BULL. AUSTRAL. MATH. SOC. 52A21, 46B04
VOL. 51 (1994) [87-101]

# ON THE AVERAGE DISTANCE PROPERTY IN FINITE DIMENSIONAL REAL BANACH SPACES

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The average distance Theorem of Gross implies that for each $N$-dimensional real Banach space $E$ ($N \geqslant 2$) there is a unique positive real number $r(E)$ with the following property: for each positive integer $n$ and for all (not necessarily distinct) $x_1, x_2, \ldots, x_n$ in $E$ with $\|x_1\| = \|x_2\| = \ldots = \|x_n\| = 1$, there exists an $x$ in $E$ with $\|x\| = 1$ such that

$$\frac{1}{n}\sum_{i=1}^{n}\|x_i - x\| = r(E).$$

In this paper we prove that if $E$ has a 1-unconditional basis then $r(E) \leqslant 2-(1/N)$ and equality holds if and only if $E$ is isometrically isomorphic to $\mathbb{R}^n$ equipped with the usual 1-norm.

## 1. INTRODUCTION

In 1964 Gross published the following surprising result:

**THEOREM.** *Let $(X, d)$ be a compact connected metric space. Then there is a unique positive real number $r(X, d)$ with the following property: for each positive integer $n$ and for all (not necessarily distinct) $x_1, x_2, \ldots, x_n$ in $X$, there exists an $x$ in $X$ such that*

$$\frac{1}{n}\sum_{i=1}^{n} d(x_i, x) = r(X, d).$$

For a proof of this Theorem see [2]. A survey of contributions to this topic is given in [1].

REMARK 1.

(a) In the situation of Gross's Theorem we say that $(X, d)$ has the average distance property with rendezvous number $r(X, d)$.

(b) Graham Elton first generalised Gross's Theorem in the following sense (for a proof see [1]):
Let $(X, d)$ be a compact connected metric space and $M^1(X)$ be the set of

Received 28 March 1994

all regular Borel probability measures on $X$, then $r(X, d)$ is the unique positive real number with the following property: for each $\mu \in M^1(X)$ there exists an $x$ in $X$ such that

$$\int_X d(x, y)\, d\mu(y) = r(X, d).$$

Moreover there are $\mu$, $\nu$ in $M^1(X)$ with

$$\int_X d(x, y)\, d\nu(y) \leqslant r(X, d) \leqslant \int_X d(x, y)\, d\mu(y),$$

for all $x$ in $X$.

(c) $(D(X))/2 \leqslant r(X, d) < D(X)$, with $D(X)$ the diameter of $X$. For a proof see Theorem 2 in [2].

## 2. Basic definitions and notation

For a real Banach space $E$ let $S = \{x \in E \mid \|x\| = 1\}$ denote the unit sphere of $E$. For $n \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$ let $\ell^p(n)$ denote $\mathbb{R}^n$ with the usual $p$-norm.

Recall that an $n$-dimensional real Banach space $E$ has a 1-unconditional basis $a_1, \ldots, a_n$ in $E$ if

$$\left\| \sum_{i=1}^n \alpha_i a_i \right\| = \left\| \sum_{i=1}^n |\alpha_i|\, a_i \right\|, \quad \text{for all } \alpha_1, \ldots, \alpha_n \text{ in } \mathbb{R}.$$

This is equivalent to

$$\left\| \sum_{i=1}^n \alpha_i a_i \right\| \leqslant \left\| \sum_{i=1}^n \beta_i a_i \right\|,$$

for all $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$ in $\mathbb{R}$ with $|\alpha_i| \leqslant |\beta_i|$ for all $i = 1, 2, \ldots, n$.

It is easy to see $a_1, \ldots, a_n$ is a 1-unconditional basis of $E$ if and only if $(a_1)/(\|a_1\|), \ldots, (a_n)/(\|a_n\|)$ is a 1-unconditional basis of $E$.

Simple arguments show that if $a_1, \ldots, a_n$ is a 1-unconditional basis of $E$, then its dual basis $f_1, \ldots, f_n \in E'$ $\left(f_i(a_j) = \delta^i_j\right)$ is a 1-unconditional basis of $E'$, the dual space of $E$ and moreover both of them are Auerbach bases:

$$\max_i |\alpha_i| \leqslant \left\| \sum_{i=1}^n \alpha_i a_i \right\| \leqslant \sum_{i=1}^n |\alpha_i|, \; \max_i |\beta_i| \leqslant \left\| \sum_{i=1}^n \beta_i f_i \right\| \leqslant \sum_{i=1}^n |\beta_i|,$$

for all $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$ in $\mathbb{R}$.

For $x = \sum_{i=1}^{n} \alpha_i a_i$ in $E$ and $f = \sum_{i=1}^{n} \beta_i f_i$ in $E'$ we simply write $x = (\alpha_1, \ldots, \alpha_n)$ and $f = [\beta_1, \ldots, \beta_n]$.

In [4] it is said that a real Banach space $E$ of arbitrary dimension has the average distance property with rendezvous number $r(E)$ if Gross's Theorem holds for the unit sphere $S$ of $E$ equipped with the norm induced metric:
There is a unique positive real number (called $r(E)$) such that: for each $n$ in $\mathbb{N}$ and for all (not necessarily distinct) $x_1, x_2, \ldots, x_n$ in $S$ there exists an $x$ in $S$ such that

$$\frac{1}{n}\sum_{i=1}^{n} \|x_i - x\| = r(E).$$

Remark 2. Each $n$-dimensional real Banach space ($n \geqslant 2$) has the average distance property, since in this case $S$ is compact and connected. For example in [3] Morris and Nickolas proved that

$$r\left(\ell^2(n)\right) = \frac{2^{n-1}\left[\Gamma\left(\frac{n}{2}\right)\right]^2}{\sqrt{\pi}\Gamma\left(\frac{2n-1}{2}\right)}, \quad \text{for all} \quad n \geqslant 2.$$

In [4] it is shown that

$$r\left(\ell^1(n)\right) = 2 - \frac{1}{n}, \quad r(\ell^\infty(n)) = \frac{3}{2}, \quad \text{for all } n \geqslant 2.$$

Looking at infinite dimensional real Banach spaces we have for example:
The Hilbert space $\ell^2$ of absolutely square summable real sequences has the average distance property with rendezvous number $\sqrt{2}\left(= \lim_{n\to\infty} r\left(\ell^2(n)\right)\right)$, and $\ell^1$ the space of all absolutely summable real sequences fails to have the desired property. (For a more detailed discussion see [4]).

