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WEAKLY CONTINUOUS ACCRETIVE OPERATORS IN GENERAL BANACH SPACES

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Global wellposedness theorems are established for a class of abstract Cauchy initial value problems and a class of abstract Volterra equations which have a linear semigroup as a convolution kernel. These existence theorems are used to show that a class of nonlinear operators and a class of perturbed linear operators are m-accretive. The m-accretiveness results are used in turn to represent solutions to the differential and integral equations.

1. INTRODUCTION

We shall be concerned with conditions sufficient to guarantee that a weakly continuous accretive operator on a general Banach space X be m-accretive and conditions which guarantee that an additive perturbation of a linear m-accretive operator by a weakly continuous accretive operator be m-accretive. The author has settled these questions for reflexive Banach spaces and the work herein may be considered as a continuation of work in [**8, 9**].

The question of m-accretiveness of a nonlinear accretive operator A is intimately connected with the question of global existence of solutions to the nonlinear Cauchy initial value problem $\dot{u}(t) + Au(t) = 0$, $u(0) = x$. Roughly speaking one establishes a local existence result, uses the accretiveness of A to extend the local theory and then employs the global theory to prove m-accretiveness. In this sense it is fair to say that the question of the m-accretiveness for an operator is predicated upon local existence theory. Local theory for abstract ordinary differential equation with weakly continuous vector fields has been considered in a variety of publications, see, [**3, 4, 21, 22**]. The local result we shall use appears in [**4**]. The most general and recent investigations of the local theory appear in [**18, 19**] and the interested reader is referred to these papers for a historical discussion of the question.

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2. Definitions and Preliminaries

In what follows X will denote a Banach space having norm, $\|\cdot\|$. The dual space of X will be denoted by X^* and the pairing between X and X^* is designated by $\langle , \rangle$. The symbols $\to$ and $\rightharpoonup$ mean strong and weak convergence respectively with lim and w-lim distinguishing the strong and weak limits. Unless otherwise specified all operators on X will be nonlinear.

Definition 2.1: A *strongly continuous semigroup* is a one parameter family of mappings $\{S(t) \mid t \geqslant 0\}$ of X to itself having the properties that:

(i) $S(0) = I$;

(ii) $S(t+s) = S(t)S(s)$ for $s, t \geqslant 0$;

(iii) $\lim_{t\to 0} \|S(t)x - x\| = 0$ for all $x \in X$.

The semigroup is said to be *nonexpansive* if $\|S(t)x - S(t)y\| \leqslant \|x-y\|$ for $t \geqslant 0$, $x, y \in X$. The *infinitesimal generator* A of $\{S(t) \mid t \geqslant 0\}$ is defined by the relation $Ax = \lim_{h\to 0} (S(h)x - x)/h$ and the *weak infinitesimal generator* $A_w x = x - \lim_{h\to 0} (S(h)x - x)/h$.

We remark that the notions of weak infinitesimal generator and infinitesimal generator coincide for the case of a strongly continuous semigroup of bounded linear operators [**19**, p.43]. We now make precise our notion of an accretive operator.

Definition 2.2: An operator $A\colon X \to X$ is said to be *accretive* provided that $\|(x+\lambda Ax) - (y + \lambda Ay)\| \geqslant \|x-y\|$ for all $x, y \in X$ and $\lambda \geqslant 0$. An accretive operator is said to be *m-accretive* provided that $R(I + \lambda A) = X$ for all $\lambda \geqslant 0$.

It is known, see [**12**] that accretiveness is equivalent to the statement that $\operatorname{Re}\langle Ax - Ay, f\rangle \geqslant 0$ for $x, y \in D(A)$ and $f \in F(x-y)$ where $F\colon X \to 2^{X^*}$ is the duality map. We recall that by definition the duality map is the unique multiple valued mapping from X to X^* with $D(f) = X$ such that $f \in F(x)$ if and only if $\langle x, f\rangle = \|x\|^2 = \|f\|^2$. If A is an m-accretive operator A has no proper accretive extension, however not every maximal accretive operator is m-accretive. We remark that multi-valued accretive operators arise in a natural way in the study of semigroups of nonexpansive operators, however in the work at hand we shall limit our discussion to the single-valued case.

An operator $A\colon X \to X$ is said to be *weakly continuous* if it is continuous from the weak topology of X to the weak topology of X. Such an operator will map weakly convergent sequences to weakly convergent sequences. However, we point out that a sequentially weakly continuous operator need not be weakly continuous.

