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LINEAR INDEPENDENCE OF VALUES OF THE q-EXPONENTIAL AND RELATED FUNCTIONS

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Abstract

We establish the linear independence of values of the q-analogue of the exponential function and its derivatives at specified algebraic arguments, when q is a Pisot–Vijayaraghavan number. We also deduce similar results for cognate functions, such as the Tschakaloff function and certain generalised q-series.

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1. Introduction

For any complex number q with $|q| > 1$, the q-analogue of the exponential function is defined by the absolutely convergent series

$$E_q(x) := 1 + \sum_{n=1}^{\infty} \frac{x^n}{[n]_q!},$$

where $[n]_q = q^n - 1$ and $[n]_q! = (q^n - 1)(q^{n-1} - 1) \cdots (q - 1)$. Similarly, the q-analogue of the logarithm is given by

$$L_q(x) := \sum_{n=1}^{\infty} \frac{x^n}{[n]_q} \quad \text{for } |x| < |q|.$$

The analogy between the classical functions and their q-analogues is driven by the limit

$$\lim_{q \to 1^+} \frac{q^n - 1}{q - 1} = n.$$

Unlike the classical exponential and logarithm functions, their q-counterparts are related by the differential relation

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$$L_q(x) = x\frac{E'_q(-x)}{E_q(-x)} \quad \text{for } |x| < |q|.$$

For more details, we refer the reader to [7, Section 6]. These functions appear in various contexts in combinatorics and number theory and are interesting functions in their own right.

The value at $x = 1$ of the q-logarithm function is of particular importance, as $L_q(1) = \zeta_q(1)$, where

$$\zeta_q(s) := \sum_{n=1}^{\infty} \frac{n^{s-1}}{[n]_q},$$

is the q-analogue of the Riemann zeta-function (see [6]). The value $\zeta_q(1)$,

$$\zeta_q(1) = \sum_{n=1}^{\infty} \frac{1}{q^n - 1},$$

is often referred to as the q-harmonic series.

We examine the arithmetic nature and linear independence properties of certain special values of these functions. Recall that a real algebraic integer ω is said to be a *Pisot–Vijayaraghavan number* (abbreviated to PV number) if $\omega > 1$ and $|\omega^{(j)}| < 1$ for all other Galois conjugates $\omega^{(j)}$ of ω. Immediate examples of PV numbers are positive integers greater than one. A nontrivial example is obtained by considering β, the real root of $x^4 - x^3 - 2x^2 + 1$ with $\beta > 1$. Pisot [8] showed that, in every real algebraic number field, there exist PV numbers that generate the field. These numbers make a fundamental appearance in Diophantine approximation and have been studied extensively.

Fix an algebraic integer $q \neq 0$ and let $n_q = [\mathbb{Q}(q) : \mathbb{Q}]$. Let $\sigma_1, \sigma_2, \ldots, \sigma_{n_q}$ denote the embeddings of $\mathbb{Q}(q)$ into $\mathbb{C}$, with σ_1 being identity. Let O_q be the ring of integers of $\mathbb{Q}(q)$. For any algebraic number $\alpha \in \mathbb{Q}(q)$, the q-relative height of α, $H_q(\alpha)$, is

$$H_q(\alpha) := \prod_{l=1}^{n_q} \max\{1, |\sigma_l(\alpha)|\}.$$

Thus, if q is a PV number, then $H_q(q) = q$.

Our first theorem concerns the linear independence of values of derivatives of a certain generalised q-exponential function. Let $P(X) \in \mathbb{Z}[X]$ be a nonconstant polynomial such that $P(q^t) \neq 0$ for all $t \in \mathbb{N}$. Then the generalised q-exponential function with respect to P is given by

$$E_{q,P}(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{\prod_{t=1}^{n} P(q^t)}.$$

If $P(X) = X - 1$, then $E_{q,P}(x) = E_q(x)$, the q-exponential function. Note that $E_{q,P}(x)$ is a basic hypergeometric series, as defined in [5].

With this notation, we can state our first result.

