RECIPROCAL CONVERGENCE CLASSES FOR FOURIER SERIES AND INTEGRALS

A. P. GUINAND

Introduction. The classical result of Plancherel for Fourier cosine transforms of functions $f(x)$ of the class $L^2(0, \infty)$ states that (see **(7)** for references)

$$g(x) = \underset{T\to\infty}{\text{l.i.m.}} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^T f(t) \cos xt \, dt$$

converges in mean square to a function $g(x)$ which also belongs to $L^2(0, \infty)$, and furthermore

$$f(x) = \underset{T\to\infty}{\text{l.i.m.}} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^T g(t) \cos xt \, dt.$$

Some years ago in a series of papers **(1; 2; 3)** on summation formulae I showed that a similar symmetrical theory for narrower classes of functions and ordinary convergence of the integrals can also be developed. The relevant results can be expressed as follows:

THEOREM 1. *If $f(x)$ is the integral of its derivative and $xf'(x)$ belongs to $L^2(0, \infty)$, then*

$$\lim_{x\to\infty} f(x) = l$$

exists, $f(x) - l$ belongs to $L^2(0, \infty)$, and

$$f(x) - l = o(x^{-\frac{1}{2}})$$

as x tends to $+0$ or to $+\infty$.

Definition 1. If $f(x)$ is the integral of its derivative and $x f'(x)$ belongs to $L^2(0, \infty)$, and if the limit to which $f(x)$ tends as x tends to infinity is zero, then we say that $f(x)$ belongs to the class $S_1{}^2(0, \infty)$.

THEOREM 2. *If $f(x)$ belongs to $S_1{}^2(0, \infty)$ then for $x > 0$*

$$g(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\to\infty} f(t) \cos xt \, dt \tag{1}$$

converges, $g(x)$ also belongs to $S_1{}^2(0, \infty)$, and

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\to\infty} g(t) \cos xt \, dt. \tag{2}$$

Here we use the notation

$$\int_a^{\to\infty} = \lim_{T\to\infty} \int_a^T .$$

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That is to say that the class $S_1^2(0, \infty)$ is a subclass of $L^2(0, \infty)$, and that it can be described as a self-reciprocal convergence class for Fourier cosine transformations.

This theory has recently been extended by Miller **(5)** to cover wider subclasses of $L^2(0, \infty)$ and more general transformations. A disadvantage of Theorem 2 is that, although the result is simple and easily applied, the proof of Theorem 2 is indirect and it uses results from the Plancherel theory.

In the first part of the present paper I show how to find narrower self-reciprocal convergence classes for Fourier cosine transforms, and I give a direct proof of the Fourier inversion formula without using the Plancherel theory for one such self-reciprocal convergence class.

In the second part of the paper I prove analogues of Theorems 1 and 2 for Fourier series. I define a class $S_1^2(0, 2\pi)$ of functions $f(x)$ of period 2π and a class $\sum_1^2(-\infty, \infty)$ of sequences $\{c_n\}$ $(n = 0, \pm 1, \pm 2, \ldots,)$ which are reciprocal convergence classes for Fourier series in the sense that:

(i) if $f(x)$ belongs to $S_1^2(0, 2\pi)$ then it has a Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{3}$$

which converges for $x \not\equiv 0 \pmod{2\pi}$, and $\{c_n\}$ belongs to $\sum_1^2(-\infty, \infty)$;

(ii) if $\{c_n\}$ belongs to $\sum_1^2(-\infty, \infty)$ then $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges for all $x \not\equiv 0 \pmod{2\pi}$ and defines a function $f(x)$ belonging to $S_1^2(0, 2\pi)$.

Part I: Fourier Integrals

1. Reciprocal classes and Mellin transforms. If $f(x)$ and $g(x)$ are Fourier cosine transforms connected by the equations (1) and (2), and $\mathfrak{F}(s)$ and $\mathfrak{G}(s)$ are their Mellin transforms then, formally **(7**, p. 213**)**,

$$\mathfrak{G}(s) = \mathfrak{K}(s)\,\mathfrak{F}(1 - s) \tag{4}$$

and

$$\mathfrak{F}(s) = \mathfrak{K}(s)\,\mathfrak{G}(1 - s),$$

where

$$\mathfrak{K}(s) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \Gamma(s) \cos \tfrac{1}{2}s\pi,$$

and consequently

$$|\mathfrak{K}(\tfrac{1}{2} + it)| = 1 \tag{5}$$

for all real t.

From the L^2 theory of Mellin transforms **(7**, p. 94**)** it follows that if $f(x)$ belongs to $L^2(0, \infty)$ then $\mathfrak{F}(s)$ belongs to $\mathfrak{L}^2(-\infty, \infty)$. Hence by (4) and (5) it follows that $\mathfrak{G}(s)$ also belongs to $\mathfrak{L}^2(-\infty, \infty)$ and consequently $g(x)$ belongs to $L^2(0, \infty)$, as required by the Plancherel theory.

A similar argument can be used to show that the class $S_1^2(0, \infty)$ is self-reciprocal for Fourier cosine transformations. If $f(x)$ belongs to $S_1^2(0, \infty)$ then the Mellin transform of $x f'(x)$ exists and is

$$\text{(6)} \qquad \underset{X\to\infty}{\text{l.i.m.}} \int_{1/X}^{X} xf'(x)x^{s-1}\,dx$$

$$= \underset{X\to\infty}{\text{l.i.m.}} \left\{ [x^s f(x)]_{1/X}^{X} - s\int_{1/X}^{X} f(x)x^{s-1}\,dx \right\}$$

$$= -s\,\mathfrak{F}(s),$$

since the integrated terms vanish for $R(s) = \frac{1}{2}$ by Theorem 1. Hence $s\,\mathfrak{F}(s)$ belongs to $\mathfrak{L}^2(-\infty, \infty)$ and it follows from (4) and (5) that $s\,\mathfrak{G}(s)$ also belongs to $\mathfrak{L}^2(-\infty, \infty)$. Then, reversing the above argument, it follows that $g(x)$ belongs to $S_1^2(0, \infty)$, as required by Theorem 2.