We shall have occasion to consider two types of integrals of vector-valued functions. A function $x(\,)\colon \mathbb{R} \supseteq [a, b] \to X$ is said to be *Pettis integrable* if there exists a $y \in X$ such that $\langle y, f\rangle = \int_a^b \langle x(s), f\rangle ds$ for all $f \in X^*$. The integrand is a complexvalued function and the integral is taken in the sense of Lebesgue. In the case when the Pettis integral exists we write $y = \int_a^b x(s)ds$. A function $x(\,)\colon [a, b] \to X$ is said to be *Bochner integrable* if there exists a sequence of countable-valued functions $\{x_n(\,)\}$ converging for a.e. $s \in [a, b]$ such that $\lim_{n\to\infty} \int_a^b \|x_n(s) - x(s)\|\, ds = 0$. In this case, the Bochner integral can be defined to be $\int_a^b x(s)ds = \lim_{n\to\infty} \int_a^b x_n(s)ds$ where the integrals on the right are Pettis integrals. Every Bochner integrable function is Pettis integrable. A vector-valued function $x(\,)$ is *weakly measurable* if $\langle x(\,), f\rangle$ is measurable and it is *strongly measurable* if it is weakly measurable and almost separably valued. A function can be shown to be Bochner integrable if it is strongly measurable and the Lebesgue integral $\int_a^b \|x(s)\|\, ds$ exists. Needless to say, our discussion has been exceedingly cursory. An in-depth treatment of the notions of integration in abstract spaces appears in [**10**],

Given a weakly continuous vector field $A\colon X \to X$ we consider the following Cauchy initial value problem

$$\dot{u}(t) + Au(t) = 0, \tag{2.3a}$$

$$u(0) = x. \tag{2.3b}$$

By a *weak solution* on $[0, T]$ we understand a strongly continuous, once weakly differentiable function $u\colon [0, T] \to X$ such that $u(0) = x$ and the weak derivative satisfies (2.3a). For a *strong solution* we shall require additionally that the function be absolutely continuous and have a strong derivative for a.e. $t \in [0, T]$. We point out the obvious lack of parity in the notions of weak and strong solution and hope that it will not lead to confusion. Because our solutions turn out to be both weak solutions and strong solutions we use the same symbol $\dot{u}(t)$ to denote both the weak and the strong derivative.

Our existence theory requires the following measure of weak noncompactness which was introduced in [**6**].

DEFINITION 2.4: Let U be a bounded nonempty subset of a Banach space X. The *measure of weak noncompactness* of U, $\beta(U)$ is defined by

$$\beta(U) = \inf\{t \geqslant 0 \mid (\exists V \in K^w)(U \subseteq V + tB_1(0))\}.$$

Here K^w denotes the collection of weakly compact subsets of X and $B_1(0)$ is the unit ball in X.

The following lemma which appears in [**19**] asseerts that $\beta(\,)$ is a sublinear measure of noncompactness.

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Lemma 2.5. *If U and V are bounded subsets of X, the following are true:*

(1) $U \subseteq V$ *implies* $\beta(U) \leqslant \beta(V)$,

(2) $\beta(U) = \beta(\mathrm{wcl}\, U)$, *where* $\mathrm{wcl}\, U$ *denotes the weak closure of* U,

(3) $\beta(U) = 0$ *if* $\mathrm{wcl}\, U$ *is compact,*

(4) $\beta(U \cup V) = \max[\beta(U), \beta(V)]$,

(5) $\beta(\mathrm{conv}\, U) = \beta(U)$ *where* $\mathrm{conv}\, U$ *denotes the convex hull of* U,

(6) $\beta(U + V) \leqslant \beta(U) + \beta(V)$,

(7) $\beta(x + U) = \beta(U)$ *for all* $x \in X$,

(8) $\beta(\lambda U) = |\lambda|\, \beta(U)$,

(9) $\beta\left(\bigcup_{0 \leqslant t \leqslant r} tU\right) = \tau\beta(U)$.

In addition to requiring that an operator be weakly continuous our local existence theory needs the concept of a Kamke function. A continuous function $\omega\colon \mathrm{R}^+ \to \mathrm{R}^+$ is said to be a *Kamke function* provided that $u = 0$ is the only solution of the integral inequality $u(t) \leqslant \int_0^t \omega(u(s))ds$, $u(0) = 0$. The following local existence theorem appears in [**4**].

Theorem 2.6. *Let A be a weakly continuous operator on a Banach space. If there exists a Kamke function $\omega(\,)\colon \mathrm{R}^+ \to \mathrm{R}^+$ such that $\beta(AU) \leqslant \omega(\beta(U))$ for each bounded subset U then for each $x \in X$ there exists a $T_x > 0$ and $u(\,)\colon [0, T_x] \to X$ which is a weak solution to the Cauchy initial value problem*

$$\dot{u}(t) + Au(t) = 0 \quad \text{for} \quad t \in [0, Tx],$$
$$u(0) = x.$$

Strictly speaking the result in [**4**] requires that A be bounded in a neighbourhood of the initial point. However the Kamke condition we use ensures that A maps bounded subsets to bounded subsets.