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THEOREM 1.1. *Assume that q or $-q$ is a PV number. Let $P(X) = L_D X^D + \cdots + c_d X^d \in \mathbb{Z}[X]$ be a nonconstant polynomial with $P(q^t) \neq 0$ for $t \geq 1$, $d \leq D$ and $L_D c_d \neq 0$. Let $\alpha_1, \ldots, \alpha_m$ be nonzero algebraic integers in $\mathbb{Q}(q)$ satisfying*

$$|c_d|^{n_q-1} \max\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_m|\} \prod_{l=2}^{n_q} \max\{1, |\sigma_l(\alpha_1)|, |\sigma_l(\alpha_2)|, \ldots, |\sigma_l(\alpha_m)|\} < |q|^D. \quad (1.1)$$

Suppose that $\alpha_{k_1}/\alpha_{k_2}$ is not a root of unity for $1 \leq k_1, k_2 \leq m$ and $k_1 \neq k_2$. Then the numbers in the set

$$S := \{E_{q,P}^{(j)}(\alpha_k) : 1 \leq k \leq m, 0 \leq j \leq M\} \cup \{1\}$$

are linearly independent over the field $\mathbb{Q}(q)$.

The following result is an immediate corollary of this theorem.

COROLLARY 1.2. *Assume that q or $-q$ is a PV number. Let $\alpha_1, \ldots, \alpha_m$ be nonzero algebraic integers in $\mathbb{Q}(q)$ satisfying*

$$\max\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_m|\} \prod_{l=2}^{n_q} \max\{1, |\sigma_l(\alpha_1)|, |\sigma_l(\alpha_2)|, \ldots, |\sigma_l(\alpha_m)|\} < |q|.$$

Suppose that $\alpha_{k_1}/\alpha_{k_2}$ is not a root of unity for $1 \leq k_1, k_2 \leq m$ and $k_1 \neq k_2$. Then the numbers in the set

$$S := \{E_q^{(j)}(\alpha_k) : 1 \leq k \leq m, 0 \leq j \leq M\} \cup \{1\}$$

are linearly independent over the field $\mathbb{Q}(q)$.

In particular, this gives the following result about the special functions discussed earlier.

COROLLARY 1.3. *Assume that q or $-q$ is a PV number and that $\alpha \in O_q$ satisfies*

$$0 < \min\{1, |\alpha|\} H_q(\alpha) < |q|.$$

Then $E_q(\alpha), L_q(\alpha) \notin \mathbb{Q}(q)$. In particular, $\zeta_q(1)$ is irrational.

The irrationality and linear independence of the values of the q-logarithm function have been studied extensively. We refer the reader to [10] for a comprehensive history of the problem and an investigation of the values of a generalisation of the q-logarithm function. The irrationality of $\zeta_q(1)$ when q is an integer was first obtained by Erdős [4]. More recently, Tachiya [9, Theorem 2] proved that $\zeta_q(1) \notin \mathbb{Q}(q)$ when q is a PV number, which also follows from Corollary 1.3.

A special function closely related to the q-exponential function is the Tschakaloff function, given by

$$T_q(x) := 1 + \sum_{n=1}^{\infty} \frac{x^n}{q^{n(n+1)/2}}.$$

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In our notation, $T_q(x) = E_{q,I}(x)$, where $I(x) = x$. Thus, Theorem 1.1 implies the following result.

COROLLARY 1.4. *Assume that q or $-q$ is a PV number. Suppose that $\alpha \in O_q$ satisfies*

$$0 < \min\{1, |\alpha|\}H_q(\alpha) < |q|.$$

Then the numbers $1, T_q(\alpha), T_q^{(1)}(\alpha), \ldots, T_q^{(m)}(\alpha)$ are linearly independent over $\mathbb{Q}(q)$.

It was brought to our notice by the referee that Theorem 1.1 follows from [1, Corollaries 5.1 and 5.2], which require a much weaker condition on the α_k than in Theorem 1.1. In [1], Amou *et al.*, prove a general result regarding linear independence of values of solutions to q-difference equations. The techniques necessary to prove this result are involved, whereas our proof of Theorem 1.1 follows from relatively elementary considerations.

