\mathbf{A}),$$

unless y belongs to Σ.

In order not to interrupt the discussion, we shall postpone the proof of this lemma until the Appendix.

Consider the expression (6.4) first as a function of (S, a, k, y). The index S of course ranges over a finite set, while k ranges over the compact set K_σ. We restrict y to a fixed compact set of representatives of the points Σ of the last lemma, with Δ taken as the support of f. The variable a intervenes through the function (6.5). Since H_S maps $(A^\infty_S \cap G^0(\mathbf{A})^1)$ isomorphically onto $\mathfrak{a}^G_S$, Lemma 4.1 tells us that (6.5) vanishes for all a outside a compact set which depends continuously on k and y. Since k and y range over compact sets, we see that (6.4) vanishes for (a, k, y) outside a compact set which is independent of m. Now consider (6.4) as a function of $m \in M_S(F)\backslash M_S(\mathbf{A})^1$. We immediately recognize the integrand in the analogue of formula (2.1) for

$$J^{M_S, T_\sigma}_{\text{unip}}(\Phi^T_{S,a,k,y}).$$

In particular, (6.4) is an integrable function of m. In fact, it follows from the proof of Theorem 7.1 of [**1**] that the integral over m of the absolute value of (6.4) is bounded by

$$\|\Phi^T_{S,a,k,y}\|,$$

where $\|\cdot\|$ is a continuous semi-norm on $C_c^\infty(M_S(\mathbf{A})^1)$. Since this is a continuous function of (a, k, y), (6.4) is absolutely integrable over y, S, k, a and m. We have thus shown that all our changes of variables are valid. We have, in addition, proved that $J_o^T(f)$ equals

$$(6.6)\qquad |\iota^G(\sigma)|^{-1}\int_{G_\sigma(\mathbf{A})\backslash G(\mathbf{A})}\sum_{S\supset P_{1\sigma}}$$
$$\times\left(\int_{K_\sigma}\int_{A_S^\infty\cap G^0(\mathbf{A})^1} J_{\text{unip}}^{M_S,T_\sigma}(\Phi_{S,a,k,y}^T)\,da\,dk\right)dy.$$

The distribution J_o is the value at $T = T_0$ of J_o^T, while $J_{\text{unip}}^{M_S}$ is the value at $T = T_0$ of $J_{\text{unip}}^{M_S,T_\sigma}$. Setting $T = T_0$ in (6.6), we obtain

$$J_o(f)$$
$$= |\iota^G(\sigma)|^{-1}\int_{G_\sigma(\mathbf{A})\backslash G^0(\mathbf{A})}\sum_{S\supset P_{1\sigma}}\left(\int\int J_{\text{unip}}^{M_S}(\Phi_{S,a,k,y}^{T_0})\,da\,dk\right)dy.$$

The integration in (a, k), being over a compact set, can be taken inside $J_{\text{unip}}^{M_S}$. The contribution from the integral over a is just the integral of (6.5) at $T = T_0$, which equals

$$(6.7)\qquad \int_{\mathfrak{a}_S^G}\Gamma_S^G(X, \mathscr{Y}_S^{T_0}(k, y))\,dX.$$

Now, for any $Q \in \mathscr{F}_S(M_1)$,

$$Y_Q^{T_0}(k, y) = -H_Q(ky) + T_1,$$

where

$$T_1 = T_0 - T_{0\sigma}.$$

Set

$$v_Q(\lambda, ky, T_1) = e^{\lambda(-H_Q(ky)+T_1)},\quad \lambda \in i\mathfrak{a}_Q^*,$$

so that

$$v_Q'(ky, T_0) = \int_{\mathfrak{a}_Q^G}\Gamma_Q^G(X, -H_Q(ky) + T_1)\,dX,$$

in the notation of Section 4. Then by Lemma 4.1, (6.7) equals

$$v_S'(ky, T_1) = \sum_{Q\in\mathscr{F}_S^0(M_1)} v_Q'(ky, T_1).$$

It follows that

$$(6.8)\qquad J_o(f) = |\iota^G(\sigma)|^{-1}\int_{G_\sigma(\mathbf{A})\backslash G^0(\mathbf{A})}\sum_{\{S\in\mathscr{F}^\sigma : S\supset P_{1\sigma}\}} J_{\text{unip}}^{M_S}(\Phi_{S,y,T_1})\,dy,$$

where

$$\Phi_{S,y,T_1}(m)$$
$$= \delta_S(m)^{1/2} \int_{K_\sigma} \int_{N_S(\mathbf{A})} f(y^{-1}\sigma k^{-1} mnky) v'_S(ky, T_1) dndk,$$

for any $m \in M_S(\mathbf{A})^1$. It is clear that Φ_{S,y,T_1} is a function in $C_c^\infty(M_S(\mathbf{A})^1)$ which depends smoothly on y. We would like to apply the results of [**4**] to (6.8). In order to do so, we ought to free this formula from the dependence on the minimal parabolic subgroup $P_{1\sigma}$.

Suppose that R is any group in $\mathscr{F}^\sigma$. Then

$$R = w_s^{-1} S w_s,$$

for a standard $S \supset P_{1\sigma}$ and an element s in $W_0^{G_\sigma}$, the Weyl group of $(G_\sigma, A_{1\sigma})$. Let $\widetilde{w}_{s\sigma}$ be a representative of s in K_σ. Then if $m_S \in M_S(\mathbf{A})^1$, $\Phi_{S,y,T_1}(m_S)$ equals

$$\delta_R(m_R)^{1/2} \int_{K_\sigma} \int_{N_R(\mathbf{A})} f(y^{-1}\sigma k^{-1} m_R nky) v'_S(\widetilde{w}_{s\sigma} ky, T_1) dndk,$$

where

$$m_R = \widetilde{w}_{s\sigma}^{-1} m_S \widetilde{w}_{s\sigma}.$$

Now suppose that $Q \in \mathscr{F}_S(M_1)$. Then $P = w_s^{-1} Q w_s$ belongs to $\mathscr{F}_R(M_1)$. We can write

$$- H_Q(\widetilde{w}_{s\sigma} ky) + T_1$$
$$= -H_Q(\widetilde{w}_s ky) - H_Q(\widetilde{w}_{s\sigma} w_s^{-1}) - H_Q(w_s \widetilde{w}_s^{-1}) + T_1.$$