## 3. The results

In [4] it is proved that $r(E) \leqslant 3/2$ for all 2-dimensional real Banach spaces $E$, and $r(E) = 3/2$ if and only if $E$ is isometrically isomorphic to $\ell^1(2)$.

Further it is conjectured that $r(E) \leqslant 2 - 1/n$ holds for all $n$-dimensional real Banach spaces $E$ with $n \geqslant 2$.

In this paper we give a proof of $r(E) \leqslant 2 - 1/n$ in the case when $E$ has a 1-unconditional basis. The proof is based on the following

**Theorem 1.** *Let $E$ be a real $n$-dimensional Banach space ($n \geqslant 2$) with a 1-unconditional basis $a_1, \ldots, a_n$ ($\|a_1\| = \ldots = \|a_n\| = 1$). Then we have*

$$\frac{1}{2n}\sum_{i=1}^{n} \|x - a_i\| + \|x + a_i\| \leqslant 1 + \frac{n-1}{n}\|x\|,$$

https://doi.org/10.1017/S0004972700013927 Published online by Cambridge University Press

*for all $x$ in $E$ with $\|x\| \leqslant 1$.*

From this we obtain

**THEOREM 2.** *Let $E$ be a real $n$-dimensional Banach space $(n \geqslant 2)$ with a 1-unconditional basis $a_1, \ldots, a_n$ $(\|a_1\| = \ldots = \|a_n\| = 1)$. Then we have*

$$r(E) \leqslant 2 - \frac{1}{n};$$

*moreover $r(E) = 2 - 1/n$ if and only if $E$ is isometrically isomorphic to $\ell^1(n)$.*

REMARK 3. (a) The inequality established in Theorem 1 is sharp in the following sense: For each $E$ there is at least one $x$ (for example $x = 0$), such that equality holds. It is easy to check that for $E = \ell^1(n)$ equality holds for all $x$ with $\|x\| \leqslant 1$.

Furthermore in general the assumption $a_1, \ldots, a_n$ is a 1-unconditional basis of $E$ cannot be replaced by a weaker condition, for example $a_1, \ldots, a_n$ being an Auerbach basis of $E$:
Let $E = \ell^\infty(3)$, $a_1 = (-1, 1, 1)$, $a_2 = (1, -1, 1)$, $a_3 = (1, 1, -1)$.

It is easy to see that $a_1, a_2, a_3$ forms an Auerbach basis of $\ell^\infty(3)$ and

$$\frac{1}{6}\sum_{i=1}^{3} \|x - a_i\| + \|x + a_i\| = 2 > 1 + \frac{2}{3}.1 \text{ for } x = (1, 1, 1).$$

(b) Since the proof of Theorem 2 is based on the proof of Theorem 1, the condition that $E$ has a 1-unconditional basis is rather more technical than essential for obtaining the upper bound $r(E) \leqslant 2 - 1/n$. So the question remains:
Is it true that

$$r(E) \leqslant 2 - \frac{1}{n},$$

for all $n$-dimensional real Banach spaces $E$ $(n \geqslant 2)$?

## 4. THE PROOFS

The following Lemma collects some simple consequences of $E$ having a 1-unconditional basis:

**LEMMA 1.** *Let $E$ be an $n$-dimensional real Banach space with a 1-unconditional basis $a_1, \ldots, a_n$ $(\|a_1\| = \ldots = \|a_n\| = 1)$. Then we have*

(1) *If there is an $x = (\lambda_1, \ldots, \lambda_n)$ in $E$ such that $\lambda_i \neq 0$ for all $i = 1, 2, \ldots, n$ and $\|x\| = |\lambda_1| + \ldots + |\lambda_n|$, then we have*

$$\|(\alpha_1, \ldots, \alpha_n)\| = |\alpha_1| + \ldots + |\alpha_n|,$$

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*for all* $\alpha_1, \ldots, \alpha_n$ *in* $\mathbb{R}$.

(2) *If* $\|(1, 1, \ldots, 1)\| = 1$, *we have*

$$\|(\alpha_1, \ldots, \alpha_n)\| = \max_i |\alpha_i|,$$

*for all* $\alpha_1, \ldots, \alpha_n$ *in* $\mathbb{R}$.

(3) *Let* $n = 3$ *and* $\|(1, 1, 0)\| = \|(0, 1, 1)\| = 1$ *and* $\|(1, 1, 1)\| = 2$. *Then we have*

$$\|(\alpha_1, \alpha_2, \alpha_3)\| = \max(|\alpha_1| + |\alpha_3|, |\alpha_2|),$$

*for all* $\alpha_1, \alpha_2, \alpha_3$ *in* $\mathbb{R}$.

(4) *Let* $x = (\alpha_1, \ldots, \alpha_n)$ *in* $E$ *such that* $0 \leqslant \alpha_i \leqslant 1/2$ *for some* $1 \leqslant i \leqslant n$. *Then we have*

$$\|x - a_i\| \leqslant 1 + \|x\| - 2\alpha_i.$$

PROOF: (1) By assumption and the Hahn-Banach Theorem we get $\|[1, 1, \ldots, 1]\| = 1$ $([1, 1, \ldots, 1] \in E')$. Therefore we have $|\alpha_1| + \ldots + |\alpha_n| \leqslant \|(\alpha_1, \ldots, \alpha_n)\| \leqslant |\alpha_1| + \ldots + |\alpha_n|$, for all $\alpha_1, \ldots, \alpha_n$ in $\mathbb{R}$.