The statements so far were concerned with the independence of values of a single function and its derivatives at several arguments. We now address the question of independence of different cognate functions at the same argument. For any $M \in \mathbb{N}$ and any q with $|q| > 1$, we define an arithmetic progression analogue, $E_{q,M}(x)$, of $E_q(x)$ by

$$E_{q,M}(x) := 1 + \sum_{n=1}^{\infty} \frac{x^n}{[Mn]_q!}.$$

This is an entire function. Clearly, $E_{q,1}(x) = E_q(x)$ and

$$E_{q,M}(x^M) = 1 + \sum_{\substack{n=1 \\ n\equiv 0 \bmod M}}^{\infty} \frac{x^n}{[n]_q!}.$$

Note that $E_{q,M}$ is not a basic hypergeometric function.

For these special functions, we prove the following theorem.

THEOREM 1.5. *Assume that q or $-q$ is a PV number and that $a_1 < \cdots < a_k$ are distinct positive integers. Let $\alpha \in O_q$ be such that $1 \le |\alpha|$ and*

$$H_q(\alpha) < |q|^{a_1}. \tag{1.2}$$

Then the numbers

$$1, E_{q,a_1}(\alpha), \ldots, E_{q,a_k}(\alpha) \tag{1.3}$$

are linearly independent over the field $\mathbb{Q}(q)$.

The approach in this paper is an adaptation of the proof of [7, Theorem 1.1], which is a modification of the argument by Duverney [3]. In essence, it is similar to Fourier's proof of the irrationality of the number e. The proof of Theorem 1.1 relies on a Diophantine lemma, which is a consequence of the Skolem–Mahler–Lech theorem. The proof of Theorem 1.5 is completed using a recursive elimination argument.

2. Proof of the theorems

An important ingredient in the proofs is the following particular case of the Skolem–Mahler–Lech theorem [2, Theorem 4.3, page 124].

THEOREM 2.1. *Let $\alpha_1, \ldots, \alpha_k$ be nonzero algebraic numbers such that α_i/α_j is not a root of unity for $1 \le i < j \le k$. Let $P_1(x), \ldots, P_k(x)$ be nonzero polynomials with algebraic coefficients. Then there are only finitely many natural numbers n satisfying*

$$P_1(n)\alpha_1^n + \cdots + P_k(n)\alpha_k^n = 0.$$

This is immediate from the Skolem–Mahler–Lech theorem since the sequence $P_1(n)\alpha_1^n + \cdots + P_k(n)\alpha_k^n$ is a nondegenerate recurrence sequence if none of the α_i/α_j $(1 \le i < j \le k)$ is a root of unity.

2.1. Proof of Theorem 1.1. Let $f_j(x) := x^j E_{q,P}^{(j)}(x)$ for $0 \le j \le M$. Observe that the result follows if we show that 1 and the values $f_j(\alpha_k)$ are $\mathbb{Q}(q)$-linearly independent for $0 \le j \le M$ and $1 \le k \le m$. Indeed, suppose that ξ_0 and $\xi_{j,k}$ are algebraic numbers in $\mathbb{Q}(q)$ for $1 \le k \le m$ and $0 \le j \le M$, not all zero, such that

$$\xi_0 + \sum_{j=0}^{M} \sum_{k=1}^{m} \xi_{j,k} E_{q,P}^{(j)}(\alpha_k) = 0.$$

Then we obtain the nontrivial linear relation

$$\xi_0 + \sum_{j=0}^{M} \sum_{k=1}^{m} \frac{\xi_{j,k}}{\alpha_k^j} f_j(\alpha_k) = 0,$$

which again has coefficients in $\mathbb{Q}(q)$. Thus, it suffices to establish the linear independence of the $f_j(\alpha_k)$ over $\mathbb{Q}(q)$.

