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EXTENSIVE SUBCATEGORIES IN UNIVERSAL TOPOLOGICAL ALGEBRAS

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0. Introduction. Herrlich [7] has introduced the limit-operators to obtain every coreflective subcategory of the category Top of topological spaces and continuous maps. Using limit-operators, S. S. Hong [9] has constructed new reflective subcategories from a known extensive subcategory of a hereditary category of Hausdorff spaces and continuous maps.

In this paper, we introduce extensive operators in a category of universal topological algebras and continuous homomorphisms. For an extensive subcategory $\mathscr{B}$ of a hereditary category $\mathscr{A}$ of universal Hausdorff topological algebras and their continuous homomorphisms, let l be an extensive operator on $\mathscr{B}$ and let $\mathscr{B}_l$ be the subcategory of $\mathscr{A}$ determined by those objects of $\mathscr{A}$ which are l-closed in their $\mathscr{B}$-reflections. It is shown that $\mathscr{B}_l$ is also extensive in $\mathscr{A}$.

Observing that for any coreflective $\mathscr{C}$ of Top which is (finitely) productive, $\mathscr{C}$ induces an extensive operator on a category of (finitary resp.) universal topological algebras and continuous homomorphisms, one can associate a reflective subcategory of a certain category of universal Hausdorff topological algebras with such a coreflective subcategory of Top.

Furthermore, we observe that every bijection-coreflective subcategory of the category of universal topological algebras of a fixed type τ and continuous homomorphisms, induces an extensive operator on any category of universal topological algebras of type τ and continuous homomorphisms.

Using G_k-sets, we construct new extensive subcategories in various categories of topological groups.

For general categorical background and terminology, we refer to [**8**] and for universal algebras to [**2**] or [**5**].

In the following every subcategory of a category will be assumed to be full and isomorphism-closed.

1. Extensive subcategories.

1.1 *Definition.* A *universal topological algebra of type* $\tau = (\lambda_i)_{i \in I}$ is a triple $A = (X, (f_i)_{i \in I}, \mathscr{O})$ in which $(X, (f_i)_{i \in I})$ is a universal algebra of type τ and $\mathscr{O}$ is a topology on X such that $f_i : (X, \mathscr{O})^{\lambda_i} \to (X, \mathscr{O})$ is continuous for each $i \in I$.

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To simplify the language in this paper, universal (topological) algebras of type τ will be called (topological resp.) algebras of type τ or (topological) algebras when there is no confusion about the type.

1.2 *Definition.* A topology $\mathcal{O}$ on an algebra $(X, (f_i)_{i \in I})$ of type τ is called an *algebra topology* if $(X, (f_i)_{i \in I}, \mathcal{O})$ becomes a topological algebra.

1.3 *Definition.* Let $\mathcal{A}$ be a category of Hausdorff topological algebras of fixed type and continuous homomorphisms. A subcategory $\mathcal{B}$ of $\mathcal{A}$ is called an *extensive subcategory of* $\mathcal{A}$ if it is a reflective subcategory such that the $\mathcal{B}$-reflection $r_A : A \to \gamma A$ is a dense embedding for each $A \in \mathcal{A}$.

1.4 PROPOSITION. *Let $\mathcal{A}$ be a category of Hausdorff topological algebras of fixed type and continuous homomorphisms and $\mathcal{B}$ an extensive subcategory of $\mathcal{A}$. Then any reflective subcategory $\mathcal{C}$ of $\mathcal{A}$ containing $\mathcal{B}$ is also an extensive subcategory of $\mathcal{A}$.*

Proof. For any $\mathcal{A}$-object A, let $r_{\mathcal{B}} : A \to r_{\mathcal{B}}A$ and $r_{\mathcal{C}} : A \to r_{\mathcal{C}}A$ be reflections of A with respect to $\mathcal{B}$ and $\mathcal{C}$ respectively. Since $\mathcal{C}$ contains $\mathcal{B}$, there is a unique morphism $r : r_{\mathcal{C}}A \to r_{\mathcal{B}}A$ with $rr_{\mathcal{C}} = r_{\mathcal{B}}$. Using the fact that $r : r_{\mathcal{C}}A \to r_{\mathcal{B}}A$ is the $\mathcal{B}$-reflection of $r_{\mathcal{C}}A$, it is easy to show that $r_{\mathcal{C}}$ is again a dense-embedding. We omit the details of the proof.