Now the same procedure can be used when $\mathfrak{F}(s)$, instead of being multiplied by $-s$ as in (6), is multiplied by some other suitable function of s. For example, put

$$\Phi(1 - s) = \mathfrak{F}(s)/\Gamma(s)$$

and assume that $\Phi(s)$ belongs to $\mathfrak{L}^2(-\infty, \infty)$. Then $\Phi(s)$ is the Mellin transform of a function $\phi(x)$ belonging to $L^2(0, \infty)$. Further $\Gamma(s)$ is the Mellin transform of e^{-x} and consequently the relationship

$$\text{(7)} \qquad \mathfrak{F}(s) = \Gamma(s)\Phi(1 - s)$$

corresponds to (**7**, p. 213)

$$\text{(8)} \qquad f(x) = \int_0^\infty e^{-xt}\phi(t)\,dt.$$

From (4) and (7) we have

$$\mathfrak{G}(s) = \mathfrak{K}(s)\Gamma(1 - s)\Phi(s)$$

$$= \left\{\mathfrak{K}(s)\,\frac{\Gamma(1 - s)}{\Gamma(s)}\right\}\{\Gamma(s)\Phi(s)\}.$$

Hence, by (5), on $R(s) = \frac{1}{2}$

$$\left|\frac{\mathfrak{G}(s)}{\Gamma(s)}\right| = \left|\mathfrak{K}(s)\,\frac{\Gamma(1 - s)}{\Gamma(s)}\right|\,|\Phi(s)| = |\Phi(s)|$$

and so $\mathfrak{G}(s)/\Gamma(s)$ belongs to $\mathfrak{L}^2(-\infty, \infty)$.

Reversing the argument from (7) to (8) it follows that there is a $\psi(x)$ belonging to $L^2(0, \infty)$ for which

$$g(x) = \int_0^\infty e^{-xt}\psi(t)\,dt,$$

and we have the following result.

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THEOREM 3. *If $f(x)$ is the Laplace transform of a function of $L^2(0, \infty)$, and $g(x)$ is its Fourier cosine transform, then $g(x)$ is also the Laplace transform of a function of $L^2(0, \infty)$.*

Let us now make the following definition.

Definition 2. The function $f(x)$ is said to belong to the class $\Lambda^2\{h(x)\}$ if there exists a function $\phi(x)$ belonging to $L^2(0, \infty)$ such that

$$f(x) = \int_0^\infty h(xt)\,\phi(t)\,dt$$

for all $x > 0$.

Then Theorem 3 states that the class $\Lambda^2(e^{-x})$ is self-reciprocal for Fourier cosine transformations.

The same type of argument can be used to prove the following more general result.

THEOREM 4. *If $h(x)$ belongs to $L^2(0, \infty)$ and has a Mellin transform $\mathfrak{H}(s)$ satisfying*

$$\left|\frac{\mathfrak{H}(\frac{1}{2}+it)}{\mathfrak{H}(\frac{1}{2}-it)}\right| = 1$$

for all real t then $\Lambda^2\{h(x)\}$ is a self-reciprocal class of functions with respect to any general transformation of the Fourier type.

For general transformations see (**7**, ch. VIII).

2. Symmetrical convergence theorems by direct methods. The arguments of § 1 do not prove that the Fourier integrals (1) and (2) converge, and they use the L^2 theory of Mellin transforms. If we consider the class of functions

$$\Lambda^2(e^{-\frac{1}{2}x^2})$$

we can derive a symmetrical convergence theorem for the Fourier cosine transformation by a direct method. The result is:

THEOREM 5. *If $f(x)$ belongs to the class*

$$\Lambda^2(e^{-\frac{1}{2}x^2})$$

then

$$g(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\to\infty} f(t)\cos xt\,dt \tag{9}$$

converges for $x > 0$, $g(x)$ also belongs to

$$\Lambda^2(e^{-\frac{1}{2}x^2}),$$

and

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(10) $$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\to\infty} g(t) \cos xt\, dt$$

for $x > 0$. *Further, if, for* $x > 0$,

(11) $$f(x) = \int_0^\infty e^{-\frac{1}{2}x^2 t^2} \phi(t)\, dt$$

and

$$g(x) = \int_0^\infty e^{-\frac{1}{2}x^2 t^2} \psi(t)\, dt$$

in accordance with Definition 2 *then*

(12) $$\psi(x) = \frac{1}{x} \phi\left(\frac{1}{x}\right)$$

almost everywhere.

From Definition 2 there exists a function $\phi(x)$ of $L^2(0, \infty)$ satisfying (11). Hence

(13) $$\begin{aligned} g(x) &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(t) \cos xt\, dt \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos xt\, dt \int_0^\infty \phi(u) e^{-\frac{1}{2}u^2 t^2}\, du \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \phi(u)\, du \int_0^\infty e^{-\frac{1}{2}u^2 t^2} \cos xt\, dt, \end{aligned}$$

provided that this formal process can be justified. Now

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty e^{-\frac{1}{2}u^2 t^2} \cos xt\, dt = \frac{1}{u} e^{-\frac{1}{2}x^2/u^2},$$

so (13) becomes

(14) $$\begin{aligned} g(x) &= \int_0^\infty \phi(u) e^{-\frac{1}{2}x^2/u^2} \frac{du}{u} \\ &= \int_0^\infty \frac{1}{t} \phi\left(\frac{1}{t}\right) e^{-\frac{1}{2}x^2 t^2}\, dt \\ &= \int_0^\infty \psi(t) e^{-\frac{1}{2}x^2 t^2}\, dt \end{aligned}$$

by (12).

We can justify this process by the following three lemmas.

LEMMA 1. *If* $V_1 \geqslant V > 0$, $y > 0$, *then*

$$\left| \int_V^{V_1} e^{-\frac{1}{2}v^2} \cos yv\, dv \right| \leqslant \frac{2}{y} e^{-\frac{1}{2}V^2}.$$

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Proof.

$$\int_V^{V_1} e^{-\frac{1}{2}v^2} \cos yv\, dv = \left[\frac{\sin yv}{y} e^{-\frac{1}{2}v^2}\right]_V^{V_1} + \frac{1}{y}\int_V^{V_1} ve^{-\frac{1}{2}v^2} \sin yv\, dv.$$