Let $r_0(X) = 1$ and $r_j(X) := X(X-1)\cdots(X-j+1)$ for $1 \le j \le M$. Then

$$f_j(x) = \sum_{n=j}^{\infty} \frac{r_j(n)x^n}{\prod_{t=1}^{n} P(q^t)} = \sum_{n=1}^{\infty} \frac{r_j(n)x^n}{\prod_{t=1}^{n} P(q^t)},$$

since $r_j(n) = 0$ for $0 \le n \le j-1$. Now, suppose that λ_0 and $\lambda_{j,k} \in \mathbb{Q}(q)$ are such that

$$\lambda_0 + \sum_{j=0}^{M} \sum_{k=1}^{m} \lambda_{j,k} f_j(\alpha_k) = 0.$$

Without loss of generality, we can assume that λ_0 and the $\lambda_{j,k}$ are algebraic integers. For $1 \le k \le m$, let $A_k(X) := \sum_{j=0}^{M} \lambda_{j,k} r_j(X)$. Then, from the definition of $E_{q,P}(x)$,

$$\widetilde{\lambda_0} + \sum_{n=1}^{\infty} \frac{\sum_{k=1}^{m} A_k(n)\alpha_k^n}{\prod_{t=1}^{n} P(q^t)} = 0,$$

where $\widetilde{\lambda_0} = \lambda_0 + \sum_{k=1}^{m} \lambda_{0,k}$.

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Let N be a sufficiently large positive integer. We truncate the infinite sum above at N and clear denominators to obtain

$$X_N := \widetilde{\lambda_0}\prod_{t=1}^{N}P(q^t) + \sum_{n=1}^{N}\Big(\sum_{k=1}^{m}A_k(n)\alpha_k^n\Big)\prod_{t=n+1}^{N}P(q^t) = -\prod_{t=1}^{N}P(q^t)\sum_{n=N+1}^{\infty}\frac{\sum_{k=1}^{m}A_k(n)\alpha_k^n}{\prod_{t=1}^{n}P(q^t)}. \tag{2.1}$$

Then $X_N \in \mathcal{O}_q$. Moreover, the right-hand side of (2.1) can be written as

$$\begin{aligned}\prod_{t=1}^{N}&P(q^t)\sum_{n=N+1}^{\infty}\frac{\sum_{k=1}^{m}A_k(n)\alpha_k^n}{\prod_{t=1}^{n}P(q^t)}\\ &= \frac{\sum_{k=1}^{m}A_k(N+1)\alpha_k^{N+1}}{P(q^{N+1})} + \frac{1}{P(q^{N+1})}\sum_{n=2}^{\infty}\frac{\sum_{k=1}^{m}A_k(N+n)\alpha_k^{N+n}}{\prod_{t=N+2}^{N+n}P(q^t)}.\end{aligned} \tag{2.2}$$

For simplicity of notation, let

$$\boldsymbol{\alpha} := \max\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_m|\}.$$

From the triangle inequality and the fact that each $A_k(X)$ is a polynomial of degree M, for all $\nu > 0$,

$$\Big|\sum_{k=1}^{m}A_k(\nu)\alpha_k^{\nu}\Big| \le \boldsymbol{\alpha}^{\nu}\sum_{k=1}^{m}|A_k(\nu)| \ll \nu^M\boldsymbol{\alpha}^{\nu}.$$

Also, since $|P(q^t)| \sim |q|^{tD}$ for t sufficiently large, the second term on the right-hand side of (2.2) is

$$\ll \frac{\boldsymbol{\alpha}^{N+1}}{|P(q^{N+1})|}\sum_{n=2}^{\infty}(n+N)^M\cdot\Big(\frac{\boldsymbol{\alpha}}{|q|^{DN}}\Big)^{n-1}\cdot|q|^{-D(n^2+n-2)/2}.$$

This infinite series converges absolutely as $|q| > 1$ and the terms decay exponentially. Applying these bounds to the expression in (2.2) gives

$$|X_N| \ll \frac{\boldsymbol{\alpha}^{N+1}}{|P(q^{N+1})|}N^M, \tag{2.3}$$

where the implied constant depends on q, the α_k and the coefficients $\lambda_{j,k}$.