We point out that in the course of the proof one shows that $u(\,)$ is Lipschitz. Hence it is of strong bounded variation and by virtue of [**10**, Theorem 3.8.6], $u(\,)$ is strongly differentiable for a.e. $t \in [0, T_x]$. When it exists, the strong derivative equals the weak derivative and we therefore obtain strong solutions to the Cauchy problem. We feel that requirements similar to local boundedness and the Kamke condition are needed for local existence when dealing with weakly continuous vector fields. It has been shown that weak continuity of the vector field is not sufficient to guarantee local existence in general Banach spaces [**1**]. Weakly continuous functions need not be locally bounded in general Banach spaces. Moreover the existence proofs rely on the extraction of weakly convergent approximating sequences. The Kamke condition ensures this extraction. It is well-known that closed and bounded subsets of nonreflexive spaces need not be weakly sequentially compact.

We shall also need a weak Arzela-Ascoli Theorem which is proved in [4].

THEOREM 2.7. *Let J be a weakly equicontinuous family of functions from an interval $[a, b]$ to X. If $\{x_n(\ )\} \subseteq J$ is a sequence such that for each $t \in [a, b]$, $\{x_n(t)\}$ is weakly precompact, then there exists a subsequence $\{x_m(\ )\}$ which converges weakly uniformly on $[a, b]$ to a weakly continuous function $x(\ )$.*

3. GLOBAL EXISTENCE RESULTS

In this section we obtain global existence for our Cauchy initial value problem by imposing the additional requirement that A be accretive and use this result to see that A is m-accretive.

THEOREM 3.1. *Let $A: X \to X$ be a weakly continuous accretive operator. If there also exists a Kamke function $\omega(\ ): \mathbb{R}^+ \to \mathbb{R}^+$ such that $\beta(AU) \leqslant \omega(\beta(U))$ for each bounded $U \subseteq X$, then for $x \in X$ there exists a strong solution to*

$$\dot{u}(t) + Au(t) = 0 \quad \text{for } t \in (0, \infty), \tag{3.2a}$$

$$u(0) = x. \tag{3.2b}$$

Moreover if $u(\ ,x)$ and $u(\ ,y)$ are strong solutions with initial points x, y respectively

$$\|u(t, x) - u(t, y)\| \leqslant \|x - y\| \quad \text{for all } t \geqslant 0. \tag{3.3}$$

PROOF: The local result guarantees the existence of a solution in a small time interval and using standard arguments of ordinary differential equations we obtain a maximal interval of existence $[0, T^*)$. We assume for the sake of contradiction that $T^* < \infty$. If $t, t+h \in [0, T^*)$ then by a lemma appearing in [**11**] we have

$$\frac{d}{dt}\|u(t+h) - u(t)\|^2 = -\operatorname{Re}\langle Au(t+h) - Au(t), f(t)\rangle \quad \text{for a.e. } t$$

where $f(t) \in F(u(t+h) - u(t))$ and $F(\ ): X \to 2^{X^*}$ is the duality map. Using the accretiveness of A we have

$$\frac{d}{dt}\|u(t+h) - u(t)\|^2 \leqslant 0 \quad \text{for a.e. } t.$$

The inequality may be integrated to produce

$$\|u(t+h) - u(t)\| \leqslant \|u(h) - u(0)\| = \|u(h) - x\|$$

and we use the strong continuity of $u(\ )$ to compute $\lim_{t \to T^*} u(t) = u(T^*)$. The local theory allows us to continue the solution past T^* and thereby contradict the maximality of T^*.

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To obtain the final assertion we differentiate $\|u(t, x) - u(t, y)\|^2$ and use the accretivity of A to observe

$$\frac{d}{dt}\|u(t, x) - u(t, y)\|^2 \leqslant 0 \quad \text{for a.e. } t.$$

We obtain the inequality by integration.

If $x \in X$ and $t \geqslant 0$ we define $S(t): X \to X$ by setting

$$S(t)x = u(t, x). \tag{3.4}$$

□

Clearly $S(\,)x$ is continuous ion t; uniqueness implies that $S(t)S(s) = S(t+s)x$; $S(0) = I$. Moreover, $\|S(t)x - S(t)y\| \leqslant \|x - y\|$. Thus, we associate a strongly continuous semigrouup of nonexpansive mappings with solutions to the Cauchy initial value problem.

Our next result insures that our accretive operators are m-accretive.