(2) $\max_i |\alpha_i| \leqslant \|(\alpha_1, \ldots, \alpha_n)\| \leqslant \max_i |\alpha_i| \, \|(1, 1, \ldots, 1)\| = \max_i |\alpha_i|$, for all $\alpha_1, \ldots, \alpha_n$ in $\mathbb{R}$.

(3) From $\|(1, 1, 0)\| = \|(0, 1, 1)\| = 1$ it follows that $\beta_1 + \beta_2 \leqslant 1$ and $\beta_2 + \beta_3 \leqslant 1$ for all $[\beta_1, \beta_2, \beta_3] \in E'$ with $\|[\beta_1, \beta_2, \beta_3]\| = 1$. By the Hahn-Banach Theorem there is a $f \in E'$, $\|f\| = 1$ such that $f((1, 1, 1)) = 2$. Therefore we have $f = [1, 0, 1]$. So $\|(\alpha_1, 0, \alpha_3)\| = |\alpha_1| + |\alpha_3|$ for all $\alpha_1, \alpha_3$ in $\mathbb{R}$. From this and part (2) it remains to show that

$$\|(\alpha_1, \alpha_2, \alpha_3)\| = \max(\alpha_1 + \alpha_3, \alpha_2) \text{ for all } \alpha_1, \alpha_2, \alpha_3 > 0.$$

If $\alpha_1 + \alpha_3 \geqslant \alpha_2$ we get $\|(\alpha_1, \alpha_2, \alpha_3)\| \leqslant \|(\alpha_1, \alpha_1, 0)\| + \|(0, \alpha_3, \alpha_3)\| = \alpha_1 + \alpha_3$ by part (2).

If $\alpha_1 + \alpha_3 < \alpha_2$ we get $\|(\alpha_1, \alpha_2, \alpha_3)\| \leqslant \alpha_2 \|((\alpha_1)/(\alpha_1 + \alpha_3), (\alpha_1)/(\alpha_1 + \alpha_3), 0)\| + \alpha_2 \|(0, (\alpha_3)/(\alpha_1 + \alpha_3), (\alpha_3)/(\alpha_1 + \alpha_3))\| = \alpha_2$ by part (2).

On the other hand $\|(\alpha_1, \alpha_2, \alpha_3)\| \geqslant \|(\alpha_1, 0, \alpha_3)\| = \alpha_1 + \alpha_3$ and of course $\|(\alpha_1, \alpha_2, \alpha_3)\| \geqslant \alpha_2$.

(4) $\|x - a_i\| = \|(\alpha_1, \ldots, 1 - \alpha_i, \ldots, \alpha_n)\| \leqslant (1 - 2\alpha_i) \|a_i\| + \|x\| = 1 + \|x\| - 2\alpha_i$. ∎

PROOF OF THEOREM 1: Let $f(x) = (1/(2n)) \sum_{i=1}^{n} \|x - a_i\| + \|x + a_i\|$ for all $x$ in $E$. It is easy to see that $f((\alpha_1, \ldots, \alpha_n)) = f((|\alpha_1|, \ldots, |\alpha_n|))$ for all $\alpha_1, \ldots, \alpha_n$ in $\mathbb{R}$.

https://doi.org/10.1017/S0004972700013927 Published online by Cambridge University Press

Since $\|x - a_i\|$, $\|x + a_i\| \leqslant 1 + \|x\|$ for all $x$ in $E$ and all $i = 1, 2, \ldots, n$, it remains to show that $S((\alpha_1, \ldots, \alpha_n)) \leqslant n + (n-2)\|x\|$ for $x = (\alpha_1, \ldots, \alpha_n)$ in $E$ with $\|x\| \leqslant 1$ and $\alpha_1, \ldots, \alpha_n \geqslant 0$, where $S((\alpha_1, \ldots, \alpha_n))$ is defined as $S((\alpha_1, \ldots, \alpha_n)) = \|(1-\alpha_1, \alpha_2, \ldots, \alpha_n)\| + \ldots + \|(\alpha_1, \alpha_2, \ldots, 1-\alpha_n)\|$.

Now let $x = (\alpha_1, \ldots, \alpha_n)$ in $E$, $\|x\| \leqslant 1$ and $\alpha_1, \ldots, \alpha_n \geqslant 0$. Note that $\|x\| \leqslant 1$ implies $\alpha_1, \ldots, \alpha_n \leqslant 1$.

Let $\tau = \alpha_1 + \ldots + \alpha_n$ and $s = S((\alpha_1, \ldots, \alpha_n))$. We consider five cases:

(1) $\tau = \|x\|$

By the triangle inequality we get $s \leqslant n(1+\tau) - 2\tau = n + (n-2)\|x\|$.

(2) $\tau > \|x\|$

(a) There are at least three coordinates of $x$ greater or equal to $1/2$. Without loss of generality let $\alpha_1, \alpha_2, \alpha_3 \geqslant 1/2$.

Then we have $\|x - a_i\| \leqslant \|x\|$ since $1 - \alpha_i \leqslant \alpha_i$ for $i = 1, 2, 3$. So we get $s \leqslant 3\|x\| + (n-3)(1 + \|x\|) \leqslant n - 1 + (n-2)\|x\|$, since $\|x\| \leqslant 1$. Therefore $s < n + (n-2)\|x\|$.

(b) Without loss of generality let $\alpha_1, \alpha_2 \geqslant 1/2$ and $\alpha_3, \ldots, \alpha_n < 1/2$.