Hence

$$\left|\int_V^{V_1} e^{-\frac{1}{2}v^2} \cos yv\, dv\right| \leqslant \frac{1}{y}(e^{-\frac{1}{2}V^2} + e^{-\frac{1}{2}V_1^2}) + \frac{1}{y}\int_V^{V_1} ve^{-\frac{1}{2}v^2}\, dv$$

$$= \frac{1}{y}(e^{-\frac{1}{2}V^2} + e^{-\frac{1}{2}V_1^2}) + \frac{1}{y}(e^{-\frac{1}{2}V^2} - e^{-\frac{1}{2}V_1^2})$$

$$= \frac{2}{y} e^{-\frac{1}{2}V^2}.$$

LEMMA 2. *If T, x, δ are positive real numbers and $\phi(x)$ belongs to $L^2(0, \infty)$ then*

$$\left|\int_T^{\infty} \cos xt\, dt \int_0^{\delta} e^{-\frac{1}{2}u^2t^2}\phi(u)du\right| \leqslant \frac{2^{\frac{1}{2}}\pi^{\frac{1}{4}}}{xT^{\frac{1}{2}}}\left\{\int_0^{\delta} |\phi(u)|^2 du\right\}^{\frac{1}{2}}. \tag{15}$$

Proof. Consider

$$\left|\int_T^{T_1} \cos xt\, dt \int_0^{\delta} e^{-\frac{1}{2}u^2t^2}\, \phi(u)\, du\right| \tag{16}$$

where $T_1 > T > 0$. Since $\phi(x)$ belongs to $L^2(0, \infty)$ it follows that $\phi(x)$ belongs to $L(0, \delta)$ for any finite δ, and that (16) converges absolutely. Hence (16) is equal to

$$\left|\int_0^{\delta} \phi(u)\, du \int_T^{T_1} e^{-\frac{1}{2}u^2t^2} \cos ut\, dt\right| \tag{17}$$

$$= \left|\int_0^{\delta} \phi(u)\frac{du}{u}\int_{Tu}^{T_1u} e^{-\frac{1}{2}v^2} \cos\frac{xv}{u}\, dv\right|$$

$$\leqslant \frac{2}{x}\int_0^{\delta} |\phi(u)| e^{-\frac{1}{2}T^2u^2}\, du$$

by Lemma 1. By Schwarz's inequality (17) is less than or equal to

$$\frac{2}{x}\left\{\int_0^{\delta} |\phi(u)|^2\, du\right\}^{\frac{1}{2}}\left\{\int_0^{\delta} e^{-T^2u^2}\, du\right\}^{\frac{1}{2}} \tag{18}$$

$$\leqslant \frac{2}{x}\left\{\int_0^{\delta} |\phi(u)|^2\, du\right\}^{\frac{1}{2}}\left\{\int_0^{\infty} e^{-T^2u^2}\, du\right\}^{\frac{1}{2}}$$

$$= \frac{2^{\frac{1}{2}}\pi^{\frac{1}{4}}}{xT^{\frac{1}{2}}}\left\{\int_0^{\delta} |\phi(u)|^2\, du\right\}^{\frac{1}{2}}.$$

Making T and T_1 tend to infinity it follows that the double integral in (15) converges, and (15) follows from (18) if we keep T fixed and make T_1 tend to infinity.

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LEMMA 3. *If x is real and positive, and $\phi(x)$ belongs to $L^2(0, \infty)$ then*

$$\int_0^{\to\infty} \cos xt\,dt \int_0^\infty \phi(u) e^{-\frac{1}{2}u^2 t^2} du \tag{19}$$

converges and is equal to

$$\int_0^\infty \phi(u)\,du \int_0^\infty e^{-\frac{1}{2}u^2 t^2} \cos xt\,dt. \tag{20}$$

Proof. The inversion of order of integration

$$\int_\delta^\infty \phi(u)\,du \int_0^\infty e^{-\frac{1}{2}u^2 t^2} \cos xt\,dt = \int_0^\infty \cos xt\,dt \int_\delta^\infty \phi(u)\, e^{-\frac{1}{2}u^2 t^2}\,du$$

is justified by absolute convergence since

$$\begin{aligned}\int_\delta^\infty |\phi(u)|du \int_0^\infty |e^{-\frac{1}{2}u^2 t^2} \cos xt| dt &\leqslant \int_\delta^\infty |\phi(u)|du \int_0^\infty e^{-\frac{1}{2}u^2 t^2}\,dt \\ &= (2\pi)^{\frac{1}{2}} \int_\delta^\infty |\phi(u)| \frac{du}{u} \\ &\leqslant (2\pi)^{\frac{1}{2}} \left\{\int_\delta^\infty |\phi(u)|^2\,du\right\}^{\frac{1}{2}} \left\{\int_\delta^\infty \frac{du}{u^2}\right\}^{\frac{1}{2}} \\ &= \left(\frac{2\pi}{\delta}\right)^{\frac{1}{2}} \left\{\int_\delta^\infty |\phi(u)|^2\,du\right\}^{\frac{1}{2}}.\end{aligned}$$

Also

$$\begin{aligned}\int_0^\delta |\phi(u)| e^{-\frac{1}{2}x^2/u^2} \frac{du}{u} &\leqslant \left\{\int_0^\delta |\phi(u)|^2\,du\right\}^{\frac{1}{2}} \left\{\int_0^\delta e^{-x^2/u^2} \frac{du}{u^2}\right\}^{\frac{1}{2}} \\ &= \left\{\int_0^\delta |\phi(u)|^2\,du\right\}^{\frac{1}{2}} \left\{\int_{1/\delta}^\infty e^{-x^2 v^2}\,dv\right\}^{\frac{1}{2}} \\ &\leqslant \left\{\int_0^\delta |\phi(u)|^2\,du\right\}^{\frac{1}{2}} \left\{\int_0^\infty e^{-x^2 v^2}\,dv\right\}^{\frac{1}{2}} \\ &= 2^{-\frac{1}{2}} \pi^{\frac{1}{4}} x^{-\frac{1}{2}} \left\{\int_0^\delta |\phi(u)|^2\,du\right\}^{\frac{1}{2}}.\end{aligned} \tag{21}$$