Theorem 3.5. *If A satisfies the conditions of Theorem 3.1, then A is m-accretive.*

Proof: It has been shown in [**16**] that if $R(I + \lambda_0 A) = X$ for some $\lambda_0 > 0$ then $R(I + \lambda A) = X$ for all $\lambda > 0$. Consequently it will be sufficient to show that for any $a \in X$ there exists an $x_0 \in S$ so that $Ax_0 + x_0 = a$. We introduce an operator $A_1 = A - a$. Clearly A_1 is weakly continuous, is locally bounded and accretive. Because $\beta(A_1 U) = \beta(AU - a) = \beta(AU) \leqslant \omega(\beta(U))$, the existence of solutions to $\dot u(t) + A_1 u(t) = 0$; $u(0) = x$ is assured. It is trivial to see that $e^{-t}u(t) = v(t)$ gives the unique solution to $\dot v(t) + A_2 v(t) = 0$, $v(0) = x$ when $A_2 = A_1 + I$. If $\{S_2(t) \mid t \geqslant 0\}$ is the semigroup associated with the solutions we differentiate $\|S_2(t)x - S_2(t)y\|^2$ and use the accretiveness of A_1 to observe that

$$\begin{aligned}\frac{d}{dt}\|S_2(t)x - S_2(t)y\|^2 &\leqslant 2\,\mathrm{Re}\langle (A_1 + I)S_2(t)x - (A_1 + I)S_2(t)y, f(t)\rangle \\ &\leqslant -2\,\|S_2(t)x - S_2(t)y\|^2.\end{aligned}$$

We may integrate and observe

$$\|S_2(t)x - S_2(t)y\| \leqslant e^{-t}\,\|x - y\|.$$

If $t_0 > 0$, the Banach Fixed Point Theorem guarantees a unique fixed point $S_2(t_0)x_0 = x_0$. To see that $S_2(t)x_0 = x_0$ for all $t > 0$ we observe that $S_2(t)x_0 = S_2(t)S_2(t_0)x_0 = S_2(t_0)S_2(t)x_0$. Because $S_2(t)x_0$ is seen to be a fixed point of $S_2(t_0)$ we know that $S_2(t)x_0 = x_0$. Thus x_0 is a rest point of $\{S_2(t) \mid t \geqslant 0\}$ and we have

$$0 = \frac{d}{dt}S_2(t)s_0\,|_{t=0} = -A_2 x_0 = 0$$

and hence $Ax_0 + x_0 = a$ and we reach our intended conclusion. ∎

We obtain immediately a representation of solutions to (3.2 a-b).

COROLLARY. *If A satisfies the condition of Theorem 3.1 and $\{S(t) \mid t \geqslant 0\}$ is the semigroup associated with solutions to the Cauchy initial value problem, then for all $t > 0$ and $x \in X$*

$$S(t)x - \lim_{n\to\infty}\left(I + \frac{t}{n}A\right)^{-n} x$$

and the convergence is uniform on compact subintervals of $\mathbf{R}$.

PROOF: This result is an immediate consequence of the m-accretiveness of A and the Crandall-Liggett Representation Theorem for nonlinear semigroups, [5]. ∎

4. Additive Perturbation

In this section we consider the perturbation of linear accretive operators by weakly continuous operators. More specifically we provide conditions which guarantee their sum is m-accretive. Whereas the key to the m-accretiveness of a weakly continuous operator was an abstract differential equation, the key to the perturbation result will be an abstract Volterra equation with a linear semigroup acting as a convolution kernel. If A and B are two operators whose domains are subsets of X, we define their sum $A + B$ by the equation $(A + B)x = Ax + Bx$ for $x \in D(A \cap B)$.

For the remainder of this section A shall denote a linear m-accretive operator and $\{T(t) \mid t \geqslant 0\}$ shall denote the semigroup of nonexpansive linear transformations generated by $-A$. Hence

$$T(t)x = \lim_{n\to\infty}\left(I + \frac{t}{n}A\right)^{-n} x \quad \text{for all } x \in X, t \geqslant 0$$

and if $x \in D(A)$

$$\frac{d}{dt}T(t)x + AT(t)x = 0 \quad \text{for } t \geqslant 0.$$

B shall be a weakly continuous locally bounded accretive operator satisfying the hypotheses of Theorem 3.1. Because A has been shown to be m-accretive we know that $J_\lambda x = (I + \lambda B)^{-1}x$ is a nonexpansive operator on X and it is not hard to show that $\|J_{\lambda_1}x - J_{\lambda_2}x\| \leqslant (\lambda_1 - \lambda_2)\|Bx\|$. We can define the so-called *Yosida approximations*

$$B_n x = n\big(I - J_{1/n}\big)x,$$

It is immediate that B_n is everywhere defined, Lipschitz and accretive. A simple computation shows that $B_n x = BJ_{1/n}x$ and $\|B_n x\| \leqslant \|Bx\|$.