(Recall that $\widetilde{w}_s$ is a representative of s in K.) By Lemma 1.1 of [**2**] and the definition of H_P, this equals

$$-sH_P(ky) + T_{0\sigma} - sT_{0\sigma} - T_0 + sT_0 + T_1$$
$$= -sH_P(ky) + sT_1,$$

modulo a vector which is orthogonal to $\mathfrak{a}_Q$. Consequently,

$$-H_Q(\widetilde{w}_{s\sigma} ky) + T_1 = s(-H_P(ky) + T_1),$$

from which it follows that

$$v'_S(\widetilde{w}_{s\sigma} ky, T_1) = v'_R(ky, T_1).$$

We have therefore shown that

$$\Phi_{S,y,T_1}(m_S) = \Phi_{R,y,T_1}(m_R).$$

From this it follows easily that

$$(6.9) \qquad J_{\text{unip}}^{M_S}(\Phi_{S,y,T_1}) = J_{\text{unip}}^{M_R}(\Phi_{R,y,T_1}).$$

(See the remark at the end of Section 2 of [**2**].)

Our progress to this point may be summarized as follows.

LEMMA 6.2. $J_0(f) =$

$$|\iota^G(\sigma)|^{-1}\int_{G_\sigma(\mathbf{A})\backslash G^0(\mathbf{A})}\left(\sum_{R\in\mathscr{F}^\sigma}|W_0^{M_R}|\,|W_0^{G_\sigma}|^{-1}J_{\mathrm{unip}}^{M_R}(\Phi_{R,y,T_1})\right)dy.$$

Proof. Recall that $W_0^{G_\sigma}$ and $W_0^{M_R}$ are the Weyl groups of (G_σ, A_1) and (M_R, A_1) respectively. The number of groups in $\mathscr{F}^\sigma$ which are conjugate to a given R equals $|W_0^{M_R}|\,|W_0^{G_\sigma}|^{-1}$. The lemma then follows from (6.8) and (6.9).

7. Relation with weighted orbital integrals. Lemma 6.2 is our main step. We shall combine it with results from **[4]** and **[5]** to obtain a formula for $J_0(f)$ in terms of weighted orbital integrals.

Suppose that S is a finite set of valuations of F. Set

$$G(F_S)^1 = G(F_S) \cap G(\mathbf{A})^1,$$

where

$$F_S = \prod_{v\in S} F_v.$$

A weighted orbital integral is a distribution

$$f \to J_M(\gamma, f), \quad f \in C_c^\infty(G(F_S)^1),$$

on $G(F_S)^1$ which is associated to an $M \in \mathscr{L}$ and an orbit γ of $M^0(F_S)$ in $M(F_S) \cap G(F_S)^1$. We shall use a descent formula from **[4]**, which we recall in a form applicable here. Suppose that σ is a semisimple element in $M(F)$. Set

$$D^G(\sigma) = \det(1 - \mathrm{Ad}(\sigma))_{\mathfrak{g}/\mathfrak{g}_\sigma},$$

where $\mathfrak{g}$ and $\mathfrak{g}_\sigma$ are the Lie algebras of G and G_σ. Then $D^G(\sigma)$ belongs to F^*. As in **[5]**, we write

$$(\mathscr{U}_{M_\sigma}(F))_{M_\sigma,S}$$

for the finite set of unipotent $M_\sigma(F_S)$ conjugacy classes which meet $M_\sigma(F)$. If

$$u \in (\mathscr{U}_{M_\sigma}(F))_{M_\sigma,S},$$

the element $\gamma = \sigma u$ represents an $M^0(F_S)$ orbit in $M(F_S) \cap G(F_S)^1$. For any $f \in C_c^\infty(G(F_S)^1)$, Corollary 8.7 of **[4]** asserts that

(7.1) $J_M(\gamma, f)$

$$= |D^G(\sigma)|_S^{1/2} \int_{G_\sigma(F_S)\backslash G^0(F_S)} \left(\sum_{R \in \mathscr{F}^\sigma(M_\sigma)} J_{M_\sigma}^{M_R}(u, \Phi_{R,y,T_1}) \right) dy,$$

where

$$|D^G(\sigma)|_S = \prod_{v \in S} |D^G(\sigma)|_v,$$

and Φ_{R,y,T_1} is the function in $C_c^\infty(M_R(F_S)^1)$ defined by a formula analogous to that of Section 6. That is,

(7.2) $\Phi_{R,y,T_1}(m)$

$$= \delta_R(m)^{1/2} \int_{K_\sigma} \int_{N_R(F_S)} f(y^{-1}\sigma k^{-1} mnky) v'_R(ky, T_1) dndk,$$

for any $y \in G^0(F_S)$, $T_1 \in \mathfrak{a}_0$ and $m \in M_R(F_S)^1$.

Suppose that S contains the Archimedean valuations. Then we embed $C_c^\infty(G(F_S)^1)$ in $C_c^\infty(G(\mathbf{A})^1)$ by taking the product of a given function in $C_c^\infty(G(F_S)^1)$ with the characteristic function of $\prod_{v \notin S} (K_v^+ \cap G(F_v))$. In this way $C_c^\infty(G(F_S)^1)$ is to be regarded as a closed subspace of $C_c^\infty(G(\mathbf{A})^1)$. Any function in $C_c^\infty(G(\mathbf{A})^1)$ belongs to $C_c^\infty(G(F_S)^1)$ for some such S.

Let $\mathfrak{o}$, σ and M_1 be fixed as in the last section. Let S_0 be a finite set of valuations of F, containing the Archimedean places, such that for any v not in S_0 the following four conditions are satisfied.

(i) $|D^G(\sigma)|_v = 1$.

(ii) $K_v \cap G_\sigma(F_v) = K_{\sigma v}$.

(iii) $\sigma K_v \sigma^{-1} = K_v$.

(iv) If $y_v \in G^0(F_v)$ is such that $y_v^{-1}\sigma \mathscr{U}_{G_\sigma}(F_v) y_v$ meets σK_v, then y_v belongs to $G_\sigma(F_v)K_v$.