Hence

$$\begin{aligned}&\int_0^{\to\infty} \cos xt\,dt \int_0^\infty \phi(u)\, e^{-\frac{1}{2}u^2 t^2}\,du - \int_0^\infty \phi(u)\,du \int_0^\infty e^{-\frac{1}{2}u^2 t^2} \cos xt\,dt \\ &= \int_0^{\to\infty} \cos xt\,dt \int_0^\infty \phi(u)\, e^{-\frac{1}{2}u^2 t^2}\,du - \int_0^\delta \phi(u)\,du \int_0^\infty e^{-\frac{1}{2}u^2 t^2} \cos xt\,dt \\ &\qquad - \int_\delta^\infty \phi(u)\,du \int_0^\infty e^{-\frac{1}{2}u^2 t^2} \cos xt\,dt \\ &= \int_0^{\to\infty} \cos xt\,dt \int_0^\delta \phi(u)\, e^{-\frac{1}{2}u^2 t^2}\,du - \int_0^\delta \phi(u)\,du \int_0^\infty e^{-\frac{1}{2}u^2 t^2} \cos xt\,dt \\ &= \int_0^T \cos xt\,dt \int_0^\delta \phi(u)\, e^{-\frac{1}{2}u^2 t^2}\,du + \int_T^{\to\infty} \cos xt\,dt \int_0^\delta \phi(u)\, e^{-\frac{1}{2}u^2 t^2}\,du \\ &\qquad - \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \int_0^\delta \phi(u)\, e^{-\frac{1}{2}x^2/u^2} \frac{du}{u}.\end{aligned} \tag{22}$$

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Now

(23)
$$\left|\int_0^T \cos xt\,dt \int_0^\delta \phi(u)\,e^{-\frac{1}{2}u^2t^2}\,du\right|$$
$$\leqslant \int_0^T dt \int_0^\delta |\phi(u)|\,du$$
$$\leqslant T\left\{\int_0^\delta |\phi(u)|^2\,du\right\}^{\frac{1}{2}}\left\{\int_0^\delta du\right\}^{\frac{1}{2}}$$
$$= T\delta^{\frac{1}{2}}\left\{\int_0^\delta |\phi(u)|^2\,du\right\}^{\frac{1}{2}}.$$

Hence, by Lemma 2, (21), (22), and (23)
$$\left|\int_0^{\to\infty} \cos xt\,dt \int_0^\infty \phi(u)\,e^{-\frac{1}{2}u^2t^2}\,du - \int_0^\infty \phi(u)\,du \int_0^\infty e^{-\frac{1}{2}u^2t^2}\cos xt\,dt\right|$$
$$\leqslant \left\{T\delta^{\frac{1}{2}} + \frac{2^{\frac{1}{2}}\pi^{\frac{1}{4}}}{xT^{\frac{1}{2}}} + 2^{-\frac{1}{2}}\pi^{\frac{1}{4}}x^{-\frac{1}{2}}\right\}\left\{\int_0^\delta |\phi(u)|^2\,du\right\}^{\frac{1}{2}}.$$

This can be made arbitrarily small by choosing T first and then making δ sufficiently small, and Lemma 3 follows.

Proof of Theorem 5. Lemma 3 justifies the result (14). Further $\psi(x)$ belongs to $L^2(0, \infty)$ since
$$\int_0^\infty |\psi(x)|^2\,dx = \int_0^\infty \frac{1}{x^2}\left|\phi\left(\frac{1}{x}\right)\right|^2 dx$$
$$= \int_0^\infty |\phi(u)|^2\,du.$$

Hence $g(x)$, defined by (9), belongs to $\Lambda^2(e^{-\frac{1}{2}x^2})$. Then repeating the preceding argument
$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\to\infty} g(t)\cos xt\,dt = \int_0^\infty \frac{1}{t}\,\psi\left(\frac{1}{t}\right)e^{-\frac{1}{2}x^2t^2}\,dt$$
$$= \int_0^\infty \phi(t)\,e^{-\frac{1}{2}x^2t^2}\,dt$$
$$= f(x)$$

since (12) implies that for almost all x
$$\phi(x) = \frac{1}{x}\,\psi\left(\frac{1}{x}\right).$$

This completes the proof of Theorem 5.

Part II: Fourier Series

3. The class of functions $S_1^2(0, 2\pi)$.

Definition 3. If $f(x)$ is a periodic function of period 2π, is the integral of its

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derivative, and $(\sin \frac{1}{2} x) f'(x)$ belongs to $L^2(0, 2\pi)$, then we say that $f(x)$ belongs to the class $S_1{}^2(0, 2\pi)$.

Definition 4. If $f(x)$ is a periodic function of period 2π, and is such that there exists a function $\phi(x)$ of $L^2(0, 2\pi)$ for which

$$f(x) = \operatorname{cosec} \tfrac{1}{2} x \int_0^x \phi(t)\, dt \tag{24}$$

and

$$\int_0^{2\pi} \phi(t)\, dt = 0, \tag{25}$$

then we say that $f(x)$ belongs to the class $S_1{}^2[0, 2\pi]$.

These definitions give two ways of characterizing the same class of functions. Properties of this class of functions are given by the following theorem.

THEOREM 6. *The classes $S_1{}^2(0, 2\pi)$ and $S_1{}^2[0, 2\pi]$ are identical, and all functions $f(x)$ of either class belong to $L^2(0, 2\pi)$. Also $x^{\frac{1}{2}} f(x)$ and $x^{\frac{1}{2}} f(2\pi - x)$ both tend to zero as $x \to +0$.*

This result is analogous to results given by **(4)**.

To prove the result we use Lemmas 4, 5, and 6.

LEMMA 4. *If $f(x)$ belongs to $S_1{}^2[0, 2\pi]$ then $x^{\frac{1}{2}} f(x)$ and $x^{\frac{1}{2}} f(2\pi - x)$ both tend to zero as $x \to +0$, and $f(x)$ belongs to $L^2(0, 2\pi)$.*

Proof. As $x \to +0$, by (24) and Schwarz's inequality

$$\begin{aligned}|f(x)|^2 &\leqslant \operatorname{cosec}^2 \tfrac{1}{2} x \left\{\int_0^x |\phi(t)|^2\, dt\right\}\left\{\int_0^x dt\right\}\\ &= o(x^{-1}).\end{aligned}$$

Hence $x^{\frac{1}{2}} f(x) \to 0$ as $x \to +0$. Further

$$\begin{aligned}f(2\pi - x) &= \operatorname{cosec} \tfrac{1}{2} x \int_0^{2\pi - x} \phi(t)\, dt\\ &= -\operatorname{cosec} \tfrac{1}{2} x \int_{2\pi - x}^{2\pi} \phi(t)\, dt\end{aligned}$$

by (25), and a similar argument shows that $x^{\frac{1}{2}} f(2\pi - x) \to 0$ as $x \to +0$. Now let $0 < a < b < 2\pi$, put

$$\phi_1(x) = \int_0^x \phi(t)\, dt,$$

and suppose that $f(x)$ is real. Then

$$\int_a^b \{f(x)\}^2\, dx = \int_a^b \operatorname{cosec}^2 \tfrac{1}{2} x\, \{\phi_1(x)\}^2\, dx. \tag{26}$$