The following lemma concerns the action of semigroup of linear nonexpansive operators on the measure of weak noncompactness.

LEMMA 4.1. *If U is a closed and bounded subset of a Banach space X and $\{T(t) \mid t \geqslant 0\}$ is a semigroup of nonexpansive linear operators on X then $\beta(T(t)U) \leqslant \beta(U)$.*

PROOF: If V is a weakly compact subset of X and $s \geqslant 0$ are such that $U \subset V + sB_1(0)$ then $T(t)U \subseteq T(t)V + sT(t)B_1(0) \subseteq T(t)V + sB_1(0)$. Because $T(t)V$ is the weakly continuous image of a weakly compact set it is weakly compact and our desired result follows from the definition of the measure of weak noncompactness. □

Our next theorem concerns the well-posedness of an abstract semilinear Volterra integral equation.

THEOREM 4.2. *Let A be a linear m-accretive operator from a Banach space X to itself and let $\{T(t) \mid t \geqslant 0\}$ be the linear semigroup generated by $-A$. If B is a weakly continuous operator satisfying the hypotheses of Theorem 3.1 and $x \in X$, then for all $t \geqslant 0$ there exists a unique solution to*

$$u(t, x) = T(t)x - \int_0^t T(T-s)Bu(s, x)ds. \tag{4.3}$$

Moreover

$$\|u(t, x) - u(t, y)\| \leqslant \|x - y\| \qquad \text{for } x, y \in X. \tag{4.4}$$

PROOF: We begin by assuming $x \in D(A)$. Because B_n is continuous we can use the theory of continuous nonlinear perturbations developed in [**22**]. The operator $A + B_n$ is m-accretive and for any $T > 0$ there exists a unique solution

$$u_n(t) = T(t)x - \int_0^t T(t-s)B_n u_u(s)ds. \tag{4.5}$$

Moreover $u_n(t)$ has the exponential representation

$$u_n(t) = \lim_{m \to \infty} \left(I + \frac{t}{m}(A + B_n)\right)^{-m_x}.$$

We use this representation to show that $u_n(t)$ is uniformly Lipschitz continuous on

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$[0, T]$. For $t, \tau \in [0, T]$ we see that

$$\begin{aligned}
&\left\|\left(I+\frac{t}{m}(A+B_n)\right)^{-m}x-\left(I+\frac{\tau}{m}(A+B_n)\right)^{-m}x\right\| \\
&\leqslant \sum_{j=1}^{m}\left\{\left\|\prod_{i=1}^{m-j+1}\left(I+\frac{t}{m}(A+B_n)\right)^{-1}\prod_{i=1}^{j-1}\left(I+\frac{\tau}{m}(A+B_n)\right)^{-1}x\right.\right. \\
&\qquad\left.\left.-\prod_{i=1}^{m-j}\left(I+\frac{t}{m}(A+B_n)\right)^{-1}\prod_{i=1}^{j}\left(I+\frac{\tau}{m}(A+B_n)\right)^{-1}x\right\|\right\} \\
&\leqslant \sum_{j=1}^{m}\left\{\left|\frac{t}{m}-\frac{\tau}{m}\right|\left\|(A+B_n)\prod_{i=1}^{m-j+1}\left(I+\frac{t}{m}(A+B_n)\right)^{-1}\right.\right. \\
&\qquad\left.\left.-\prod_{i=1}^{j}\left(I+\frac{\tau}{m}(A+B_n)\right)^{-1}x\right\|\right\} \\
&\leqslant |t-\tau|\,\|Ax+B_nx\| \leqslant |t-\tau|\,(\|Ax\|+\|Bx\|).
\end{aligned}$$

Because $(I+t/m(A+B_n))^{-m}x$ converges uniformly to $u_n(t)$ we have

$$\|u_n(t)-u_n(\tau)\| \leqslant |t-\tau|\,(\|Ax\|+\|Bx\|).$$

Because $u_n(0)=x$, the sequence $\{x_n(\,)\}$ is uniformly bounded.

We set $p(t) = \beta(\{u_n(t)\})$ where $\beta(\,)$ is the weak measure of noncompactness. By virtue of [**2**, Lemma 13, 2.1] we know that $p(\,)$ is Lipschitz continuous and hence differentiable for a.e. $t \in [0, T]$. Let $t \in [0, t]$ be a Lebesgue point of $p(\,)$. If $\varepsilon > 0$ select $\delta > 0$ so that $|z-p(t)| < \delta$ implies $|\omega(z)-\omega(p(t))| < \varepsilon$ where $\omega(\,)$ is the Kamke function of Theorem 3.1. Let K_h denote $\{u_n(s) \mid t \leqslant s \leqslant t+h < T\}$. From [**19**, Lemma 2.2] there exists $\hat{t} \in [t, t+h]$ so that

$$\beta(K_h) = \sup_{s\in[t,t+h]} \beta(\{u_n(s)\}) = p(\hat{t}).$$

We let $L_h = \{J_{1/n}u_n(\,)\}$. Then $\beta(L_h) = \beta(K_h)$ is a consequence of the fact that $\|J_{1/n}u_n(\,)-u_n(t)\|_\infty \to 0$ which follows readily from the observation that $\|J_{1/n}u_n(t)-u_n(t)\| = 1/n\,\|B_nu_n(t)\| \leqslant 1/n\,\|Bu_n(t)\|$ and that the Kamke condition implies B maps bounded subsets of X to bounded subsets of X.