It is clear that the first three conditions can be made to hold; the fourth is a consequence of Lemma 6.1. Let S be a finite set of valuations which contains S_0, and take $f \in C_c^\infty(G(\mathbf{A})^1)$ to be in (the image of) $C_c^\infty(G(F_S)^1)$. The results of Section 6 tell us that the function

$$y \to \sum_{R \in \mathscr{F}^\sigma} |W_0^{M_R}| \, |W_0^{G_\sigma}|^{-1} J_{\text{unip}}^{M_R}(\Phi_{R,y,T_1}), \quad y \in G^0(\mathbf{A}),$$

is left $G_0(\mathbf{A})$-invariant. It vanishes unless

$$y = y_S y',$$

with $y_S \in G_\sigma(F_S)\backslash G^0(F_S)$, and y' an element in

$$\prod_{v \notin S} (K_{\sigma v}\backslash K_v) = \prod_{v \notin S} ((K_v \cap G_\sigma(F_v))\backslash K_v)$$

$$\cong \prod_{v \notin S} (G_\sigma(F_v)\backslash G_\sigma(F_v)K_v).$$

(This follows from conditions (ii) and (iv) above and the fact that $J^{M_R}_{\text{unip}}$ annihilates any function that vanishes on the unipotent set in $M_R(\mathbf{A})^1$.) If $k = k_S k'$, with

$$k_S \in \prod_{v \in S} K_{\sigma v} \quad \text{and} \quad k' \in \prod_{v \notin S} K_{\sigma v},$$

then for any such y

$$v'_R(ky, T_1) = v'_R(k_S y_S, T_1).$$

It follows that Φ_{R,y,T_1}, defined in Section 6 as a function in $C_c^\infty(M_R(\mathbf{A})^1)$, actually equals Φ_{R,y_S,T_1} and is defined by (7.2) as a function in the subspace $C_c^\infty(M_R(F_S)^1)$. The formula of Lemma 6.2 becomes

$$J_0(f) = |\iota^G(\sigma)|^{-1} \int_{G_\sigma(F_S)\backslash G^0(F_S)} \left(\sum_{R \in \mathscr{F}^\sigma} |W_0^{M_R}|\,|W_0^{G_\sigma}|^{-1} J^{M_R}_{\text{unip}}(\Phi_{R,y,T_1}) \right) dy.$$

Let

$$\mathscr{L}^\sigma = \mathscr{L}^{G_\sigma}(M_{1\sigma})$$

denote the set of Levi subgroups of G_σ which contain $M_{1\sigma}$. We apply the main result (Corollary 8.3) of **[5]** to write $J^{M_R}_{\text{unip}}$ as a linear combination of weighted orbital integrals. We obtain

$$J^{M_R}_{\text{unip}}(\Phi_{R,y,T_1}) = \sum_{\{L \in \mathscr{L}^\sigma : L \subset M_R\}} |W_0^L|\,|W_0^{G_\sigma}|^{-1} \times \sum_{u \in (\mathscr{U}_L(F))_{L,S}} a^L(S, u) J_L^{M_R}(u, \Phi_{R,y,T_1}),$$

for complex numbers $a^L(S, u)$. Consequently, $J_0(f)$ equals

$$|\iota^G(\sigma)|^{-1} \int_{G_\sigma(F_S)\backslash G^0(F_S)} \sum_{L \in \mathscr{F}^\sigma} \sum_{R \in \mathscr{F}^\sigma(L)} |W_0^L|\,|W_0^{G_\sigma}|^{-1} \times \sum_{u \in (\mathscr{U}_L(F))_{L,S}} a^L(S, u) J_L^{M_R}(u, \Phi_{R,y,T_1}) dy,$$

where $\mathscr{F}^\sigma(L)$ is the set of elements in $\mathscr{F}^\sigma$ which contain L.

Now, let $\mathscr{L}^0_\sigma(M_1)$ be the set of $M \in \mathscr{L}(M_1)$ such that $A_M = A_{M_\sigma}$. If L is any group in $\mathscr{L}^\sigma$ and M is the centralizer in G of A_L, then M belongs to $\mathscr{L}^0_\sigma(M_1)$ and $M_\sigma = L$. It follows that $M \to M_\sigma$ is a bijection from $\mathscr{L}^0_\sigma(M_1)$ onto $\mathscr{L}^\sigma$. We rewrite our formula for $J_0(f)$ as the sum over

$$M \in \mathscr{L}^0_\sigma(M_1) \quad \text{and} \quad u \in (\mathscr{U}_{M_\sigma}(F))_{M_\sigma,S}$$

of the product of

$$|\iota^G(\sigma)|^{-1}|W_0^{M_\sigma}|\,|W_0^{G_\sigma}|^{-1}a^{M_\sigma}(S, u)$$

with

$$(7.3) \qquad \int_{G_\sigma(F_S)\backslash G^0(F_S)} \left(\sum_{R \in \mathscr{F}^\sigma(M_\sigma)} J^{M_R}_{M_\sigma}(u, \Phi_{R,y,T_1}) \right) dy.$$

By our choice of S, $|D^G(\sigma)|_v = 1$ for each v not in S. Therefore

$$|D^G(\sigma)|_S = \prod_{v \in S} |D^G(\sigma)|_v \cdot \prod_{v \notin S} |D^G(\sigma)|_v = 1$$

by the product formula. In other words, (7.3) can be multiplied by $|D^G(\sigma)|_S^{1/2}$ without changing its value. Consequently (7.3) equals $J_M(\sigma u, f)$ by the descent formula (7.1).

We have proved

LEMMA 7.1. *There is a finite set S_0 of valuations of F, which contains the Archimedean places, such that for any finite $S \supset S_0$ and $f \in C_c^\infty(G(F_S)^1)$,*

$$J_0(f) = |\iota^G(\sigma)|^{-1} \sum_{M \in \mathscr{L}^0_\sigma(M_1)} |W_0^{M_\sigma}|\,|W_0^{G_\sigma}|^{-1}$$

$$\times \sum_{u \in (\mathscr{U}_{M_\sigma}(F))_{M_\sigma,S}} a^{M_\sigma}(S, u) J_M(\sigma u, f).$$

8. The main theorem. In Lemma 7.1 we have what is essentially our final formula for $J_0(f)$. However, it will be more useful if we rewrite it in a way that does not depend on a distinguished element σ in $\mathfrak{o}$.