Since $(1-\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{i=2}^{n}(2\alpha_i - (\alpha_i)/(\alpha_1))a_i + (1/(\alpha_1) - 1)x$ we get $\|(1-\alpha_1, \alpha_2, \ldots, \alpha_n)\| \leqslant \sum_{i=2}^{n} \alpha_i(2 - 1/(\alpha_1)) + (1/(\alpha_1) - 1)\|x\|$ and therefore

$$\|(1-\alpha_1, \alpha_2, \ldots, \alpha_n)\| \leqslant 2(\tau - \alpha_1) - \frac{\tau - \alpha_1}{\alpha_1} + \left(\frac{1}{\alpha_1} - 1\right)\|x\|.$$

The same argument leads to

$$\|(\alpha_1, 1-\alpha_2, \ldots, \alpha_n)\| \leqslant 2(\tau - \alpha_2) - \frac{\tau - \alpha_2}{\alpha_2} + \left(\frac{1}{\alpha_2} - 1\right)\|x\|.$$

Hence we get

$$s \leqslant 4\tau - 2(\alpha_1 + \alpha_2) - \frac{\tau - \alpha_1}{\alpha_1} - \frac{\tau - \alpha_2}{\alpha_2} + \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} - 2\right)\|x\| + \sum_{i=3}^{n} 1 + \|x\| - 2\alpha_i$$

by Lemma 1 part (4). Therefore

$$\begin{aligned} s &\leqslant 2\tau + (n-4)\|x\| + n - (\tau - \|x\|)\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) \\ &\leqslant 2\tau + (n-4)\|x\| + n - (\tau - \|x\|)2 = n + (n-2)\|x\|. \end{aligned}$$

(c) Without loss of generality let $\alpha_1 \geqslant 1/2$ and $\alpha_2, \ldots, \alpha_n < 1/2$.

The proof of case (2)(b) shows that

$$\|(1-\alpha_1, \alpha_2, \ldots, \alpha_n)\| \leqslant 2(\tau-\alpha_1) - \frac{\tau-\alpha_1}{\alpha_1} + \left(\frac{1}{\alpha_1}-1\right)\|x\|.$$

So by Lemma 1 part (4) we get

$$\begin{aligned} s &\leqslant 2(\tau-\alpha_1) - \frac{\tau-\alpha_1}{\alpha_1} + \left(\frac{1}{\alpha_1}-1\right)\|x\| + \sum_{i=2}^{n} 1 + \|x\| - 2\alpha_i \\ &= n + (n-2)\|x\| - (\tau - \|x\|)\frac{1}{\alpha_1} < n + (n-2)\|x\|. \end{aligned}$$

(d) $\alpha_1, \ldots, \alpha_n < 1/2$.
By Lemma 1 part (4) we get

$$s \leqslant (1+\|x\|)n - 2\tau < n + (n-2)\|x\|.$$

□

Now let $x = (\alpha_1, \ldots, \alpha_n)$ in $S$. We say that $x$ is of Type I if $|\alpha_1| + \ldots + |\alpha_n| = 1$. If there are $i_1$, $i_2$ in $\{1, 2, \ldots, n\}$, such that $i_1 \neq i_2$, $|\alpha_{i_1}| = |\alpha_{i_2}| = 1$ and $\alpha_i = 0$ for all $i$ in $\{1, 2, \ldots, n\} \setminus \{i_1, i_2\}$, we say that $x$ is of Type II.

Furthermore $x = (\alpha_1, \ldots, \alpha_n)$ is a typical element of Type I if $\alpha_1, \ldots, \alpha_k > 0$, $\alpha_{k+1} = \ldots = \alpha_n = 0$ for some $1 \leqslant k \leqslant n$ and $\alpha_1 + \ldots + \alpha_n = 1$.

A typical element of Type II is the vector $(1, 1, 0, \ldots, 0)$. We formulate the second part of the following Lemma for typical elements of Type I and Type II. By renumbering the indices and changing the signs of the coordinates you get the analogous results for arbitrary elements of Type I and Type II.

**Lemma 2.** *Let $f$ be defined as in the proof of Theorem 1 and let $A = \{x \in S \mid f(x) = 2 - 1/n\}$. Then we have*

(1) *$x$ in $A$ implies $x$ is of Type I or Type II.*
(2) *$x$ in $A$ is of Type I implies*

$$\|(\beta_1, \ldots, \beta_k, 0, \ldots, 0)\| = \sum_{i=1}^{k} |\beta_i|$$

*and*

$$\|(\beta_1, \ldots, \beta_k, 0, \ldots, 0, \beta_j, 0, \ldots, 0)\| = \sum_{i=1}^{k} |\beta_i| + |\beta_j|$$

https://doi.org/10.1017/S0004972700013927 Published online by Cambridge University Press

*for all* $\beta_1, \ldots, \beta_k, \beta_j$ *in* $\mathbb{R}$ *and all* $k+1 \leqslant j \leqslant n$, *for* $x = (\alpha_1, \ldots, \alpha_k, 0, \ldots, 0)$ *a typical element of Type I.*

(3) $x$ *in* $A$ *and* $x$ *is of Type II implies*

$$\|(\beta_1, \beta_2, 0, \ldots, 0)\| = \max(|\beta_1|, |\beta_2|)$$

*for all* $\beta_1$, $\beta_2$ *in* $\mathbb{R}$ *and*

$$\|(1, 1, 1, 0, \ldots, 0)\| = \ldots = \|(1, 1, 0, \ldots, 0, 1)\| = 2,$$

*for* $x = (1, 1, 0, \ldots, 0)$ *a typical element of Type II.*

PROOF: Let $x = (\alpha_1, \ldots, \alpha_n)$ in $A$. Since $f((\alpha_1, \ldots, \alpha_n)) = f((|\alpha_1|, \ldots, |\alpha_n|))$ for all $\alpha_1, \ldots, \alpha_n$ in $\mathbb{R}$ we can assume without loss of generality that $\alpha_1, \ldots, \alpha_n \geqslant 0$.

