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INTERPOLATION BY LINEAR SUMS OF HARMONIC MEASURES

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1. **Introduction.** Let α be an open arc on the unit circle,

$$\alpha=\{e^{i\phi}:\phi_1<\phi<\phi_2,\ \phi_2-\phi_1\le 2\pi\},$$

and for $z=re^{i\theta}$, $0\le r<1$, let

$$\omega(z;\alpha)=\omega(z;\phi_1,\phi_2)=\frac{1}{2\pi}\int_{\phi_1}^{\phi_2}\frac{1-r^2}{1-2r\cos(t-\theta)+r^2}\,dt. \tag{1.1}$$

The function $\omega(z;\alpha)$ is called the harmonic measure of the arc α with respect to the unit disc, (Nevanlinna *2*); it is harmonic and bounded in the unit disc and possesses (Fatou) boundary values 1 and 0 at interior points of α and the complementary arc β respectively. In this article, we consider linear sums of the form

$$S(z)=\sum_{l=1}^{m}x_l\omega(z;\alpha_l), \tag{1.2}$$

where $\alpha_1,\alpha_2,\ldots,\alpha_m$ are mutually disjoint open arcs on the unit circle and $x_1,\ldots,x_m$ are complex numbers. If we regard the arcs $\alpha_1,\alpha_2,\ldots,\alpha_m$ as fixed, and choose m distinct points $z_1,\ldots,z_n$ in the unit disc, we may be able to find n complex numbers $y_1,\ldots,y_n$ with the property that there are *no* solutions of the system

$$\sum_{l=1}^{m}x_l\omega(z_k;\alpha_l)=y_k,\qquad k=1,2,\ldots,n. \tag{1.3}$$

This will certainly be the case if $m<n$, however, the configuration of arcs and points may render the solution of (1.3) impossible even when the number of arcs is considerably larger than the number of points of interpolation.

For an example, let $\alpha_1,\alpha_2,\ldots,\alpha_{n-1}$ $(n>1)$ be mutually disjoint open arcs on the upper half of the unit circle and let $\alpha_{k+(n-1)}$ be the reflection of α_k in the real axis, $k=1,2,\ldots,n-1$. If $z_1,z_2,\ldots,z_n$ are distinct points on the interval $(-1,1)$, then we have $\omega(z_k;\ \alpha_l)=\omega(z_k;\ \alpha_{l+(n-1)})$ for $k=1,2,\ldots,n$, and $l=1,2,\ldots,n-1$.

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Hence for any numbers $x_1, \ldots, x_{2(n-1)}$, we have

$$\sum_{l=1}^{2(n-1)} x_l\omega(z_k; \alpha_l) = \sum_{l=1}^{n-1} (x_l + x_{l+(n-1)})\omega(z_k; \alpha_l)$$

for $k=1, 2, \ldots, n$. Thus there is some n-tuple of complex numbers $(y_1, y_2, \ldots, y_n)$ for which the system

$$\sum_{l=1}^{2(n-1)} x_l\omega(z_k; \alpha_l) = y_k, \qquad k = 1, 2, \ldots, n$$

has no solution.

That the above is indeed the extreme example is the content of the following.

THEOREM 1. *Let $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$, $(n \geq 1)$ be mutually disjoint open arcs on the unit circle and let $z_1, z_2, \ldots, z_n$ be disjoint points in $|z|<1$. Then if $y_1, y_2, \ldots, y_n$ are any n complex numbers, there are $2n-1$ corresponding complex numbers $x_1, x_2, \ldots, x_{2n-1}$ which satisfy*

$$\sum_{l=1}^{2n-1} x_l\omega(z_k; \alpha_l) = y_k, \qquad k = 1, 2, \ldots, n. \tag{1.4}$$

Proof. Let $\alpha_l = \{e^{i\phi} : \phi_1^l < \phi < \phi_2^l\}$ with the available assumption that for some real R_0, we have $R_0 \leq \phi_1^l < \phi_2^l \leq R_0 + 2\pi$, for $l=1, 2, \ldots, 2n-1$. Let $z_k = r_k \exp(i\theta_k)$, $1 \leq k \leq n$; we shall show that the matrix

$$\left(\int_{\phi_1{}^l}^{\phi_2{}^l} \frac{1-r_k^2}{1-2r_k\cos(\theta_k - t) + r_k^2}\, dt\right) \tag{1.5}$$

possesses n linearly independent columns.