1.5 *Definition.* Let $\mathcal{A}$ be a category of Hausdorff topological algebras of fixed type and continuous homomorphisms. An operator l which associates with every pair (A, S), where A is an $\mathcal{A}$-object and S is a subalgebra of A, a subalgebra $l_A S$ of A is said to be an *extensive operator on* $\mathcal{A}$, if l satisfies the following conditions:

(1) If S is a subalgebra of A, then $S \subseteq l_A S \subseteq \mathrm{cl}_A S$, where cl_A denotes the closure operator on the underlying space of A.

(2) If S and T are subalgebras of an $\mathcal{A}$-object A and $S \subseteq T$ then $l_A S \subseteq l_A T$.

(3) If $f : A \to B$ is an $\mathcal{A}$-morphism and S is a subalgebra of A, then $f(l_A S) \subseteq l_B f(S)$.

(4) If S is a subalgebra of an $\mathcal{A}$-object A, then $l_A(l_A S) = l_A S$.

A subalgebra S of A is said to be *l-closed* in A if $l_A S = S$.

Notation. The category of algebras of fixed type τ and their homomorphisms will be denoted by $\mathcal{A}(\tau)$. For a subcategory $\mathcal{T}$ of Top and a subcategory $\mathcal{A}$ of $\mathcal{A}(\tau)$, the category of topological algebras whose underlying algebras belong to $\mathcal{A}$ and whose underlying spaces belong to $\mathcal{T}$, and their continuous homomorphisms will be denoted by $\mathcal{T}\mathcal{A}$.

1.6 PROPOSITION. *Every bijection-coreflective subcategory $\mathcal{C}$ of* Top$\mathcal{A}(\tau)$ *induces an extensive operator on a subcategory of* Top$\mathcal{A}(\tau)$.

Proof. Let $\mathcal{B}$ be a subcategory of Top$\mathcal{A}(\tau)$. For any $A \in \mathcal{B}$, let $c_A : cA \to A$ be the $\mathcal{C}$-coreflection. Since we may assume that c_A is an identity

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map on A, for any subalgebra S of A, define $l_A S = \mathrm{cl}_{cA} S$, where cl_{cA} is the closure operator of the underlying space of cA. Obviously, $\mathrm{cl}_{cA} S$ is again a subalgebra of A and $(l_A)_{A \in \mathscr{B}}$ is an extensive operator on $\mathscr{B}$.

1.7 THEOREM. *Let $\mathscr{B}$ be an extensive subcategory of a hereditary category $\mathscr{A}$ of Hausdorff topological algebras of fixed type and continuous homomorphisms and let l be an extensive operator on $\mathscr{B}$. Then the subcategory $\mathscr{B}_l$ determined by those objects in $\mathscr{A}$ which are l-closed in their $\mathscr{B}$-reflections, is also an extensive subcategory of $\mathscr{A}$.*

Proof. For any $\mathscr{A}$-object A, let $r_A : A \to rA$ be the $\mathscr{B}$-reflection of A such that A is a dense subalgebra of rA and r_A is the natural embedding. Let $r^l A$ be the subalgebra of rA with $l_{rA} A$ as its underlying set. Since $\mathscr{A}$ is hereditary, $r^l A$ belongs to $\mathscr{A}$. It is easy to show that the natural embedding $j : r^l A \to rA$ is actually $\mathscr{B}$-reflection of $r^l A$. Since $r^l A$ is l-closed in its $\mathscr{B}$-reflection rA, $r^l A$ belongs to $\mathscr{B}_l$.