(1) Note that Theorem 1 implies $f(y) \leqslant 2 - 1/n$ for all $y$ in $S$. A detailed look at the proof shows that equality on $S$ is attained only in case (1) and case (2)(b). Case (1) leads to the fact that $x$ is of Type I. The estimates in case (2)(b) lead to $\alpha_1 = \alpha_2 = 1$. It remains to show that $\alpha_3 = \ldots = \alpha_n = 0$.

The proof of Theorem 1 and $x$ in $A$ imply $\sum_{i=1}^{n} \|x - a_i\| = 2n - 2$. Since $\|x - a_1\|$, $\|x - a_2\| \leqslant 1$ we get $\|x - a_1\| = \|x - a_2\| = 1$ and $\|x - a_i\| = 2$ for all $i = 3, \ldots, n$. Lemma 1 part (4) implies $\|x - a_i\| \leqslant 2 - 2\alpha_i$ for all $i = 3, \ldots, n$. Therefore $\alpha_3 = \ldots = \alpha_n = 0$.

(2) The proof of Theorem 1 and $x$ in $A$ again imply $\sum_{i=1}^{n} \|x - a_i\| = 2n - 2$ and $\|x + a_i\| = 2$ for all $i = 1, 2, \ldots, n$. The assumptions on $x$ and Lemma 1 part (1) verify the assertions.

(3) Since $x = (1, 1, 0, \ldots, 0)$ in $S$ and $\|x + a_i\| = 2$ for $i = 1, 2, \ldots, 3$ ($x$ in $A$ and the proof of Theorem 1 once again) we are done by Lemma 1 part (2). □

PROOF OF THEOREM 2: Let $f$ be defined as in the proof of Theorem 1. By Gross's Theorem there is an $x$ in $S$ such that $f(x) = r(E)$. By Theorem 1 we have $f(x) \leqslant 2 - 1/n$ and therefore $r(E) \leqslant 2 - 1/n$. It is easy to check that for $E = \ell^1(n)$ we have $f(x) = 1 + ((n-1)/n)\|x\|$ for all $x$ in $\ell^1(n)$ with $\|x\| \leqslant 1$. Hence we get $f(x) = 2 - 1/n$ for all $x$ in $S$ and by Gross's Theorem $r(\ell^1(n)) = 2 - 1/n$.

Now let $E$ be an arbitrary $n$-dimensional real Banach space with a 1-unconditional basis $a_1, \ldots, a_n$ in $S$ and $r(E) = 2 - 1/n$. It remains to show that $E$ is isometrically isomorphic to $\ell^1(n)$. In [4] it is shown that $r(E) = 3/2$ implies that $E$ is isometrically isomorphic to $\ell^1(2)$ for all 2-dimensional real Banach spaces $E$. So we can assume that $n \geqslant 3$.

By Remark 1 part (b) there is a regular Borel probability measure $\mu$ on $S$ such that

$$\int_S \|x - y\| \, d\mu(y) \geqslant 2 - \frac{1}{n} \text{ for all } x \text{ in } S.$$

By definition of $f$ we get

$$\int_S f(y)d\mu(y) \geqslant 2 - \frac{1}{n}.$$

As in Lemma 2 let $A = \{y \in S \mid f(y) = 2 - 1/n\}$. By Theorem 1 we have $f(y) \leqslant 2 - 1/n$ for all $y$ in $S$, and therefore we get $\mu(A) = 1$. Lemma 2 part (1) quaranteed that $A = B \cup C$ where $B$ consists of Type I elements of $A$ and $C$ consists of Type II elements of $A$. Of course we have $B \cap C = \emptyset$.

CASE 1. $\mu(C) = 0$.

Take some $\varepsilon > 0$ such that $\|(\varepsilon, \dots, \varepsilon)\| = 1$ and let $z = (\varepsilon, \dots, \varepsilon)$. Furthermore let

$$g(y) = \frac{\|z - y\| + \|z + y\|}{2} \quad \text{for all } y \text{ in } S.$$

If $\varepsilon \geqslant 1/2$ it follows that $\|z - a_i\| \leqslant \|z\| = 1$ for all $i = 1, 2, \dots, n$. So we have $g(a_i)$, $g(-a_i) \leqslant 3/2$ for all $i = 1, 2, \dots, n$. Since $g$ is a convex function and Type I elements are included in the convex hull of $a_1, -a_1, \dots, a_n, -a_n$ we have $g(b) \leqslant 3/2$ for all $b$ in $B$. Since

$$\int_B g(y)d\mu(y) \geqslant 2 - \frac{1}{n}$$

we get a contradiction to $n \geqslant 3$. So it follows that

$$\varepsilon < \frac{1}{2}.$$

By Lemma 1 part (4) we have $\|z - a_i\| \leqslant 2 - 2\varepsilon$ for all $i = 1, 2, \dots, n$ and therefore $g(b) \leqslant 2 - \varepsilon$ for all $b$ in $B$. Hence $\varepsilon \leqslant 1/n$. Since $1 = \|z\| \leqslant n\varepsilon$, we get $\varepsilon = 1/n$. Now Lemma 1 part (1) quarantees that $E$ is isometrically isomorphic to $\ell^1(n)$.

CASE 2. $\mu(C) > 0$.