We now estimate the size of conjugates of X_N. Since $\pm q$ is a PV number, $|\sigma_l(q)| < 1$ for $2 \le l \le n_q$. From the expression for X_N in (2.1), for all $n \ge 0$,

$$\sigma_l(X_N) = \sigma_l(\widetilde{\lambda_0})\prod_{t=1}^{N}P(\sigma_l(q)^t) + \sum_{n=1}^{N}\Big(\sum_{k=1}^{m}\sigma_l(A_k(n))\sigma_l(\alpha_k)^n\Big)\prod_{t=n+1}^{N}P(\sigma_l(q)^t).$$

Observe that

$$\Big|\prod_{t=n+1}^{N}P(\sigma_l(q^t))\Big| = \Big|c_d\Big(\prod_{t=n+1}^{N}\sigma_l(q^t)\Big)^d\Big|^{N-n}\prod_{t=n+1}^{N}\Big|1+\cdots+\frac{L_D}{c_d}(\sigma_l(q^t))^{D-d}\Big|.$$

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Since $|\sigma_l(q)| < 1$ for $2 \le l \le n_q$, the series $\sum_{t=1}^{\infty} (\sigma_l(q^t))^s$ is absolutely convergent for $1 \le s \le D - d$. Thus, the infinite product

$$\prod_{t=1}^{\infty} \left| 1 + \cdots + \frac{L_D}{c_d} (\sigma_l(q^t))^{D-d} \right|$$

is convergent and

$$\left| \prod_{t=n+1}^{N} P(\sigma_l(q)^t) \right| \ll |c_d|^{N-n} \prod_{t=n+1}^{N} |(\sigma_l(q^t))^{d(N-n)}| \ll |c_d|^{N-n},$$

again since $|\sigma_l(q)| < 1$ for $2 \le l \le n_q$. By these observations,

$$|\sigma_l(X_N)| \ll |c_d|^N \left(1 + \sum_{n=1}^{N} |c_d|^{-n} \sum_{k=1}^{m} |\sigma_l(A_k(n))| |\sigma_l(\alpha_k)|^n \right).$$

Note that $c_d \in \mathbb{Z}$ so that $|c_d| \ge 1$. Now, $\sigma_l(A_k(n)) = \sum_{j=0}^{M} \sigma_l(\lambda_{j,k}) r_j(n)$, which is again a polynomial of degree M in n. Putting these bounds together, we deduce that

$$|\sigma_l(X_N)| \ll N^{M+2} |c_d|^N (\max\{1, |\sigma_l(\alpha_1)|, \ldots, |\sigma_l(\alpha_m)|\})^N. \tag{2.4}$$

As before, the implied constant only depends on q, the α_k and the $\lambda_{j,k}$.

Multiplying the absolute values of all the conjugates of X_N and the corresponding bounds in (2.3) and (2.4) gives

$$\begin{aligned} \prod_{l=1}^{n_q} |\sigma_l(X_N)| &\ll \frac{N^{n_q(M+2)-2} |c_d|^{(n_q-1)N} \boldsymbol{\alpha}^N}{|P(q^{N+1})|} \left(\prod_{l=2}^{n_q} \max\{1, |\sigma(\alpha_1)|, \ldots, |\sigma(\alpha_m)|\} \right)^N \\ &\ll N^{n_q(M+2)-2} \left(\frac{\boldsymbol{\alpha} |c_d|^{(n_q-1)} \prod_{l=2}^{n_q} \max\{1, |\sigma(\alpha_1)|, \ldots, |\sigma(\alpha_m)|\}}{|q|^D} \right)^N. \end{aligned}$$

By the hypothesis (1.1), the last bound tends to zero as $N \to \infty$. In particular,

$$\left| \prod_{l=1}^{n_q} \sigma_l(X_N) \right| < 1$$

for all N sufficiently large. Here, the left-hand side is a power of the norm of an algebraic integer (noting that $\mathbb{Q}(X_N)$ may be a strict subfield of $\mathbb{Q}(q)$). Thus, $\prod_{l=1}^{n_q} \sigma_l(X_N)$ must be a rational integer for all $N > 0$. This is only possible if $X_N = 0$ for *all* N sufficiently large.