Let M be a Levi subset of G, and let σ be an arbitrary semisimple element in $M(F)$. We shall say that σ is *F-elliptic* in M if σ commutes with a maximal torus in M^0 which is F-anisotropic modulo A_M. If σ is the fixed element of Lemma 7.1, then σ is F-elliptic in M if and only if M belongs to the set $\mathscr{L}^0_\sigma(M_1)$.

Suppose that γ is any element in $M(F)$ with semisimple Jordan component σ. If γ' is another element in $M(F)$, we shall say that γ' is *(M, S)-equivalent* to γ if there is a $\delta \in M^0(F)$ with the following two properties.

(i) σ is also the semisimple Jordan component of $\delta^{-1}\gamma'\delta$.

(ii) $\sigma^{-1}\gamma$ and $\sigma^{-1} \cdot \delta^{-1}\gamma'\delta$, regarded as unipotent elements in $M_\sigma(F_S)$, are $M_\sigma(F_S)$-conjugate.

Notice that there could be several classes u in $(\mathscr{U}_{M_\sigma}(F))_{M_\sigma,S}$ such that σu is (M, S)-equivalent to γ. The set of all such u, which we denote simply by $\{u:\sigma u \sim \gamma\}$, has a transitive action under the finite group

$$\iota^M(\sigma) = M_\sigma(F)\backslash M(F, \sigma).$$

It is, in particular, finite. Define

$$(8.1) \quad a^M(S, \gamma) = \epsilon^M(\sigma)\,|\iota^M(\sigma)|^{-1} \sum_{\{u:\sigma u \sim \gamma\}} a^{M_\sigma}(S, u),$$

where $\epsilon^M(\sigma)$ equals 1 if σ is F-elliptic in M, and is 0 otherwise. Clearly $a^M(S, \gamma)$ depends only on the (M, S)-equivalence class of γ.

Returning to our study of the class $\mathfrak{o}$, we write

$$(M(F) \cap \mathfrak{o})_{M,S}$$

for the set of (M, S)-equivalence classes in $M(F) \cap \mathfrak{o}$. It is finite. Our main result is

THEOREM 8.1. *There is a finite set S_0 of valuations of F, which contains the Archimedean places, such that for any finite set $S \supset S_0$ and any $f \in C_c^\infty(G(F_S)^1)$,*

$$(8.2) \quad J_{\mathfrak{o}}(f) = \sum_{M \in \mathscr{L}} |W_0^M|\,|W_0^G|^{-1} \sum_{\gamma \in (M(F) \cap \mathfrak{o})_{M,S}} a^M(S, \gamma) J_M(\gamma, f).$$

Proof. We will deduce (8.2) from Lemma 7.1. We let σ and M_1 be as in Lemma 7.1. Using the Jordan decomposition, we write the sum over γ in (8.2) as a double sum over semisimple classes and unipotent classes. Combine the first of these with the sum over M. We obtain a sum over the set

$$\Pi = \{ (M, \sigma_M) \},$$

in which M belongs to $\mathscr{L}$ and σ_M is a semisimple $M^0(F)$-orbit in $M(F)$ which is $G^0(F)$ conjugate to σ. The Weyl group W_0 clearly operates on Π. Now, it follows from the definitions that

$$a^{w_s M w_s^{-1}}(S, w_s \gamma w_s^{-1}) = a^M(S, \gamma)$$

for any $s \in W_0$. Moreover, we observed in **[4]** that

$$J_{w_s M w_s^{-1}}(w_s \gamma w_s^{-1}, f) = J_M(\gamma, f).$$

Therefore the sum over Π can be replaced by a sum over the orbits of W_0, provided that each summand is multiplied by the quotient of $|W_0|$ by the order of the isotropy subgroup.

Every W_0 orbit in Π contains pairs of the form (M, σ), where M is some element in $\mathscr{L}(M_1)$. Note that the isotropy group of (M, σ) in W_0 contains $W_0^{M^0}$, the Weyl group of (M^0, A_0). Suppose that (M, σ) and

$$(M', \sigma) = w_s(M, \sigma)w_s^{-1}, \quad s \in W_0,$$

are two such pairs in the same W_0-orbit. Then M' equals $w_s M w_s^{-1}$, and s has a representative in $G^0(F)$ which lies in $G^0(F, \sigma)$. Since M and M' both contain M_1, we can choose the representative of s to lie in

$$\mathrm{Norm}(A_1, G^0(F, \sigma)),$$

the normalizer of A_1 in $G^0(F, \sigma)$. Now there is an injection

$$\mathrm{Norm}(A_1, M^0(F, \sigma))\backslash\mathrm{Norm}(A_1, G^0(F, \sigma)) \to W_0^{M^0}\backslash W_0.$$

We have just proved that the coset of s modulo $W_0^{M^0}$ lies in the image. Conversely, suppose that s is any coset of $W_0^{M^0}\backslash W_0$ which belongs to the image. Then the pair $w_s(M, \sigma)w_s^{-1}$ will clearly be of the above special form. Notice that

$$\mathrm{Norm}(A_1, M^0(F, \sigma))\backslash\mathrm{Norm}(A_1, G^0(F, \sigma)) = W_0^{M^0(F,\sigma)}\backslash W_0^{G^0(F,\sigma)},$$

where

$$W_0^{G^0(F,\sigma)} = M_1^0(F, \sigma)\backslash\mathrm{Norm}(A_1, G^0(F, \sigma))$$

and $W_0^{M^0(F,\sigma)}$ is the analogous group for M. Thus, we can replace the original sum over Π by a sum over all pairs (M, σ), as long as we multiply each summand by

$$(|W_0^{M^0}|\,|W_0|^{-1})^{-1}|W_0^{M^0(F,\sigma)}|\,|W_0^{G^0(F,\sigma)}|^{-1}.$$

Since

$$|W_0^{M^0}|\,|W_0|^{-1} = |W_0^M|\,|W_0^G|^{-1},$$

we have established that the right hand side of (8.2) is equal to

$$\sum_{M\in\mathcal{L}(M_1)} |W_0^{M^0(F,\sigma)}|\,|W_0^{G^0(F,\sigma)}|^{-1}$$
$$\times \sum_{\{\gamma\in(M(F)\cap\mathfrak{o})_{M,S}:\gamma_s=\sigma\}} a^M(S, \gamma)J_M(\gamma, f).$$

We must show that this is equal to the right hand side of the formula in Lemma 7.1.