If P denotes the number of linearly independent columns in (1.5), it is clear that $P \geq 1$, so assume that P is strictly less than n. For $1 \leq h \leq P+1$, form the determinants

$$A_h = (-1)^{h+P+1} \det_{\substack{1\leq k\leq P+1 \\ 1\leq l\leq P \\ k\neq h}} \left(\int_{\phi_1{}^l}^{\phi_2{}^l} \frac{1-r_k^2}{1-2r_k\cos(\theta_k - t) + r_k^2}\, dt\right). \tag{1.6}$$

By a preliminary rearrangment of the matrix (1.5), we may guarantee that $A_{P+1} \neq 0$.

As $P<n$, it follows that

$$\sum_{h=1}^{P+1} A_h \int_{\phi_1{}^l}^{\phi_2} \frac{1-r_h^2}{1-2r_h\cos(\theta_h - t) + r_h^2}\, dt = 0 \tag{1.7}$$

for $l=1, 2, \ldots, 2n-1$. By the mean-value theorem, we then have

$$\begin{aligned} 0 &= \int_{\phi_1{}^l}^{\phi_2{}^l} \left(\sum_{h=1}^{P+1} A_h \frac{1-r_h^2}{1-2r_h\cos(\theta_h - t) + r_h^2}\right) dt \\ &= (\phi_2^l - \phi_1^l) \sum_{h=1}^{P+1} A_h \frac{1-r_h^2}{1-2r_h\cos(\theta_h - \beta_l) + r_h^2}, \end{aligned} \tag{1.8}$$

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where $\phi_1^l < \beta_l < \phi_2^l$, for $1 \leq l \leq 2n-1$.

Thus, the function

$$T(\beta) = \sum_{h=1}^{P+1} A_h \frac{1-r_h^2}{1-2r_h\cos(\theta_h-\beta)+r_h^2}, \tag{1.9}$$

has $2n-1$ *distinct* zeros in $(R_0, R_0+2\pi)$. From (1.9) we may write

$$T(\beta) = \frac{N(\beta)}{D(\beta)},$$

where $N(\beta)$ is a trigonometric polynomial in β, of degree at most P, and where $D(\beta)$ never vanishes. Since $T(\beta_l)=0$ for $l=1, 2, \ldots, 2n-1$, we must have $T(\beta)=0$ for all real β.

If $P(z)$ is a polynomial which vanishes at the points z_h, $1 \leq h \leq P$ and satisfies $P(z_{P+1})=1$, it is seen that

$$A_{P+1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} T(\beta)P(e^{i\beta})\,d\beta = 0$$

which is a contradiction. The conclusion of Theorem 1 follows.

Theorem 1 remains true if the arcs α_l are replaced by sets E_l of positive measure with $E_l \subset \alpha_l$, $1 \leq l \leq 2n-1$. In fact, the intermediate value property actually allows us to write

$$\int_{E_l}\left(\sum_{h=1}^{P+1} A_h \frac{1-r_h^2}{1-2r_h\cos(\theta_h-t)+r_h^2}\right)dt = m(E_l)\sum_{h=1}^{P+1} A_h \frac{1-r_h^2}{1-2r_h\cos(\theta_h-\beta_l)+r_h^2},$$

where $\exp(i\beta_l) \in \alpha_l$, $1 \leq l \leq 2n-1$, in place of (1.8).

2. **Multiply connected regions.** The proof of Theorem 1 may be extended to give an analogous result for finitely connected regions.