Assume that there are two elements $c_1 = (\alpha_1, \dots, \alpha_n)$ and $c_2 = (\beta_1, \dots, \beta_n)$ in $C$, such that there are $i_1$, $i_2$, $i_3$ in $\{1, 2, \dots, n\}$ with $|\alpha_{i_1}| = |\beta_{i_1}| = 1$, $|\alpha_{i_2}| = |\beta_{i_3}| = 1$, $|\alpha_{i_3}| = |\beta_{i_2}| = 0$ and $|\alpha_i| = |\beta_i| = 0$ for all $i$ in $\{1, 2, \dots, n\} \setminus \{i_1, i_2, i_3\}$. Without loss of generality let $c_1 = (1, 1, 0, \dots, 0)$ and $c_2 = (0, 1, 1, 0, \dots, 0)$. Lemma 2 part (3) and Lemma 1 part (3) imply that

$$\|(\alpha_1, \alpha_2, \alpha_3, 0, \dots, 0)\| = \max(|\alpha_1| + |\alpha_3|, |\alpha_2|)$$

for all $\alpha_1, \alpha_2, \alpha_3$ in $\mathbb{R}$.

Now define $d_1 = (1/2, 0, 1/2, 0, \dots, 0)$, $d_2 = a_2$, $d_3 = (-1/2, 0, 1/2, 0, \dots, 0)$ and $d_i = a_i$ for all $i = 4, \dots, n$.

Furthermore let

$$h(y) = \frac{1}{2n}\sum_{i=1}^{n} \|y - d_i\| + \|y + d_i\| \text{ for all } y \text{ in } S.$$

Note that $h((\alpha_1, \dots, \alpha_n)) = h((|\alpha_1|, \dots, |\alpha_n|))$ for all $\alpha_1, \dots, \alpha_n$ in $\mathbb{R}$.

Easy calculations show that $h(a_1)$, $h(a_2)$, $h(a_3) \leqslant 2 - 2/n$ and $h(a_i) \leqslant 2 - 1/n$ for all $i = 4, 5, \dots, n$. Therefore we get $h(b) \leqslant 2 - 1/n$ for all $b$ in $B$. Moreover it follows immediately that $h(c_1)$, $h(c_2) \leqslant 2 - 3/2n$. It is easy to check that $h(c) \leqslant 2 - 1/n$ for all $c$ in $C$. (Note that $C$ is finite and $(1, 0, 1, 0, \dots, 0)$ is not in $C$ since $\|(1, 0, 1, 0, \dots, 0)\| = 2$.)

For example let

$$c = (1, 0, 0, 1, 0, \dots, 0):$$
$$\|c - d_1\| = \left\|\left(\frac{1}{2}, 0, \frac{1}{2}, 1, 0, \dots, 0\right)\right\| \leqslant \left\|\left(\frac{1}{2}, 0, 0, 1, 0, \dots, 0\right)\right\| + \frac{1}{2} = \frac{3}{2}$$

by Lemma 2 part (3).

An analogous estimation leads to $\|c + d_3\| \leqslant 3/2$, so we have

$$h(c) \leqslant \frac{1}{2n}\left(\frac{3}{2} + 2 + 2 + 2 + 2 + \frac{3}{2} + 1 + 2 + 4(n-4)\right) = 2 - \frac{1}{n}.$$

Summing up we have

$$h(a) \leqslant 2 - \frac{1}{n} \text{ for all } a \text{ in } A$$

and

$$h((\sigma_1, \sigma_2, 0, \dots, 0)),\ h((0, \sigma_1, \sigma_2, 0, \dots, 0)) \leqslant 2 - \frac{3}{2n}$$

for all $|\sigma_1| = |\sigma_2| = 1$.

Since $d_1, \dots, d_n$ in $S$ we get

$$\int_A h(y)d\mu(y) \geqslant 2 - \frac{1}{n},$$

and therefore $\mu(\{(\sigma_1, \sigma_2, 0, \dots, 0), (0, \sigma_1, \sigma_2, 0, \dots, 0), |\sigma_1| = |\sigma_2| = 1\}) = 0$.

Therefore, and since Lemma 2 part (2), (3) quarantees that each $b$ in $B$ and $c$ in $C$ have no coordinate unequal to zero in common, we can assume without loss of generality that

$$C = \{\sigma_1 a_1 + \sigma_2 a_2, \sigma_1 a_3 + \sigma_2 a_4, \dots, \sigma_1 a_{2k-1} + \sigma_2 a_{2k}, |\sigma_1| = |\sigma_2| = 1\}$$

for some $1 \leqslant k \leqslant n/2$ and that $B$ is included in the subspace generated by $a_{2k+1}, \ldots, a_n$.

For convenience let $x_1 = a_1$, $y_1 = a_2, \ldots, x_k = a_{2k-1}$, $y_k = a_{2k}$, $z_1 = a_{2k+1}, \ldots, z_s = a_n$; $s = n - 2k$. Furthermore let $E_1$ be the subspace generated by $x_1, y_1, \ldots, x_k, y_k$ and $E_2$ be the subspace generated by $z_1, \ldots, z_s$ and

$$S_1 = \{x \in E_1 \mid \|x\| = 1\}, \quad S_2 = \{x \in E_2 \mid \|x\| = 1\}.$$

Since $C = A \cap S_1$ and $B = A \cap S_2$ we have $\mu(C) = \mu(S_1)$ and $\mu(B) = \mu(S_2)$.

Our next aim is to show that $\mu(C) = (2k)/n$. We consider two cases:

(i) $\mu(B) = 0$.

Since $1 = \mu(A) = \mu(C) = \mu(S_1)$ we get

$$\int_{S_1} \|x - y\| \, d\mu(y) \geqslant 2 - \frac{1}{n} \text{ for all } x \text{ in } S.$$

By Remark 1 part (b) there is some $x$ in $S_1$ such that

$$\int_{S_1} \|x - y\| \, d\mu(y) = r(E_1).$$

Now Theorem 1 implies $r(E_1) \leqslant 2 - 1/(2k)$, and therefore we get $n = 2k$. Hence $\mu(C) = \mu(A) = 1 = (2k)/n$.