Integrating by parts (26) becomes

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$$[-2 \cot \tfrac{1}{2}x\{\phi_1(x)\}^2]_a^b + 4 \int_a^b \cot \tfrac{1}{2} x \, \phi(x)\phi_1(x) \, dx.$$

As $a \to +0$ and $b \to 2\pi - 0$, this is

$$o(1) + 4 \int_a^b \cos \tfrac{1}{2} x \, \phi(x) f(x) \, dx$$

$$\leqslant o(1) + 4\left\{\int_a^b \cos^2 \tfrac{1}{2} x \, |\phi(x)|^2 \, dx\right\}^{\frac{1}{2}} \left\{\int_a^b |f(x)|^2 \, dx\right\}^{\frac{1}{2}}$$

$$\leqslant o(1) + 4\left\{\int_0^{2\pi} |\phi(x)|^2 \, dx\right\}^{\frac{1}{2}} \left\{\int_a^b |f(x)|^2 \, dx\right\}^{\frac{1}{2}}.$$

Dividing by

$$\left\{\int_a^b |f(x)|^2 \, dx\right\}^{\frac{1}{2}}$$

and taking the limit as $a \to +0$, $b \to 2\pi - 0$, we have

$$\left\{\int_0^{2\pi} |f(x)|^2 \, dx\right\}^{\frac{1}{2}} \leqslant 4\left\{\int_0^{2\pi} |\phi(x)|^2 \, dx\right\}^{\frac{1}{2}},$$

and hence $f(x)$ belongs to $L^2(0, 2\pi)$, as required. If $f(x)$ is complex the result follows by splitting into real and imaginary parts.

LEMMA 5. *If $f(x)$ belongs to $S_1{}^2(0, 2\pi)$ then $x^{\frac{1}{2}} f(x)$ and $x^{\frac{1}{2}} f(2\pi - x)$ both tend to zero as $x \to +0$, and $f(x)$ belongs to $L^2(0, 2\pi)$.*

Proof. By Definition 3 there exists a function $\psi(x) = \sin \tfrac{1}{2}x f'(x)$, belonging to $L^2(0, 2\pi)$, such that

$$f(\pi) - f(x) = \int_x^\pi \operatorname{cosec} \tfrac{1}{2} t \, \psi(t) \, dt. \tag{27}$$

Suppose that $f(\pi) = 0$ and consider the behaviour of $f(x)$ as $x \to +0$. Choose $0 < \delta < \pi$ so that

$$\int_0^\delta |\psi(t)|^2 \, dt \leqslant \epsilon.$$

Then for $0 < x \leqslant \delta$

$$|f(x)| \leqslant \int_\delta^\pi \operatorname{cosec} \tfrac{1}{2} t |\psi(t)| dt + \int_x^\delta \operatorname{cosec} \tfrac{1}{2} t |\psi(t)| dt$$

$$\leqslant \int_\delta^\pi \operatorname{cosec} \tfrac{1}{2} t |\psi(t)| dt + \left\{\int_x^\delta |\psi(t)|^2 \, dt\right\}^{\frac{1}{2}} \left\{\int_\delta^x \operatorname{cosec}^2 \tfrac{1}{2} t \, dt\right\}^{\frac{1}{2}}$$

$$\leqslant \operatorname{cosec} \tfrac{1}{2} \delta \int_0^\pi |\psi(t)| dt + \epsilon^{\frac{1}{2}}(2 \cot \tfrac{1}{2}x - 2 \cot \tfrac{1}{2}\delta)^{\frac{1}{2}}.$$

Hence as $x \to +0$

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$$\begin{aligned} x^{\frac{1}{2}}f(x) &= O(x^{\frac{1}{2}}) + O(\epsilon^{\frac{1}{2}})(x \cot \tfrac{1}{2}x - x \cot \tfrac{1}{2}\delta)^{\frac{1}{2}} \\ &= O(x^{\frac{1}{2}}) + O(\epsilon^{\frac{1}{2}}) \\ &= o(1) \end{aligned}$$

since $x \cot \frac{1}{2} x \to 2$ as $x \to 0$.

A similar argument also shows that $x^{\frac{1}{2}} f(2\pi - x) \to 0$ as $x \to +0$.

Now suppose that $0 < a < \pi$, and that $f(x)$ is real, and put

$$f_1(x) = \int_0^x f(t)\, dt.$$

Hence $f_1(x) = o(x^{\frac{1}{2}})$ as $x \to +0$. Also

$$\begin{aligned} |f(x)| &= \left| \int_x^\pi \operatorname{cosec} \tfrac{1}{2} t\, \psi(t)\, dt \right| \\ &\leqslant \left\{ \int_x^\pi |\psi(t)|^2\, dt \right\}^{\frac{1}{2}} \left\{ \int_x^\pi \operatorname{cosec}^2 \tfrac{1}{2} t\, dt \right\}^{\frac{1}{2}} \\ &\leqslant \left\{ \int_0^\pi |\psi(t)|^2\, dt \right\}^{\frac{1}{2}} \left\{ 2 \cot \tfrac{1}{2} x \right\}^{\frac{1}{2}}, \end{aligned}$$

whence $f_1(x)$ is bounded for the whole interval $(0, \pi)$.

Now let $0 < a < \pi$ and consider

$$\begin{aligned} \int_a^\pi |f(x)|^2\, dx &= \int_a^\pi f(x)\, dx \int_x^\pi \operatorname{cosec} \tfrac{1}{2} t\, \psi(t)\, dt \\ &= \Big[f_1(x)\, f(x) \Big]_a^\pi + \int_a^\pi f_1(x) \operatorname{cosec} \tfrac{1}{2} x\, \psi(x)\, dx. \end{aligned}$$

As $x \to \pi - 0$, $f_1(x)$ is bounded and $f(x) \to f(\pi) = 0$, and as $x \to +0$

$$f_1(x) f(x) = o(x^{\frac{1}{2}}) o(x^{-\frac{1}{2}}) = o(1).$$

Hence

$$\begin{aligned} (28) \quad \int_a^\pi \{f(x)\}^2 dx &= o(1) + \int_a^\pi f_1(x) \operatorname{cosec} \tfrac{1}{2} x\, \psi(x)\, dx \\ &\leqslant o(1) + \left\{ \int_a^\pi \left| f_1(x) \operatorname{cosec} \tfrac{1}{2} x \right|^2 dx \right\}^{\frac{1}{2}} \left\{ \int_a^\pi |\psi(x)|^2\, dx \right\}^{\frac{1}{2}}. \end{aligned}$$