THEOREM 2. *Let Ω be a region in the complex plane whose boundary consists of mutually disjoint Jordan curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{2(n+m)-3}$, $(n \geq 1)$ be mutually disjoint open connected subsets of $\Gamma = \sum_{P=1}^{m} \Gamma_P$ and let $\omega(z; \alpha_l)$ be the harmonic measure of α_l with respect to Ω, $1 \leq l \leq 2(n+m)-3$. Then for any n distinct points $z_1, z_2, \ldots, z_n$ chosen from Ω and any set of n complex numbers $y_1, y_2, \ldots, y_n$, the system of equations*

$$\sum_{l=1}^{2(n+m)-3} x_l\omega(z_k; \alpha_l) = y_k, \qquad (1 \leq k \leq n) \tag{2.1}$$

possesses a solution.

Proof. It may be assumed that the curves $\Gamma_1, \ldots, \Gamma_m$ are analytic. With this assumption, it suffices to show, as before, that the matrix

$$\left(\int_{\alpha_l} \frac{\partial G}{\partial \eta}(z_k; t)\,|dt|\right), \qquad 1 \leq k \leq n, \tag{2.2}$$

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$1 \le l \le 2(n+m)-3$ has n linearly independent columns. In (2.2), $G(z_k; t)$ is the Green's function for Ω with pole at z_k and $\partial/\partial\eta$ denotes the derivative with respect to the exterior normal to Ω.

For $t \in \Gamma$ define

$$P(t) = \sum_{k=1}^{m} A_k \frac{\partial G}{\partial \eta}(z_k; t); \tag{2.3}$$

we need only show that $P(t)$ may have at most $2(n+m)-4$ zeros on Γ if it is not identically zero on Γ. Form the Riemann surface R which is the Schotky double of Ω, [1]. Topologically, R is two copies of Ω with corresponding boundary points identified, so R has genus $g=m-1$. Each function $G(z_k; t)$, may be extended harmonically to all of R. The function

$$H(t) = \left(\sum_{k=1}^{n} A_k G(z_k; t)\right), \tag{2.4}$$

is then harmonic on all of R except for at most $2n$ logarithmic singularities. The differential $dH=((\partial H/\partial x)-i\partial H/\partial y)\,dt$ therefore has at most $2(m+n)-4$ zeros on the whole surface R if H is not constantly zero on R. If $t_0 \in \Gamma$, and $P(t_0)=0$, then $\partial H(t_0)/\partial x - i\partial H(t_0)/\partial y=0$ since the tangential derivative (with respect to Γ) of $H(t)$ is identically zero on Γ. Thus $P(t)$ may have at most $2(m+n)-4$ zeros on Γ if H is not identically zero. If H is identically zero, each A_k must be zero, which is made clear by allowing t to approach z_k. With this information, the proof of Theorem 2 is complete.

Theorem 2 may stand improvement in the case of high connectivity even if the trivial instance of one interpolation point is disregarded. However, Theorem 2 is best possible for two points of interpolation and arbitrary m. We only describe an example for $m=n=2$; for larger m, the pictures will then be clear. For Ω, take an annulus bounded by circles of radius $R>1$ and $1/R$ and choose points $z_1=1$ and $z_2=-1$. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the upper and lower semi-circles of $|z|=R$, $|z|=1/R$ respectively. It is then clear that $\omega(z_k; \alpha_l)=c$ for $k=1, 2$ and $l=1, 2, 3, 4$. Thus the five arcs called for by Theorem 2 are necessary for unrestricted interpolation.

It should be mentioned that Walsh [3, 4] has studied the location of critical points of sums such as (1.2) and (2.4). It is evident that existing and further results in this area may be readily applied to the problem considered here. The author would thank John Fay and Kurt Strebel for helpful conversations on these matters, and the National Research Council of Canada for support at Carleton University in the Summer of 1973.

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