We have noted that σ is F-elliptic in M, $M \in \mathcal{L}(M_1)$, if and only if M belongs to $\mathcal{L}_\sigma^0(M_1)$. It follows from the definition (8.1) that the right hand side of the formula in Lemma 7.1 equals

$$\sum_{M\in\mathcal{L}(M_1)} |W_0^{M_\sigma}|\,|W_0^{G_\sigma}|^{-1}|\iota^M(\sigma)|\,|\iota^G(\sigma)|^{-1}$$
$$\times \sum_{\{\gamma\in(M(F)\cap\mathfrak{o})_{M,S}:\gamma_s=\sigma\}} a^M(S, \gamma)J_M(\gamma, f).$$

To complete the proof of the theorem, we have only to verify that

$$(8.5)\quad |W_0^{M_\sigma}|\,|W_0^{G_\sigma}|^{-1}|\iota^M(\sigma)|\,|\iota^G(\sigma)|^{-1} = |W_0^{M^0(F,\sigma)}|\,|W_0^{G^0(F,\sigma)}|^{-1}.$$

Recall that

$$\iota^G(\sigma) = G_\sigma(F)\backslash G^0(F, \sigma).$$

Every coset in $G_\sigma(F)\backslash G^0(F, \sigma)$ has an element which normalizes A_1, so there is a surjective map

$$G_\sigma(F)\backslash G^0(F, \sigma) \to W_0^{G_\sigma}\backslash W_0^{G^0(F,\sigma)}.$$

The kernel is the subgroup of cosets having representatives that act trivially on A_1. It is isomorphic to

$$G_\sigma(F) \cap M_1\backslash G^0(F, \sigma) \cap M_1 = M_{1\sigma}(F)\backslash M_1^0(F, \sigma) = \iota^{M_1}(\sigma).$$

It follows that

$$|W_0^{G_\sigma}|\,|\iota^G(\sigma)| = |W_0^{G^0(F,\sigma)}|\,|\iota^{M_1}(\sigma)|.$$

Similarly,

$$|W_0^{M_\sigma}|\,|\iota^M(\sigma)| = |W_0^{M^0(F,\sigma)}|\,|\iota^{M_1}(\sigma)|.$$

This establishes (8.5) and completes the proof of our theorem.

Suppose that $\gamma \in \mathfrak{o}$ is semisimple. Then by (8.1),

$$a^G(S, \gamma) = \epsilon^G(\gamma)\,|\iota^G(\gamma)|^{-1}a^{G_\sigma}(S, 1).$$

Combined with Corollary 8.5 of **[4]**, this immediately gives a simple formula for $a^G(S, \gamma)$. We state it separately as a theorem, since it will be important for future applications.

THEOREM 8.2. *Suppose that γ is a semisimple element in $\mathfrak{o}$. Then for any finite set $S \supset S_0$, $a^G(S, \gamma)$ equals*

$$|G_\gamma(F)\backslash G(F, \gamma)|^{-1}\ \mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbf{A})^1)$$

if γ is F-elliptic, and is 0 *otherwise.*

Remarks. 1. In this paper we have not normalized the invariant measures. However, we have implicitly assumed that they satisfy any required compatability conditions. If γ is F-elliptic (in G), the measure on $G_\gamma(\mathbf{A})^1$ implicit in Theorem 8.2 must be compatible with the measure used to define the orbital integral $J_G(\gamma, f)$. The orbital integral relies on a choice of measure on $G_\gamma(F_S)\backslash G^0(F_S)$, and since $S \supset S_0$, this amounts to a choice of measure on

$$G_\gamma(\mathbf{A})\backslash G^0(\mathbf{A}) \cong G_\gamma(\mathbf{A}) \cap G^0(\mathbf{A})^1\backslash G^0(\mathbf{A})^1 = G_\gamma(\mathbf{A})^1\backslash G^0(\mathbf{A})^1.$$

The measure on $G_\gamma(\mathbf{A})^1$ used to define this quotient measure must be the same as the one above.

2. Suppose that the class $\mathfrak{o}$ consists entirely of semisimple elements. Then Theorems 8.1 and 8.2 provide a closed formula for $J_{\mathfrak{o}}(f)$. (See Proposition 5.3.6 of **[6]**.) In the special case that $\mathfrak{o}$ is unramified (in the sense of **[1]**), the formula is easily seen to reduce to (8.7) of **[1]**.

3. Theorem 8.2 is probably sufficient for many applications of the trace formula. This is fortunate, since if γ is not semisimple, any general formula for $a^G(S, \gamma)$ is likely to be quite complicated. If $G = GL_2$ and γ is principal unipotent, a formula for $a^G(S, \gamma)$ is implicit in term (v) on page 516 of [**9**]. For GL_3 there are formulas for $a^G(S, \gamma)$ which can be extracted from [**7**].

9. The fine $\mathfrak{o}$-expansion. Set

$$J(f) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}(f), \quad f \in C_c^{\infty}(G(\mathbf{A})^1),$$

the left hand side of the trace formula. It is a distribution, which of course also equals

$$\sum_{\chi \in \mathscr{X}} J_{\chi}(f)$$

(the right hand side of the trace formula). In some situations it is easier to take $J(f)$ as a single entity, without worrying about which terms come from a given $\mathfrak{o}$. For convenience, we shall restate Theorem 8.1 as a formula for $J(f)$.

We need to know that only finitely many $\mathfrak{o}$ intervene for a given f.

LEMMA 9.1. *Suppose that Δ is a compact subset of $G(\mathbf{A})^1$. Then there are only finitely many classes $\mathfrak{o} \in \mathcal{O}$ such that the set*

$$\mathrm{ad}(G^0(\mathbf{A}))\mathfrak{o} = \{x^{-1}\gamma x : x \in G^0(\mathbf{A}), \gamma \in \mathfrak{o}\}$$

meets Δ.

We shall prove this lemma in the Appendix.