Now

$$\begin{aligned} f_1(x) &= \int_0^x f(t)\, dt \\ &= \int_0^x dt \int_t^\pi \operatorname{cosec} \tfrac{1}{2} u\, \psi(u)\, du \\ &= \int_0^x \operatorname{cosec} \tfrac{1}{2} u\, \psi(u)\, du \int_0^u dt + \int_x^\pi \operatorname{cosec} \tfrac{1}{2} u\, \psi(u)\, du \int_0^x dt \\ &= \int_0^x u \operatorname{cosec} \tfrac{1}{2} u\, \psi(u)\, du - x f(x) \end{aligned}$$

by (27). Hence

$$(29)\quad f_1(x)\operatorname{cosec}\tfrac{1}{2}x = \operatorname{cosec}\tfrac{1}{2}x\int_0^x u\operatorname{cosec}\tfrac{1}{2}u\,\psi(u)\,du - x\operatorname{cosec}\tfrac{1}{2}x\,f(x).$$

Now

$$2 \leqslant x\operatorname{cosec}\tfrac{1}{2}x \leqslant \pi$$

for $0 \leqslant x \leqslant \pi$. Hence $x\operatorname{cosec}\frac{1}{2}x\,\psi(x)$ belongs to $L^2(0, \pi)$, and by Lemma 4

$$(30)\qquad \operatorname{cosec}\tfrac{1}{2}x\int_0^x u\operatorname{cosec}\tfrac{1}{2}u\,\psi(u)\,du$$

belongs to $L^2(0, \pi)$.

Substituting (29) in (28) and using Minkowski's inequality we have

$$\int_a^\pi \{f(x)\}^2\,dx \leqslant o(1) + \left[\left\{\int_a^\pi \left|\operatorname{cosec}\tfrac{1}{2}x\int_0^x u\operatorname{cosec}\tfrac{1}{2}u\psi(u)du\right|^2 dx\right\}^{\frac{1}{2}} + \left\{\int_a^\pi \pi^2|f(x)|^2\,dx\right\}^{\frac{1}{2}}\right] \times \left\{\int_a^\pi |\psi(x)|^2\,dx\right\}^{\frac{1}{2}}$$

$$\leqslant o(1) + \left[\left\{\int_0^\pi \left|\operatorname{cosec}\tfrac{1}{2}x\int_0^x u\operatorname{cosec}\tfrac{1}{2}u\,\psi(u)\,du\right|^2 dx\right\}^{\frac{1}{2}} + \pi\left\{\int_a^\pi |f(x)|^2\,dx\right\}^{\frac{1}{2}}\right] \times \left\{\int_0^\pi |\psi(x)|^2\,dx\right\}^{\frac{1}{2}}$$

since we know that (30) and $\psi(x)$ both belong to $L^2(0, \pi)$. That is,

$$(31)\qquad \int_a^\pi |f(x)|^2\,dx \leqslant A + B\left\{\int_a^\pi |f(x)|^2\,dx\right\}^{\frac{1}{2}}$$

where A and B are constants independent of a. Now unless $f(x)$ vanishes almost everywhere in $(0, \pi)$ we can find an a_1 and a k such that $0 < a_1 < \pi$ and

$$\left\{\int_{a_1}^\pi |f(x)|^2\,dx\right\}^{\frac{1}{2}} > k > 0.$$

From (31)

$$\left\{\int_a^\pi |f(x)|^2\,dx\right\}^{\frac{1}{2}} \leqslant A\left\{\int_a^\pi |f(x)|^2\,dx\right\}^{-\frac{1}{2}} + B \leqslant \frac{A}{k} + B$$

for $a < a_1$. Hence

$$\left\{\int_0^\pi |f(x)|^2\,dx\right\}^{\frac{1}{2}} \leqslant \frac{A}{k} + B$$

and $f(x)$ belongs to $L^2(0, \pi)$. Combining the above with a similar argument for the interval $(\pi, 2\pi)$ we find that $f(x)$ belongs to $L^2(0, 2\pi)$.

If $f(\pi)$ is not zero the above argument shows that $f(x) - f(\pi)$ belongs to $L^2(0, 2\pi)$, so $f(x)$ belongs to $L^2(0, 2\pi)$.

Lastly, if $f(x)$ is complex the result of Lemma 5 follows by splitting into real and imaginary parts.

LEMMA 6. *The classes* $S_1^2(0, 2\pi)$ *and* $S_1^2[0, 2\pi]$ *are identical.*

Proof. With $\phi(x)$ and $\psi(x)$ as in Lemmas 4 and 5 we have

$$f(x) = \operatorname{cosec} \tfrac{1}{2} x \int_0^x \phi(t)\, dt = f(\pi) - \int_x^\pi \operatorname{cosec} \tfrac{1}{2} t\, \psi(t)\, dt. \tag{32}$$

By differentiation

$$\phi(x) = \sin \tfrac{1}{2}x\, f'(x) + \tfrac{1}{2} \cos \tfrac{1}{2}x\, f(x)$$

and

$$\psi(x) = \sin \tfrac{1}{2}x\, f'(x)$$

almost everywhere in $(0, 2\pi)$. Hence

$$\phi(x) = \psi(x) + \tfrac{1}{2} \cos \tfrac{1}{2}x\, f(x) \tag{33}$$

almost everywhere in $(0, 2\pi)$.

Now if $f(x)$ belongs to $S_1^2\ [0, 2\pi]$ this means that $\phi(x)$ belongs to $L^2(0, 2\pi)$, and, by Lemma 4, so does $f(x)$. Hence from (33) $\psi(x)$ also belongs to $L^2(0, 2\pi)$; that is, $f(x)$ belongs to $S_1^2(0, 2\pi)$.