(ii) $\mu(B) > 0$.

$$\int_A \|x - y\| \, d\mu(y) \geqslant 2 - \frac{1}{n} \text{ for all } x \text{ in } S$$

implies

$$\mu(C) \int_{S_1} \|x - y\| \, d\frac{\mu}{\mu(C)}(y) + \mu(B) \int_{S_2} \|x - y\| \, d\frac{\mu}{\mu(B)}(y) \geqslant 2 - \frac{1}{n}$$

for all $x$ in $S$.

By Remark 1 part (b) and Theorem 1 we get some $x_1$ in $S_1$ such that

$$\int_{S_1} \|x_1 - y\| \, d\frac{\mu}{\mu(C)}(y) = r(E_1) \leqslant 2 - \frac{1}{2k}.$$

The same argument leads to some $x_2$ in $S_2$ such that

$$\int_{S_2} \|x_2 - y\| \, d\frac{\mu}{\mu(B)}(y) = r(E_2) \leqslant 2 - \frac{1}{s}, \quad \text{if } s \geqslant 2.$$

If $s = 1$ we have

$$\min\left(\int_{S_2} \|z_1 - y\| \, d\frac{\mu}{\mu(B)}(y),\ \int_{S_2} \|z_1 + y\| \, d\frac{\mu}{\mu(B)}(y)\right) \leqslant 1 = 2 - \frac{1}{s},$$

since $S_2 = \{z_1, -z_1\}$ implies $\mu/(\mu(B)) = \lambda\delta_{z_1} + (1-\lambda)\delta_{-z_1}$ for some $0 \leqslant \lambda \leqslant 1$, where $\delta_x$ denotes the measure concentrated on $x$.

Now for $x = x_1$ and $x = x_2(z_1, -z_1)$ in

$$\int_A \|x - y\| \, d\mu(y)$$

we obtain $(2 - 1/(2k))\mu(C) + 2\mu(B)$ and $2\mu(C) + (2 - 1/s)\mu(B)$ are greater or equal to $2 - 1/n$. Since $1 = \mu(A) = \mu(B) + \mu(C)$ it follows immediately that

$$\mu(C) = \frac{2k}{n}.$$

Now assume that there is some $x$ in $S$ such that

$$\int_{S_1} \|x - y\| \, d\frac{\mu}{\mu(C)}(y) < 2 - \frac{1}{2k}.$$

Then we get

$$\left(2 - \frac{1}{2k}\right)\mu(C) + 2\mu(B) > 2 - \frac{1}{n},$$

which is a contradiction to

$$\mu(C) = \frac{2k}{n}\ (\mu(B) = \frac{s}{n},\ n = 2k + s).$$

Hence

$$\int_{S_1} \|x - y\| \, d\frac{\mu}{\mu(C)}(y) = \int_C \|x - y\| \, d\frac{\mu}{\mu(C)}(y) \geqslant 2 - \frac{1}{2k}$$

for all $x$ in $S$.

Since $C$ is finite, there are some $\lambda_c \geqslant 0$, $\sum_{c \in C} \lambda_c = 1$ such that

$$\sum_{c \in C} \lambda_c \|x - c\| = \int_C \|x - y\| \, d\frac{\mu}{\mu(C)}(y) \geqslant 2 - \frac{1}{2k} \text{ for all } x \text{ in } S.$$

Note that $\|c - c'\| = 2$ for all $c \neq c'$, $c$ and $c'$ in $C$ by Lemma 2 part (3). So by $\sum_{c \in C} \lambda_c \|c' - c\| \geqslant 2 - 1/(2k)$ for all $c'$ in $C$ and $|C| = 4k$, we obtain $\lambda_c = 1/(4k)$ for all $c$ in $C$.

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Summing up we get

$$\frac{1}{2n}\sum_{c\in C}\|x-c\|+\frac{s}{n}\int_B\|x-y\|\,d\mu(y)\geqslant 2-\frac{1}{n}\text{ for all } x \text{ in } S.$$

Let $\overline{C}=\{x_1+y_1, x_1-y_1, \ldots, x_k+y_k, x_k-y_k\}$ then we obtain formula (*):

$$(*)\qquad \frac{1}{2n}\sum_{c\in\overline{C}}\|x-c\|+\|x+c\|+\frac{s}{2n}\int_B\|x-y\|+\|x+y\|\,d\mu(y)\geqslant 2-\frac{1}{n}$$

for all $x$ in $S$.

Now let $V=\{\sigma=(\sigma_1,\ldots,\sigma_k), |\sigma_1|=\ldots=|\sigma_k|=1\}$ and identify $V$ with the set of all vertices of the $k$-dimensional cubic graph $Q_k$. Remember that two vertices of $Q_k$ are neighbours in $Q_k$ if and only if their coordinates differ in exactly one position. For each $\sigma$ in $V$ find some $\varepsilon_\sigma>0$, such that

$$x_\sigma=\varepsilon_\sigma\sum_{i=1}^{k}(\sigma_i+1)x_i-(\sigma_i-1)y_i+\varepsilon_\sigma\sum_{i=1}^{s}z_i$$

is in $S$. In the case $s=0$ leave the second sum. Since $\|x_\sigma\|=1$ we get $2\varepsilon_\sigma\leqslant 1\leqslant(2k+s)\varepsilon_\sigma$ and therefore $1/n\leqslant\varepsilon_\sigma\leqslant 1/2$.

Find $\sigma_0$ in $V$ such that $\min_{\sigma\in V}\varepsilon_\sigma=\varepsilon_{\sigma_0}$. Without loss of generaligy (transpose $x_i$ and $y_i$) we can assume that $\sigma_0=(1,1,\ldots,1)$.