Now suppose that Δ is a compact neighborhood of 1 in $G(\mathbf{A})^1$. There is certainly a finite set S of valuations of F, which contains the Archimedean places, such that Δ is the product of a compact neighborhood of 1 in $G(F_S)^1$ with the characteristic function of $\prod_{v \notin S} K_v$. We shall write S_{Δ}^0 for the minimal such set. Let $C_{\Delta}^{\infty}(G(\mathbf{A})^1)$ denote the space of functions in $C_c^{\infty}(G(\mathbf{A})^1)$ which are supported on Δ, and set

$$C_{\Delta}^{\infty}(G(F_S)^1) = C_{\Delta}^{\infty}(G(\mathbf{A})^1) \cap C_c^{\infty}(G(F_S)^1),$$

for any finite set $S \supset S_{\Delta}^0$.

THEOREM 9.2. *Given a compact neighborhood Δ of 1 in $G(\mathbf{A})^1$ we can find a finite set $S_{\Delta} \supset S_{\Delta}^0$ of valuations of F such that for any finite $S \supset S_{\Delta}$, and any $f \in C_{\Delta}^{\infty}(G(F_S)^1)$,*

$$J(f) = \sum_{M \in \mathscr{L}} |W_0^M| \, |W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}} a^M(S, \gamma) J_M(\gamma, f).$$

Proof. We shall apply Theorem 8.1 to the definition

$$J(f) = \Sigma J_{\mathfrak{o}}(f).$$

Let $\mathcal{O}_\Delta$ be the finite set of classes $\mathfrak{o}$ such that $\mathrm{ad}(G^0(\mathbf{A}))\mathfrak{o}$ meets Δ. We define S_Δ to be the union of S_Δ^0 with the sets $S_{\mathfrak{o}}$ given by Theorem 8.1, as $\mathfrak{o}$ ranges over $\mathcal{O}_\Delta$. Take any finite set $S \supset S_\Delta$, and let $f \in C_\Delta^\infty(G(F_S)^1)$. Since $J_{\mathfrak{o}}$ annihilates any function which vanishes on $\mathrm{ad}(G^0(\mathbf{A}))\mathfrak{o}$, we obtain

$$J(f) = \sum_{\mathfrak{o}\in\mathcal{O}_\Delta} J_{\mathfrak{o}}(f)$$

$$= \sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1} \sum_{\mathfrak{o}\in\mathcal{O}_\Delta}\ \sum_{\gamma\in(M(F)\cap\mathfrak{o})_{M,S}} a^M(S,\gamma)J_M(\gamma,f).$$

Now suppose that γ is any element in $(M(F))_{M,S}$. Then γ is contained in a unique $\mathfrak{o} \in \mathcal{O}$. It is a consequence of Theorem 5.2 of **[4]** that the orbital integral $J_M(\gamma, f)$ equals 0 if f vanishes on $\mathrm{ad}(G^0(\mathbf{A}))\mathfrak{o}$. In particular, $J_M(\gamma, f)$ vanishes unless $\mathfrak{o}$ belongs to $\mathcal{O}_\Delta$. It follows that

$$J(f) = \sum_{M\in\mathcal{L}} |W_0^M|\,|W_0^G|^{-1} \sum_{\gamma\in(M(F))_{M,S}} a^M(S,\gamma)J_M(\gamma,f),$$

as required.

Appendix. We still owe the proofs of Lemmas 6.1 and 9.1. We shall establish them by global means, using reduction theory and some familiar arguments from the derivation of the trace formula. Viewed in this way, the lemmas are rather closely related.

Let $G(F)'$ be the set of elements in $G(F)$ which belong to no proper parabolic subset of G which is defined over F. We fix a minimal parabolic subgroup P_0^0 of G^0 with Levi component M_0. For convenience, write 0 for any subscript or superscript where our notation would normally call for P_0^0. In particular, $H_0 = H_{P_0^0}$, $\Delta_0 = \Delta_{P_0^0}$, and $N_0 = N_{P_0^0}$. If T is any point in $\mathfrak{a}_0$, set

$$A(T) = \{a \in A_0^\infty \cap G^0(\mathbf{A})^1 : \alpha(H_0(a) - T) > 0,\ \alpha \in \Delta_0\}.$$

LEMMA A.1. *For any compact subset Γ of $G(\mathbf{A})^1$ and any $T \in \mathfrak{a}_0$, there is a compact subset $A(T)_\Gamma$ of $A_0^\infty \cap G^0(\mathbf{A})^1$ with the following property. If a is a point in $A(T)$ such that $a^{-1}\gamma a$ belongs to Γ for some $\gamma \in G(F)'$, then a lies in $A(T)_\Gamma$.*

Proof. Let ϵ be a fixed element in $G(F)$ which normalizes both P_0^0 and M_0. Then ϵ normalizes A_0 and therefore acts on $\mathfrak{a}_0$. Any element in $G(F)'$ can be written

https://doi.org/10.4153/CJM-1986-009-4 Published online by Cambridge University Press

(A.1) $\gamma = \nu w_s \pi \epsilon, \quad \nu \in N_0(F), \pi \in P_0^0(F), s \in W_0,$

by the Bruhat decomposition. Let ρ_0 be the vector in $\mathfrak{a}_0^*$, defined as usual by

$$\delta_0(a)^{1/2} = e^{\rho_o(H_0(a))}, \quad a \in A_0.$$

We claim that there is a constant c_Γ, depending only on Γ, with the property that

$$(\rho_0 - s\rho_0)(H_0(a)) \leqq c_\Gamma,$$

for any $a \in A(T)$ and γ as in (A.1) such that $a^{-1}\gamma a$ belongs to Γ. To see this, note that $\epsilon\rho_0 = \rho_0$. This implies the existence of a finite dimensional representation Λ of G^+, defined over F, with the highest weight a positive multiple of ρ_0. Let ϕ be a highest weight vector for Λ. By choosing a height function for Λ, and computing the component of the vector $\Lambda(a^{-1}\gamma a)\phi$ in the direction of $\Lambda(w_s)\phi$, (as for example on p. 944 of [**1**]), we see that the claim follows.