Conversely if $f(x)$ belongs to $S_1^2(0, 2\pi)$ then by Lemma 5 it also belongs to $L^2(0, 2\pi)$ and $\psi(x)$ belongs to $L^2(0, 2\pi)$. Hence from (33) $\phi(x)$ also belongs to $L^2(0, 2\pi)$. Also by (32)

$$\int_0^x \phi(t)\, dt = \sin \tfrac{1}{2}\, x\, f(x).$$

By Lemma 5 $x^{\frac{1}{2}} f(2\pi - x) \to 0$ as $x \to +0$. Hence

$$\int_0^{2\pi} \phi(t)\, dt = \lim_{x \to +0} \{\sin \tfrac{1}{2}(2\pi - x)\, f(2\pi - x)\}$$
$$= 0.$$

That is, $f(x)$ belongs to $S_1^2[0, 2\pi]$, and this completes the proof of Lemma 6. Combining Lemmas 4, 5, and 6 we have Theorem 6.

We also require the following result to connect $S_1^2(0, 2\pi)$ with Fourier Series in § 5.

THEOREM 7. *The class* $S_1^2(0, 2\pi)$ *is identical with the class of functions* $f(x)$ *of period* 2π *which can be expressed in the form*

$$f(x) = \frac{1}{1 - e^{-ix}} \int_0^x \chi(t)\, dt, \tag{34}$$

where $\chi(x)$ *belongs to* $L^2(0, 2\pi)$ *and*

$$\int_0^{2\pi} \chi(t)\, dt = 0. \tag{35}$$

Proof. By (24), (25), (34), and (35) we require that

$$\frac{1}{1 - e^{-ix}} \int_0^x \chi(t)\, dt = \operatorname{cosec} \tfrac{1}{2} x \int_0^x \phi(t)\, dt \tag{36}$$

where

$$\int_0^{2\pi} \phi(t)\, dt = \int_0^{2\pi} \chi(t)\, dt = 0$$

and $\phi(x)$ and $\chi(x)$ belong to $L^2(0, 2\pi)$. Now (36) gives

$$\int_0^x \chi(t)\, dt = 2i\, e^{-\frac{1}{2}ix} \int_0^x \phi(t)\, dt \tag{37}$$

and hence

$$\chi(x) = 2i\, e^{-\frac{1}{2}ix}\, \phi(x) + e^{\frac{1}{2}ix} \int_0^x \phi(t)\, dt \tag{38}$$

almost everywhere in $(0, 2\pi)$. If $\phi(x)$ belongs to $L^2(0, 2\pi)$, so does

$$\int_0^x \phi(t)\, dt,$$

and hence from (38), $\chi(x)$ belongs to $L^2(0, 2\pi)$. A similar argument shows that $\phi(x)$ belongs to $L^2(0, 2\pi)$ if $\chi(x)$ does.

Finally if we put $x = 2\pi$ in (37) we have

$$\int_0^{2\pi} \chi(t)\, dt = -2i \int_0^{2\pi} \phi(t)\, dt.$$

Hence the vanishing of either of these integrals implies the vanishing of the other.

4. The class of sequences $\sum_1^2(-\infty, \infty)$

THEOREM 8. *If* $\{c_n\}$, $(n = 0, 1, 2, \ldots,)$ *is a sequence of complex numbers such that the series*

$$\sum_{n=1}^{\infty} n^2 |c_n - c_{n+1}|^2$$

is convergent, then

(i) c_n *tends to a finite limit* l *as* $n \to \infty$, *and*

$$c_n - l = o(n^{-\frac{1}{2}}), \tag{39}$$

(ii) *the series*

$$\sum_{n=0}^{\infty} |c_n|^2$$

converges.

Proof of (i). If $m > n > 1$ then

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(40)
$$|c_n - c_m| = \left| \sum_{r=n}^{m-1} (c_r - c_{r+1}) \right|$$
$$\leqslant \left\{ \sum_{r=n}^{m-1} r^2 |c_r - c_{r+1}|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{r=n}^{m-1} r^{-2} \right\}^{\frac{1}{2}}$$
$$= o(n^{-\frac{1}{2}}).$$

Hence, by the principle of convergence, c_n tends to a finite limit l as $n \to \infty$, and making $m \to \infty$ in (40) we have (39).

LEMMA 7. *If* $\{a_n\}$, $(n = 1, 2, 3, \ldots,)$ *is any sequence of complex numbers and* N *a positive integer then*

$$6 \sum_{n=1}^{N-1} n^2 |a_n - a_{n+1}|^2 + 2N|a_N|^2 \geqslant \sum_{n=1}^{N} |a_n|^2.$$

Proof of Lemma 7. We have

$$6n^2|a_n - a_{n+1}|^2 + (2n+1)|a_{n+1}|^2$$
$$= (2n^2 - 2n)|a_n - a_{n+1}|^2 + \{2n|a_n - a_{n+1}| - |a_{n+1}|\}^2 + 2n\{|a_n - a_{n+1}| + |a_{n+1}|\}^2$$
$$\geqslant 2n\{|a_n - a_{n+1}| + |a_{n+1}|\}^2$$
$$\geqslant 2n|a_n|^2$$

since $2n^2 - 2n \geqslant 0$ for all integers n and

$$|a_n - a_{n+1}| + |a_{n+1}| \geqslant |a_n|.$$

Hence

$$6n^2|a_n - a_{n+1}|^2 + 2(n+1)|a_{n+1}|^2 - 2n|a_n|^2 \geqslant |a_{n+1}|^2,$$

and the lemma follows on summing over $n = 0, 1, 2, \ldots, N-1$.

Proof of (ii). If we put $a_n = c_n - l$ then by (39) $N|a_N|^2$ tends to zero as $N \to \infty$. Hence

$$6 \sum_{n=1}^{\infty} n^2 |c_n - c_{n+1}|^2 \geqslant \sum_{n=1}^{\infty} |c_n - l|^2$$

and the latter series converges.

Definition 5. If $\{c_n\}$, $(n = 0, \pm 1, \pm 2, \ldots,)$ is a sequence of complex numbers such that

$$\sum_{n=-\infty}^{\infty} n^2 |c_n - c_{n+1}|^2$$

converges, and if the limits to which c_n tends as $n \to \pm\infty$ are both zero, then we say that the sequence $\{c_n\}$ belongs to the class $\sum_1{}^2(-\infty, \infty)$.