Now we can conclude that the natural embedding $r_A{}^l : A \to r^l A$ is the $\mathscr{B}_l$-reflection. For any $\mathscr{B}_l$-object B and any $\mathscr{A}$-morphism $f : A \to B$, there exists a unique $\mathscr{B}$-morphism $\bar{f} : rA \to rB$ with $\bar{f} r_A = r_B$. Since $\bar{f}(r^l A) = \bar{f}(l_{rA} A) \subseteq l_{rB} \bar{f}(A) \subseteq l_{rB} B = B$, the map $g : r^l A \to B$ which is a restriction and corestriction of $\bar{f}$ to $r^l A$ and B respectively, is a continuous homomorphism, and $g r_A{}^l = f$. Noting that $r_A{}^l$ is a dense embedding, g with $g r_A{}^l = f$ is unique. This completes the proof.

2. *k*-complete topological algebras. Herrlich has shown that every coreflective subcategory $\mathscr{C}$ of the category Top generates an idempotent limit operator $l(\mathscr{C})$ (see [**7**]) and for any $X \in$ Top, $l(\mathscr{C})_X$ is precisely the closure operator on the $\mathscr{C}$-coreflection of X. Furthermore, every idempotent limit-operator l generates a coreflective subcategory $\mathscr{C}(l)$ of Top and $\mathscr{C}(l)$ is determined topological spaces whose l-closed subsets are closed.

2.1 PROPOSITION. *Let l be an idempotent limit-operator. If the coreflective subcategory $\mathscr{C}(l)$ of* Top *generated by l is (finitely) productive, then l induces an extensive operator on any category of (finitary resp.) Hausdorff topological algebras and continuous homomorphisms.*

Proof. Let $\mathscr{A}$ be a category of (finitary resp.) Hausdorff typological algebras of type $\tau = (\lambda_i)_{i \in I}$. For an $\mathscr{A}$-object $A = (X, (f_i)_{i \in I}, \mathscr{O})$, let $1_X : (X, \mathscr{O}_l) \to (X, \mathscr{O})$ be the $\mathscr{C}(l)$-coreflection of $(X, \mathscr{O})$. For any $i \in I$, since $(X, \mathscr{O}_l)^{\lambda_i} \in \mathscr{C}(l)$, there is a unique continuous map $g_i : (X, \mathscr{O}_l)^{\lambda_i} \to (X, \mathscr{O}_l)$ such that the diagram

$$\begin{array}{ccc} (X, \mathscr{O}_l)^{\lambda_i} & \xrightarrow{\;1_x^{\lambda_i}\;} & (X, \mathscr{O})^{\lambda_i} \\ g_i \big\downarrow & & \big\downarrow f_i \\ (X, \mathscr{O}_l) & \xrightarrow[\;1_x\;]{} & (X, \mathscr{O}), \end{array}$$

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commutes. Obviously, g_i must be equal to f_i as set maps. Hence $\mathscr{O}_l$ is also an algebra topology of $(X, (f_i)_{i\in I})$. For any subalgebra S of A, let $\overline{l}_A S = (l(\mathscr{C}(l)))_A S$. Since the closure of a subalgebra of a topological algebra is again a subalgebra, $\overline{l}$ is an extensive operator on A.

2.2 *Definition.* Let k be an infinite cardinal, and let X be a topological space. A subset of X is called a *G_k-set* if it is an intersection of fewer than k open subsets of X.

It is clear that the G_k-sets of a topological space $(X, \mathscr{O})$ form a basis for a topology on X. We denote the new topology by $\mathscr{O}_k$. Since the inverse image of a G_k-set under a continuous map is again a G_k-set, the closure operator l_x^k on $(X, \mathscr{O}_k)$ gives rise to an idempotent limit operator $l^k = (l_x^k)_{x\in\mathrm{Top}}$.