Given γ as in (A.1), let $P_1^0 \supset P_0^0$ be the smallest standard parabolic subgroup of G^0 which contains w_s. As with P_0^0 above, write 1 for any subscript or superscript where our notation would normally call for P_1^0. Thus, $\mathfrak{a}_1$ equals $\mathfrak{a}_{P_1^0}$, $\mathfrak{a}_0^1$ is the orthogonal complement of $\mathfrak{a}_1$ in $\mathfrak{a}_0$, and Δ_0^1 is the set of roots in Δ_0 which vanish on $\mathfrak{a}_1$. If a is any point in $A(T)$ such that $a^{-1}\gamma a$ belongs to Γ, we set

(A.2) $H_0(a) = X + Y, \quad X \in \mathfrak{a}_0^1, \ Y \in \mathfrak{a}_1^G.$

(The superscript G denotes the orthogonal complement of $\mathfrak{a}_G$.) We shall first show that X belongs to a compact subset of $\mathfrak{a}_0^1$ which depends only on Γ. The element w_s is contained in M_1 but in no proper parabolic subgroup of M_1. This implies that

$$\rho_0 - s\rho_0 = \sum_{\beta \in \Delta_0^1} c_\beta \beta,$$

where each c_β is a positive integer. From what we have proved above, we see that X belongs to

$$\{H \in \mathfrak{a}_0^1 : \sum c_\beta \beta(H) \leqq c_\Gamma; \ \alpha(H) \geqq \alpha(T), \ \alpha \in \Delta_0^1\}.$$

This is a compact subset of $\mathfrak{a}_0^1$ which certainly depends only on Γ.

All that remains is to show that the element Y in (A.2) lies in a compact subset of $\mathfrak{a}_1^G$. Our discussion at this point is motivated by some observations of Labesse (Lecture 4 of [**6**]). The element $a^{-1}\gamma a\epsilon^{-1}$ lies in the compact set $\Gamma\epsilon^{-1}$. But the element

$$\gamma\epsilon^{-1} = \nu w_s \pi$$

lies in $P_1^0(F)$. Therefore

$$H_1(a^{-1}\gamma a\epsilon^{-1}) = H_1(a^{-1}\epsilon a\epsilon^{-1}) = (eX - X)_1 + (eY - Y)_1,$$

where $(\cdot)_1$ denotes the projection of $\mathfrak{a}_0^G$ onto $\mathfrak{a}_1^G$, and $e = \mathrm{Ad}(\epsilon)$. Consequently, $(eY - Y)_1$ lies in a compact set. We shall show that the linear map

$$H \rightarrow (eH - H)_1, \quad H \in \mathfrak{a}_1^G,$$

is injective. Suppose that $Z \in \mathfrak{a}_1^G$ lies in the null space. Then

$$(eZ)_1 = Z_1 = Z.$$

Since e is an isometry on $\mathfrak{a}_0^G$, and $(\cdot)_1$ is an orthogonal projection, eZ must equal Z. Now the chamber in $\mathfrak{a}_1^G$ associated to P_1^0 meets the space

$$\mathfrak{b} = \{H \in \mathfrak{a}_1^G : eH = H\}$$

in an open subset. Consequently there is a parabolic subset P of G such that P^0 contains P_1^0, and such that $\mathfrak{a}_P^G = \mathfrak{b}$. But the element γ belongs to $P \cap G(F)'$. It follows that $P = G$, so that $\mathfrak{a}_P^G = \{0\}$. Thus, the point Z equals 0, and the linear map above is injective. We have shown that Y lies in a compact subset of $\mathfrak{a}_1^G$ which depends only on Γ. The proof of the lemma is complete.

COROLLARY A.2. *There is a compact subset G_Γ of $G^0(\mathbf{A})^1$ such that*

$$x^{-1}G(F)'x \cap \Gamma = \emptyset, \quad x \in G^0(\mathbf{A})^1,$$

unless x belongs to $G^0(F)G_\Gamma$.

Proof. Suppose that $x \in G^0(\mathbf{A})^1$. By reduction theory, x is congruent modulo (left translation by) $G^0(F)$ to an element

$$pak, \quad p \in \omega, a \in A(T_1), k \in K,$$

where ω is a compact subset of $N_0(\mathbf{A})M_0(\mathbf{A})^1$ and T_1 is a fixed point in $\mathfrak{a}_0$. The set

$$\omega_1 = \{a^{-1}pa : a \in A(T_1), p \in \omega\}$$

is compact and so therefore is

$$\Gamma_1 = \{pkgk^{-1}p^{-1} : p \in \omega_1, k \in K, g \in \Gamma\}.$$

If $x^{-1}G(F)'x$ intersects Γ, there is an element $\gamma \in G(F)'$ such that $a^{-1}\gamma a$ belongs to Γ_1. By the lemma, a then belongs to the compact set $A(T_1)_{\Gamma_1}$. The corollary follows.

Proof of Lemma 6.1. We are given a class $\mathfrak{o} \in \mathcal{O}$ and a semisimple element σ in $\mathfrak{o}$, as well as a compact subset Δ of $G(\mathbf{A})^1$. We shall first consider the case that σ belongs to $G(F)'$. Then $\mathfrak{o}$ is just the $G^0(F)$-orbit of σ. Suppose that $y^{-1}\mathfrak{o}y$ intersects Δ for some $y \in G^0(\mathbf{A})$. By the last corollary, y is $G^0(F)$-congruent to an element in the compact set G_Δ. But there are only finitely many elements $\delta \in G_\sigma(F)\backslash G^0(F)$ such that $x^{-1}\delta^{-1}\gamma\delta x$ belongs to Δ for some $x \in G_\Delta$. Since the projection of G_Δ onto $G^0(F)\backslash G^0(\mathbf{A})^1$ is compact, we see that

$$\{y \in G_\sigma(F)\backslash G^0(\mathbf{A})^1 : y^{-1}\sigma y \cap \Delta \neq \emptyset\}$$

is a compact subset of $G_\sigma(F)\backslash G^0(\mathbf{A})^1$. Its projection onto

$$G_\sigma(\mathbf{A}) \cap G^0(\mathbf{A})^1\backslash G^0(\mathbf{A})^1 = G_\sigma(\mathbf{A})\backslash G^0(A)$$

is compact. Thus, Lemma 6.1 holds if σ belongs to $G(F)'$.