5. The convergence of Fourier series for $S_1^2(0, 2\pi)$.

THEOREM 9. *If* $\{c_n\}$ *belongs to the class* $\sum_1^2(-\infty, \infty)$ *then the series*

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{41}$$

converges for all x *not congruent to zero modulo* 2π, *and its sum defines a function* $f(x)$, *belonging to* $S_1^2(0, 2\pi)$, *of which* (41) *is the Fourier series.*

Proof. Consider the series

$$\sum_{n=1}^{\infty} c_n e^{inx}, \tag{42}$$

and put

$$n(c_n - c_{n+1}) = \chi_n.$$

Then

$$\begin{aligned} c_n &= (c_n - c_{n+1}) + (c_{n+1} - c_{n+2}) + \dots \\ &= \sum_{r=n}^{\infty} \frac{\chi_r}{r}. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{n=1}^{N} c_n e^{inx} &= \sum_{n=1}^{N} e^{inx} \sum_{r=n}^{\infty} \frac{\chi_r}{r} \\ &= \sum_{r=1}^{N} \frac{\chi_r}{r} \sum_{n=1}^{r} e^{inx} + \sum_{r=N+1}^{\infty} \frac{\chi_r}{r} \sum_{n=1}^{N} e^{inx} \\ &= \sum_{r=1}^{N} \frac{\chi_r}{r} \left\{ \frac{e^{i(r+1)x} - 1}{e^{ix} - 1} \right\} + \sum_{r=N+1}^{\infty} \frac{\chi_r}{r} \left\{ \frac{e^{i(N+1)x} - 1}{e^{ix} - 1} \right\}. \end{aligned} \tag{43}$$

Now

$$\begin{aligned} \sum_{r=N+1}^{\infty} \left| \frac{\chi_r}{r} \right| &\leqslant \left\{ \sum_{r=N+1}^{\infty} |\chi_r|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{r=N+1}^{\infty} r^{-2} \right\}^{\frac{1}{2}} \\ &= o(N^{-\frac{1}{2}}) \end{aligned}$$

since the series

$$\sum_{r=1}^{\infty} |\chi_r|^2 \tag{44}$$

converges by hypothesis. Hence by (43), if $x \not\equiv 0 \bmod 2\pi$

$$\sum_{n=1}^{N} c_n e^{inx} = \frac{1}{e^{ix} - 1} \sum_{r=1}^{N} \frac{\chi_r}{r} \{e^{i(r+1)x} - 1\} + o(N^{-\frac{1}{2}}),$$

and the series on the right is absolutely convergent. Hence the series (42) converges, and

$$\sum_{n=1}^{\infty} c_n e^{inx} = \frac{1}{e^{ix} - 1} \sum_{r=1}^{\infty} \frac{\chi_r}{r} \{e^{i(r+1)x} - 1\}. \tag{45}$$

Since the series (44) converges the series

$$\sum_{r=1}^{\infty} \chi_r e^{irx}$$

converges in mean square to a function $\chi(x)$ belonging to $L^2(0, 2\pi)$, and is the Fourier series of this function. Hence, by the Fourier series integration theorem (**6**, p. 419),

$$\int_0^x \chi(t)dt = -i \sum_{r=1}^{\infty} \frac{\chi_r}{r} (e^{irx} - 1),$$

and in particular

$$\int_0^{2\pi} \chi(t)dt = 0.$$

Hence by (45)

$$\sum_{n=1}^{\infty} c_n e^{inx} = \frac{i}{1 - e^{-ix}} \int_0^x \chi(t)\, dt + \sum_{r=1}^{\infty} \frac{\chi_r}{r}. \tag{46}$$

By Theorem 7 it follows that (46) is a function of the class $S_1^2(0, 2\pi)$. A similar argument for negative n shows that the whole series (41) converges for $x \not\equiv 0 \bmod 2\pi$, and that its sum is a function of the class $S_1^2(0, 2\pi)$.

Since $S_1^2(0, 2\pi)$ is a subclass of $L^2(0, 2\pi)$ the series (41) must be the Fourier series of its sum.

THEOREM 10. *If $f(x)$ belongs to the class $S_1^2(0, 2\pi)$ then it has a Fourier series*

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

which converges to $f(x)$ for all x not congruent to zero modulo 2π, and the sequence $\{c_n\}$ belongs to the class $\sum_1^2(-\infty, \infty)$.

Proof. By Theorem 6 the function $f(x)$ belongs to $L^2(0, 2\pi)$. Hence it has a Fourier series

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{47}$$

for which c_n tends to zero as $n \to \pm\infty$.

By Theorem 7 there exists a function $\chi(x)$ of $L^2(0, 2\pi)$ such that

$$f(x) = \frac{i}{1 - e^{-ix}} \int_0^x \chi(t)\, dt \tag{48}$$

and

$$\int_0^{2\pi} \chi(t)\, dt = 0.$$

Hence if

$$\chi(x) \sim \sum_{r=-\infty}^{\infty} \chi_r e^{irx}$$

then $\chi_0 = 0$ and

$$\sum_{r=-\infty}^{\infty} |\chi_r|^2$$

converges. By the Fourier series integration theorem and (48)

$$f(x) = \frac{1}{1 - e^{-ix}} \sum_{r=-\infty}^{\infty} \frac{\chi_r}{r} (e^{irx} - 1).$$

Hence

$$f(x)\,(1 - e^{-ix}) = \sum_{r=-\infty}^{\infty} \frac{\chi_r}{r} e^{irx} - \sum_{r=-\infty}^{\infty} \frac{\chi_r}{r} \tag{49}$$

since both of these series converge absolutely.

Now

$$f(x)\,(1 - e^{-ix}) \sim (1 - e^{-ix}) \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} (c_n - c_{n+1})\, e^{inx}.$$

By (49) and the uniqueness theorem for Fourier series of functions of $L^2(0, 2\pi)$ it follows that for $n \neq 0$

$$\frac{\chi_n}{n} = c_n - c_{n+1}.$$

Hence the series

$$\sum_{n=-\infty}^{\infty} n^2 |c_n - c_{n+1}|^2 = \sum_{n=-\infty}^{\infty} |\chi_n|^2$$

converges, and therefore the sequence $\{c_n\}$ belongs to the class $\sum_1^2(-\infty, \infty)$, as required.

By Theorem 9 the series (47) converges for $x \not\equiv 0 \bmod 2\pi$ to a function of $S_1^2(0, 2\pi)$ which must therefore be equal to $f(x)$ almost everywhere. From (34) functions of $S_1^2(0, 2\pi)$ are continuous for $x \not\equiv 0 \bmod 2\pi$, so the sum of the series (47) must be equal to $f(x)$ for all such x.

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University of Saskatchewan