A subset S of a topological space X will be called *k-closed* if and only if $l_x^k S = S$.

2.3 Proposition. *For any category $\mathscr{A}$ of finitary Hausdorff topological algebras and continuous homomorphisms, l^k induces an idempotent extensive operator on $\mathscr{A}$.*

Proof. It is enough to show that the coreflective subcategory $\mathscr{C}(l^k)$ of Top, which is generated by l^k, is finitely productive. However, using the fact that a topological space X belongs to $\mathscr{C}(l^k)$ if and only if every G_k-set of X is again open, the proof is straight forward.

It is well-known [**3**] that every toplogical group has two uniform structures which are compatible with its group topology; a topological group G with the right (left) uniform structure will be denoted by G^* ($^*G$, respectively).

In the following, every topological algebra will be assumed to be Hausdorff.

2.4 *Definition.* A topological group G is said to be *complete* if G^* (equivalently $^*G$) is complete, and a topological group G is called *completable* if every Cauchy filter in G^* is also a Cauchy filter in $^*G$.

Remark. It is known that a topological group is completable if and only if it is isomorphic with a subgroup of a complete topological group. Hence, the category RCGRP of completable topological groups and continuous homomorphisms is complete and hereditary.

Furthermore, the category CGRP of complete topological groups is extensive in RCGRP whose reflections are obviously the completion of G^* (see [**3**]).

The following definition is due to Husek [**11**].

2.5 *Definition.* Let k be an infinite cardinal. A Hausdorff uniform space X is called *k-complete* if any Cauchy filter with the k-intersection property on X is convergent.

2.6 *Definition.* Let k be an infinite cardinal. A completable topological group G is said to be *k-complete* if G^* (equivalently $^*G$) is k-complete.

The following lemma is due to S. S. Hong [**10**].

2.7 LEMMA. *A Hausdorff uniform space X is k-complete if and only if it is k-closed in its completion cX.*

2.8 COROLLARY. *A completable topological group is k-complete if and only if it is k-closed in its completion.*

The following is immediate from Proposition 2.3, Corollary 2.8 and Theorem 1.7.

2.9 THEOREM. *Let k be an infinite cardinal. The subcategory* kCGRP *of* RCGRP *determined by all k-complete topological groups is extensive in* RCGRP.

Remark. CGRP = $\aleph_0$CGRP and kCGRP $\supset$ tCGRP for infinite cardinals k and t with $k \geqq t$.

2.10 COROLLARY. *The category* kCGRP *is complete.*

Since every abelian toplogical group G is completable, one has:

2.11 THEOREM. *Let k be an infinite cardinal. The subcategory determined by k-complete abelian topological groups is extensive in the category of abelian topological groups and homomorphisms.*

Remark. (1) For any object G of RCGRP, the kCGRP-reflection of G is given by the k-closure of G in its completion cG^*.

(2) Since for any Hausdorff space $(X, \mathcal{O})$, $\mathcal{O}_k$ is discrete for some infinite cardinal k, every completable topological group is k-complete for some infinite cardinal k.

(3) It is a well-known convention that whenever we speak of the uniform structure of a topological ring, it is the uniform structure of its additive group. In particular, a topological ring A is called *complete* if the additive group of A is complete. Moreover, it is known that the subcategory determined by complete topological rings is extensive in the category HR of topological rings and continuous homomorphisms (see [**3**]). Applying our results to HR, we can conclude that the subcategory determined by k-complete topological rings is extensive in HR for each finite cardinal k. Also, one can conclude the same results for the category of topological A-modules for a topological ring A and continuous homomorphisms.

(4) Let Q be the topological additive group (ring) of rational numbers with the usual topology. Since the real line $(R, \mathcal{O})$ with the usual topology $\mathcal{O}$ is the completion of Q and $(R, \mathcal{O}_{\aleph_1})$ is discrete, Q is $\aleph_1$-complete but not complete.

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