Let h be such that

$$0 \leqslant \beta(K_h) - p(t) = |p(\hat{t}) - p(t)| < \delta.$$

From the choice of $\delta > 0$ we have $|\omega(\beta(K_h))-\omega(p(t))| < \varepsilon$ and thus $|\omega(\beta(L_h))-\omega(p(t))| < \varepsilon$. We return to the Volterra equation (4.5) and observe that

$$u_n(t+h) = T(h)u_n(t) - \int_t^{t+h} T(t+h-s)B_nu_n(s)ds.$$

Thus,

$$u_n(t+h) \in T(h)u_n(t) - h\ \overline{\text{conv}} \bigcup_{s\in[t,t+h]} T(t+h-s)B_n u_n(s)$$

and

$$\{u_n(t+h)\} \subseteq \{T(h)u_n(t) - h \underset{s\in[t,t+h]}{\overline{\text{conv}}} \{T(t+h-s)B_n u_n(s)\}\}.$$

We therefore observe that

$$\beta(\{u_n(t+h)\}) \leqslant \beta(\{T(h)u_n(t)\}) + \beta h\ \overline{\text{conv}}\{T(t+h-s)B_n u_n(s)\}$$

and from Lemma 4.1 we deduce

$$\begin{aligned}\beta(\{u_n(t+h)\}) &\leqslant \beta(\{u_n(t)\}) + h\beta(L_h)\\ &= \beta(\{u_n(t)\}) + h\beta(K_h)\\ &\leqslant \beta(\{u_n(t)\}) + h(\omega(\beta(\{u_n(t)\}) + \varepsilon)).\end{aligned}$$

Consequently

$$\frac{p(t+h)-p(t)}{h} \leqslant \omega(p(t)) + \varepsilon$$

and, because t is a Lebesgue point,

$$p'(t) \leqslant \omega(p(t)) + \varepsilon.$$

Recalling that $\varepsilon > 0$ was arbitrarily chosen,

$$p'(t) \leqslant \omega(p(t)).$$

Since $p(0) = \beta(x) = 0$, $p(t) \equiv 0$ and we use the weak Arzela-Ascoli theorem. There exists a subsequence of $\{u_n(\ )\}$ which converges weakly uniformly to a weakly continuous function $u(\ ,x)$. We relabel this convergent subsequence on $\{u_n(\ )\}$. Because $B_n u_n(t) = BJ_{1/n}u_n(t)$ and $\|J_{1/n}u_n(\ ) - u_n\|_\infty \leqslant 1/n\|Bu_n(\ )\|_\infty$, $B_n u_n(t)$ converges weakly to $Bu(t)$ for each $t \in [0, T]$. We compute the weak limit of each side of (4.5) to see that

$$u(t, x) = T(t)x - \int_0^t T(t-s)Bu(s, x)ds$$

where the integral is taken in the sense of Pettis. Because $Bu(\ )$ is the weak limit of strongly continuous functions $B_n u_n(\ )$ one can argue that $Bu(\ )$ is strongly measurable. Because $\|Bu(s)\|$ is bounded the integral exists in the sense of Bochner. If $x_1, x_2 \in D(A)$ and $u_n(\ ,x_1)$ and $u_n(\ ,x_2)$ are solutions to (4.5) emanating from x_1 and x_2 then $\|u_n(t, x_1) - u_n(t, x_2)\| \leqslant \|x_1 - x_2\|$, see [**22**, Theorem 1]. Thus

$$\|u(t, x_1) - u(t, x_2)\| \leqslant \|x_1 - x_2\|.$$

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We define $U(t)\colon D(A) \to X$ by writing $U(t)x = u(t, x)$. Because $\overline{D(A)} = X$ we can extend $\widetilde{U}(t)$ to define a nonexpansive mapping $U(t)$ of X onto itself. In this manner we define a family of nonexpansive mappings $\{U(t) \mid t \geqslant 0\}$ of X to itself. If $\{x_n\} \subseteq D(A)$ and $x_n \to x$ then $U(\ )x_n = u(\ , x_n)$ converges uniformly to $U(\ )x$ on $[0, T]$. Because $Bu(t, x_n)$ converges to $BU(t)x$ we argue that solutions to