Now suppose that σ is an arbitrary semisimple element in $G(F)$. By replacing σ with a $G^0(F)$-conjugate if necessary, we can assume that σ belongs to $M(F)'$ for a Levi subset $M \in \mathscr{F}$. (This Levi subset was denoted M_1 earlier.) Choose a parabolic subset $P \in \mathscr{P}(M)$, and write $N = N_P$. Let N_σ denote the centralizer of σ in N. Then any element in $\mathscr{U}_{G_\sigma}(\mathbf{A})$ has a $G_\sigma(\mathbf{A})$ conjugate in $N_\sigma(\mathbf{A})$. Suppose that y is any point in $G^0(\mathbf{A})$ such that

$$y^{-1}\sigma\mathscr{U}_{G_\sigma}(\mathbf{A})y \cap \Delta \neq \emptyset.$$

Then y is congruent modulo $G_\sigma(\mathbf{A})$ to an element

$$nmk, \quad n \in N(\mathbf{A}), m \in M^0(\mathbf{A}), k \in K,$$

such that

$$m^{-1}n^{-1}\sigma N_\sigma(\mathbf{A})nm \tag{A.3}$$

meets the compact set

$$\{kgk^{-1} : k \in K, g \in \Delta\}.$$

Since the set (A.3) is contained in $m^{-1}\sigma mN(\mathbf{A})$, the element $m^{-1}\sigma m$ lies in a fixed compact subset of $M(\mathbf{A})$. Applying the case we have already established (with G replaced by M), we see the projection of m onto $M_\sigma(\mathbf{A})\backslash M^0(\mathbf{A})$ lies in a compact set. We can therefore choose m to lie in a fixed compact subset of $M^0(\mathbf{A})$. But then $\sigma^{-1}n^{-1}\sigma N_\sigma(\mathbf{A})n$ intersects a fixed compact subset of $N(\mathbf{A})$. It is easy to deduce that the projection of any such n onto $N_\sigma(\mathbf{A})\backslash N(\mathbf{A})$ lies in a fixed compact set. (This follows, for example, from the proof of the integration formula

$$\int_{N_\sigma(\mathbf{A})\backslash N(\mathbf{A})} \int_{N_\sigma(\mathbf{A})} \phi(\sigma^{-1}n_2\sigma n_1 n_2^{-1})dn_1 dn_2$$
$$= \int_{N(\mathbf{A})} \phi(u)du, \quad \phi \in C_c(N(\mathbf{A})).$$

See Lemma 2.2 of [**1**] and Lemma 3.1.1 of [**6**]. Equivalently, one can argue as in the proof of Lemma 19 of [**8**].) We can therefore choose n to lie in a fixed compact subset of $N(\mathbf{A})$. Thus, our original element y is congruent modulo $G_\sigma(\mathbf{A})$ to a point in a compact subset of $G^0(\mathbf{A})$ which depends only on Δ. Lemma 6.1 follows.

Proof of Lemma 9.1. The set $G(F)'$ is a union of classes $\mathfrak{o} \in \mathcal{O}$, each one consisting of a single semisimple $G^0(F)$-orbit. We shall prove Lemma 9.1 first for these classes. Suppose for a given $\mathfrak{o}$ in $G(F)'$ that $\text{ad}(G^0(\mathbf{A}))\mathfrak{o}$ meets Δ. Then by Corollary A.2, there are elements $\gamma \in \mathfrak{o}$ and $x \in G_\Delta$ such that $x^{-1}\gamma x$ belongs to Δ. Consequently γ belongs to

$$\{d^{-1}gd:d \in G_\Delta, g \in \Delta\}.$$

This is a compact subset of $G(\mathbf{A})^1$ and contains only finitely many elements in $G(F)'$. It follows that only finitely many classes $\mathfrak{o}$ in $G(F)'$ have the property that $\text{ad}(G^0(\mathbf{A}))\mathfrak{o}$ meets Δ.

Now suppose that M is a Levi subset in $\mathcal{L}$ and that $P \in \mathcal{P}(M)$. The intersection of any class $\mathfrak{o}$ with $M(F)'$ is a (finite) union of $M^0(F)$-orbits. Applying what we have just proved to M, and using the fact that

$$G^0(\mathbf{A}) = P^0(\mathbf{A})K,$$

we see that there are only finitely many classes $\mathfrak{o} \in \mathcal{O}$ such that the set

$$\{x^{-1}\mu\nu x:\mu \in \mathfrak{o} \cap M(F)', \nu \in N_P(F), x \in G^0(\mathbf{A})\} \tag{A.4}$$

meets Δ. However,

$$(\mathfrak{o} \cap M(F)')N_P(F) = \mathfrak{o} \cap (M(F)'N_P(F)).$$

(See the remark following Lemma 2.1 of [**1**], and also Lemma 3.1.1 of [**7**].) Suppose that $\mathfrak{o} \cap M(F)'$ is not empty. Then every $G^0(F)$-orbit in $\mathfrak{o}$ intersects $M(F)'N_P(F)$. It follows that the original set $\text{ad}(G^0(\mathbf{A}))\mathfrak{o}$ is just equal to (A.4). It meets Δ for only finitely many such $\mathfrak{o}$. Since any class $\mathfrak{o} \in \mathcal{O}$ intersects $M(F)'$ for some $M \in \mathcal{L}$, Lemma 9.1 follows.

References

1. J. Arthur, *A trace formula for reductive groups I: terms associated to classes in* $G(\mathbf{Q})$, Duke Math. J. *45* (1978), 911-952.

2. ——— *The trace formula in invariant form*, Ann. of Math. *114* (1981), 1-74.

3. ——— *On a family of distributions obtained from Eisenstein series II: Explicit formulas*, Amer. J. Math. *104* (1982), 1289-1336.

4. ——— *The local behaviour of weighted orbital integrals*, in preparation.

5. ——— *A measure on the unipotent variety*, Can. J. Math. *37* (1985).

6. L. Clozel, J. P. Labesse and R. P. Langlands, *Morning seminar on the trace formula*, Lecture Notes, Institute for Advanced Study.

7. Y. Flicker, *The trace formula and base change for GL*(3), Springer Lecture Notes *927* (1982).

8. Harish-Chandra, *Harmonic analysis on reductive p-adic groups*, Lecture Notes in Math. *162* (Springer-Verlag, Berlin, 1970).

9. H. Jacquet and R. P. Langlands, *Automorphic forms on GL*(2), Springer Lecture Notes *114* (1970).

University of Toronto,
Toronto, Ontario