Let $\sigma_1,\ldots,\sigma_k$ be the neighbours of $\sigma_0$. Since

$$\begin{aligned}(1-2\varepsilon_{\sigma_0}, 1, 2\varepsilon_{\sigma_0}, 0, \ldots, 2\varepsilon_{\sigma_0}, 0, \varepsilon_{\sigma_0}, \ldots, \varepsilon_{\sigma_0})\\ =(1-2\varepsilon_{\sigma_0})(x_1+y_1)+\frac{\varepsilon_{\sigma_0}}{\varepsilon_{\sigma_1}}x_{\sigma_1}\end{aligned}$$

we get

$$\|x_1+y_1-x_{\sigma_0}\|\leqslant 1-2\varepsilon_{\sigma_0}+\frac{\varepsilon_{\sigma_0}}{\varepsilon_{\sigma_1}}.$$

A similar argument leads to

$$\|x_i+y_i-x_{\sigma_0}\|\leqslant 1-2\varepsilon_{\sigma_0}+\frac{\varepsilon_{\sigma_0}}{\varepsilon_{\sigma_i}}$$

and

$$\|x_i-y_i-x_{\sigma_0}\|\leqslant 1-2\varepsilon_{\sigma_0}+\frac{\varepsilon_{\sigma_0}}{\varepsilon_{\sigma_i}}$$

for all $i=1,2,\ldots,k$.

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Since $\varepsilon_\sigma \leqslant 1/2$ for all $\sigma$ in $V$ we get by Lemma 1 part (4)

$$\|x_{\sigma_0} - z_i\| \leqslant 2 - 2\varepsilon_{\sigma_0}$$

for all $i = 1, 2, \dots, s$.

Now for $x = x_{\sigma_0}$ in formula (*) we obtain

$$2 - \frac{1}{n} \leqslant \frac{1}{2n}\sum_{i=1}^{k} 2\left(1 - 2\varepsilon_{\sigma_0} + \frac{\varepsilon_{\sigma_0}}{\varepsilon_{\sigma_i}} + 2\right) + \frac{s}{2n}(2 - 2\varepsilon_{\sigma_0} + 2).$$

Since $\varepsilon_{\sigma_i} \geqslant \varepsilon_{\sigma_0}$ for all $i = 1, 2, \dots, k$, we get

$$2 - \frac{1}{n} \leqslant 2 - \varepsilon_{\sigma_0} \leqslant 2 - \frac{1}{n} \quad \left(\varepsilon_\sigma \geqslant \frac{1}{n} \text{ for all } \sigma \text{ in } V\right).$$

Therefore we get

$$\frac{1}{n} = \varepsilon_{\sigma_0} = \varepsilon_{\sigma_i} \text{ for all } i = 1, 2, \dots, k.$$

Now repeat this calculation for $x = x_{\sigma_1}$ in formula (*). This leads to $1/n = \varepsilon_{\sigma_1} = \varepsilon_\sigma$ for all neighbours $\sigma$ of $\sigma_1$. Then for $x = x_\tau$ for some $\tau \neq \sigma_0$ a neighbour of $\sigma_1$ and so on we obtain $\varepsilon_\sigma = 1/n$ for all $\sigma$ in $V$.

By Lemma 1 part (1) we get

$$\left\|\sum_{i=1}^{k} \frac{\alpha_i}{2}(\sigma_i + 1)x_i - \frac{\beta_i}{2}(\sigma_i - 1)y_i + \sum_{i=1}^{s}\gamma_i z_i\right\| = \sum_{i=1}^{k} \frac{|\alpha_i|}{2}|\sigma_i + 1| + \frac{|\beta_i|}{2}|\sigma_i - 1| + \sum_{i=1}^{s}|\gamma_i|$$

for all $\alpha_1, \beta_1, \dots, \alpha_k, \beta_k, \gamma_1, \dots, \gamma_s$ in $\mathbb{R}$ and all $\sigma$ in $V$.

Now let $x = (\alpha_1, \beta_1, \dots, \alpha_k, \beta_k, \gamma_1, \dots, \gamma_s)$ be an arbitrary element of $E$. Choose $\sigma$ in $V$ such that

$$\max(|\alpha_i|, |\beta_i|) = \frac{1}{2}(|\alpha_i|\,|\sigma_i + 1| + |\beta_i|\,|\sigma_i - 1|)$$

for all $i = 1, 2, \dots, k$.

It follows that

$$\|x\| \geqslant \left\|\sum_{i=1}^{k} \frac{|\alpha_i|}{2}(\sigma_i + 1)x_i - \frac{|\beta_i|}{2}(\sigma_i - 1)y_i + \sum_{i=1}^{s}|\gamma_i|\, z_i\right\| = \sum_{i=1}^{k}\max(|\alpha_i|, |\beta_i|) + \sum_{i=1}^{s}|\gamma_i|.$$

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By Lemma 1 part (2) and the triangle inequality we have

$$\|x\| \leqslant \sum_{i=1}^{k} \max(|\alpha_i|, |\beta_i|) + \sum_{i=1}^{s} |\gamma_i|,$$

and therefore we get

$$\|x\| = \sum_{i=1}^{k} \max(|\alpha_i|, |\beta_i|) + \sum_{i=1}^{s} |\gamma_i|,$$

Finally define $T\colon E \to \ell^1(n)$,

$$T\left(\sum_{i=1}^{k} \alpha_i x_i + \beta_i y_i + \sum_{i=1}^{s} \gamma_i z_i\right) = \frac{\alpha_1 + \beta_1}{2} e_1 + \frac{\alpha_1 - \beta_1}{2} e_2 + \ldots + \\ + \frac{\alpha_k + \beta_k}{2} e_{2k-1} + \frac{\alpha_k - \beta_k}{2} e_k + \sum_{i=2k+1}^{n} \gamma_i e_i,$$

where $e_1, \ldots, e_n$ denote the canonical basis of $\ell^1(n)$.

Now it follows that $T$ is an isometry from $E$ to $\ell^1(n)$ and so we are done. □

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