$$u(t, x_n) = T(t)x_n - \int_0^t T(t-s)Bu(s, x_n)ds$$

converge weakly to the equation

$$U(t)x = T(t)x - \int_0^t T(t-s)BU(s)xds.$$

The integral exists in the sense of Pettis and an argument similar to the previous one ensures that it is a Bochner integral. Thus $u(t, x) = U(t)x$ gives a solution to (4.3). Moreover because $u(t, x) = w - \lim u(t, x_n)$ one can argue that

$$\|u(t, x_1) - u(t, x_2)\| \leqslant \|x_1 - x_2\| \text{ for all } x_1, x_2 \in X, t \geqslant 0$$

and thereby establish the final assertion of theorem. ∎

We point out that the family $\{U(t) \mid t \geqslant 0\}$ is a strongly continuous semigroup of nonexpansive operators. We characterise the weak infinitesimal generator.

PROPOSITION 4.6. *If $\{U(t) \mid t \geqslant 0\}$ is the semigroup of nonexpansive operators associated with solutions to (4.3) then the weak infinitesimal of $\{U(t) \mid t \geqslant 0\}$ is $-(A+B)$.*

PROOF: We first establish that for all $x \in X$, $w - \lim\limits_{h \to 0+} \left(-1/h \int_0^h T(h-s)\right.$ $\left. Bu(s)ds\right) = -Bx$ where $u(t)$ is the solution to (4.3). We set $u(t) = \int_0^t T(t-s)$ $Bu(s, x)ds$. Previous arguments guarantee that $Bu(\ , x)$ is Bochner integrable. The weak variation of constants result of [**1**] implies if $f \in D(A^*)$ then for a.e. t

$$\frac{d}{dt}\langle v(t), f\rangle = -\langle v(t), A^* f\rangle - \langle Bu(t, x), f\rangle \tag{4.7}$$

where A^* denotes the adjoint of A. Because the right $Bu(\ , x)$ is weakly continuous the right-hand-side of (4.7) is continuous and we may observe that $\langle v(t), f\rangle$ is continuously differentiable and that $d/dt\langle v(0), f\rangle = \langle -Bx, f\rangle$. Thus $d/dt\langle v(0), f\rangle = \lim\limits_{h \to 0}\langle -1/h \int_0^h T(h-s)Bu(s, x), f\rangle = -\langle Bx, f\rangle$. The domain of A^* is dense in X^*, [**13**, Theorem 5.29] and hence the limit must hold for all $f \in X^*$. To complete the proof we observe that $w - \lim\limits_{h \to 0+} 1/h(u(h)x - x)$ exists if and only if $w - \lim\limits_{h \to 0+} 1/h(T(h)x - x)$ exists. ∎

Our next result guarantees the surjectivity of the resolvent of $A + B$ is m-accretive.

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Theorem 4.8. *If A and B satisfy the criteria of Theorem 4.2 then $A+B$ is m-accretive.*

Proof: As previously observed, it is sufficient to establish this for the case $\lambda = 1$. We let $\widehat{A} = A + I$ and for $a \in X$ we let $\widehat{B} = B = a$. The linear semigroup generated by $-\widehat{A}$ will be denoted by $\{\widehat{T}(t) \mid t \geqslant 0\}$. This previous theorem yields a unique solution to

$$u(t, x) = \widehat{T}(t)x - \int_0^t \widehat{T}(t-s)\widehat{B}u(s, x)dx \quad \text{for } x \in X \tag{4.9}$$

which may be viewed as the weak limit of a subsequence of solutions to

$$u_n(t, x) = \widehat{T}(t)x - \int_0^t \widehat{T}(t-s)\widehat{B}_n u_n(s, x)ds.$$

Moreover, from [**23**, Proposition 3.15], we have

$$\|u_n(t, x) - u_n(t, y)\| \leqslant e^{-t}\|x - y\|.$$

Thus,

$$\|u(t, x) - u(t, y)\| \leqslant \liminf \|u_n(t, x) - u_n(t, y)\| \leqslant e^{-t}\|x - y\|.$$

Denoting the semigroup associated with solutions to (4.9) by $\{\widehat{U}(t) \mid t \geqslant 0\}$ we see that for fixed $t_0 > 0$, $\widehat{U}(t_0)$ is a strict contraction. Hence by the Banach Fixed Point Theorem $U(t_0)$ has a unique fixed point $\widehat{U}(t_0)x_0 = s_0$. To see that x_0 is a rest point for the semigroup we observe that $\widehat{U}(t)x_0 = \widehat{U}(t)\widehat{U}(t_0)x_0 = \widehat{U}(t_0)\widehat{U}(t)x_0$ and hence $\widehat{U}(t)x_0$ is a fixed point of $\widehat{U}(t_0)$. Consequently $\widehat{U}(t)x_0 = x_0$. Thus $u(t, x_0) = \widehat{U}(t)x_0 = x_0$ is strongly differentiable and hence x_0 belongs to a weak infinitesimal generator of $\{\widehat{U}(t) \mid t \geqslant 0\}$. Moreover, x_0 satisfies the equation

$$x_0 = \widehat{T}(t)x_0 - \int_0^t \widehat{T}(t-s)Bx_0 ds.$$

By virtue of the previous proposition we know that $x_0 \in D\left(\widehat{A}\right)$ and we can compute the limit as $t \downarrow 0$ of the equation

$$\frac{1}{t}\left(x_0 - \widehat{T}(t)x_0\right) = -\frac{1}{t}\int_0^t \widehat{T}(t-s)Bx_0 ds$$

and obtain

$$\widehat{A}x_0 = -\widehat{B}x_0,$$

or $x_0 + Ax_0 + Bx_0 = a$ and we obtain our desired result.

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Because $A+B$ is m-accretive we can use the Crandall-Liggett Representation Theorem [**5**] for nonlinear semigroups to compute

$$V(t)x = \lim_{n\to\infty}\left(1+\frac{t}{n}(A+B)\right)^{-n} x \tag{4.10}$$

and define a semigroup of nonexpansive operators $\{V(t) \mid t \geqslant 0\}$. We conclude by showing that $\{V(t) \mid t \geqslant 0)$ is the semigroup associated with solutions to (4.3). □

PROPOSITION 4.11. *If A, B satisfy the conditions of Theorem 4.2 and $\{U(t) \mid t \geqslant 0\}$ is the semigroup of nonexpansive operators associated with solutions to (4.3) then for all $x \in X$, $t \geqslant 0$*

$$U(t)x = \lim_{n\to\infty}\left(I+\frac{t}{n}(A+B)\right)^{-n} x$$

and the limit exists uniformly on intervals $[0, T]$, $T > 0$.

PROOF: We follow [**22**, Proposition 3.18] and observe that

$$\left(I+\frac{t}{n}(A+B)\right)^{-n} x = \left(I+\frac{t}{n}A\right)^{-n} x - \frac{t}{n}\sum_{i=1}^{n}\left(I+\frac{t}{n}A\right)^{-(n-i+1)} B\left(I+\frac{t}{n}(A+B)\right)^{-i} x.$$

We define $z_n(t)$ by

$$z_n(t) = T(t)x - \sum_{i=1}^{n}\frac{t}{n}T\left(t-\frac{t(i-1)}{n}\right)BV\left(\frac{t(i-1)}{n}\right)x,$$

where $V(t)x$ is the infinite product given by formula (4.10). We observe that $z_n(t)$ converges weakly to

$$z(t) = T(t)x - \int_0^t T(t-s)BV(s)x\,ds$$

and we can argue that for any $f \in X^*$

$$\lim_{n\to\infty}\left\langle\left(I+\frac{s}{n}A\right)^{-n} y, f\right\rangle = \langle T(s)y, f\rangle$$

uniformly for $0 \leqslant s \leqslant t \leqslant T < \infty$ on a weakly compact subset of X.

If $f \in X^*$ we observe that

$$\lim_{n\to\infty}\langle\Big(I+\frac{t}{n}(A+B)\Big)^{-n}x - z_n(t),\, f\rangle = \lim_{n\to\infty}\langle\Big(I+\frac{t}{n}A\Big)^{-n}x - T(t)x,\, f\rangle$$
$$+\lim_{n\to\infty}\langle\sum_{i=1}^{n}\frac{t}{n}[\Big(I+\frac{t}{n}A\Big)^{-(n-i+1)}B\Big(I+\frac{t}{n}(A+B)\Big)^{-i}x - BV\Big(\frac{t(i-1)}{n}\Big)x],\, f\rangle$$
$$+\lim_{n\to\infty}\langle\sum_{i=1}^{n}\frac{t}{n}\{\Big(I+\frac{t}{n}A\Big)^{-(n-i+1)} - T\Big(t\frac{t(i-1)}{n}\Big)\}BV\Big(\frac{t(i-1)}{n}\Big)x,\, f\rangle$$

and conclude that

$$w-\lim\Big(I+\frac{t}{n}(A+B)\Big)^{-n}x = V(t)x = z(t)$$
$$= T(t)x - \int_0^t T(t-s)BV(s)x\,ds.$$

Because $U(t)x$ is the unique solution to (4.3) we have $U(t)x = V(t)x$. ∎

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