Therefore, there exists a natural number N_0 such that, for all $N \ge N_0$,

$$\frac{X_N}{\prod_{t=1}^{N} P(q^t)} = \widetilde{\lambda_0} + \sum_{n=1}^{N} \sum_{k=1}^{m} A_k(n) \alpha_k^n = 0.$$

Thus, considering the expression

$$\frac{X_{N+1}}{\prod_{t=1}^{N+1} P(q^t)} - \frac{X_N}{\prod_{t=1}^{N} P(q^t)},$$

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which equals zero for $N > N_0$, we obtain

$$A_1(N)\alpha_1^N + \cdots + A_m(N)\alpha_m^N = 0$$

for all $N > N_0$. As $\alpha_{k_1}/\alpha_{k_2}$ is not a root of unity, it follows from Theorem 2.1 that $A_k(N) = 0$ for $1 \le k \le m$ and all $N > N_0$. Thus, the polynomials $A_k(X)$ are identically zero. Recall that

$$A_k(X) = \sum_{j=0}^{M} \lambda_{j,k} r_j(X),$$

and $\deg r_j(X) = j$. Since the $r_j(X)$ have distinct degrees, $A_k(X)$ is identically zero if and only if $\lambda_{j,k} = 0$ for $0 \le j \le M$ and $1 \le k \le m$. This completes the proof of the theorem.

2.2. Proof of Theorem 1.5. We begin along the same lines as in the proof of Theorem 1.1.

Suppose that the numbers in (1.3) are linearly dependent over $\mathbb{Q}(q)$. Then there exist algebraic integers $\lambda_0, \lambda_1, \ldots, \lambda_k \in \mathcal{O}_q$, not all zero, such that

$$\lambda_0 + \lambda_1 E_{q,a_1}(\alpha) + \cdots + \lambda_k E_{q,a_k}(\alpha) = 0.$$

Without loss of generality, we can assume that $\lambda_1 \ne 0$. Otherwise, we can change the notation to replace a_j by a_1 for the smallest $j \le k$ for which $\lambda_j \ne 0$ and follow the argument below.

From the definition of the q-exponential function,

$$\widetilde{\lambda_0} + \sum_{n=1}^{\infty} \frac{\lambda_1 \alpha^n}{[a_1 n]_q!} + \cdots + \sum_{n=1}^{\infty} \frac{\lambda_k \alpha^n}{[a_k n]_q!} = 0, \tag{2.5}$$

where $\widetilde{\lambda_0} = \lambda_0 + \lambda_1 + \cdots + \lambda_k$. Set $d = \operatorname{lcm}\{a_1, \ldots, a_k\}$ and $d_i = d/a_i$. Choose a large positive integer N and set $N_i = Nd_i$ for $i = 1, 2, \ldots, k$. With these choices of N_i,

$$a_1 N_1 = a_2 N_2 = \cdots = a_k N_k = dN.$$

Furthermore, for all $i = 1, 2, 3, \ldots, k$,

$$\begin{aligned} \frac{[dN]_q!}{[a_i(N_i+1)]_q!} &= \frac{[a_i N_i]_q!}{[a_i(N_i+1)]_q!} \\ &= \frac{(q^{a_i N_i} - 1)\cdots(q-1)}{(q^{a_i N_i + a_i} - 1)\cdots(q-1)} = \frac{1}{(q^{Nd+a_i} - 1)\cdots(q^{Nd+1} - 1)}. \end{aligned} \tag{2.6}$$

Now truncate the ith infinite sum in (2.5) at N_i and multiply by $[dN]_q!$ to get

$$\begin{aligned} X_N &:= [dN]_q!\Big(\widetilde{\lambda_0} + \sum_{n=1}^{N_1} \frac{\lambda_1 \alpha^n}{[a_1 n]_q!} + \cdots + \sum_{n=1}^{N_k} \frac{\lambda_k \alpha^n}{[a_k n]_q!}\Big) \\ &= -[dN]_q!\Big(\sum_{n=N_1+1}^{\infty} \frac{\lambda_1 \alpha^n}{[a_1 n]_q!} + \cdots + \sum_{n=N_k+1}^{\infty} \frac{\lambda_k \alpha^n}{[a_k n]_q!}\Big). \end{aligned} \tag{2.7}$$

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Since $[dN]_q! = [a_iN_i]_q!$ for $1 \le i \le k$, X_N is an algebraic integer in O_q. We now estimate the right-hand side of (2.7). By an argument similar to the one in Theorem 1.1 and using (2.6), we deduce that

$$\left|[dN]_q! \sum_{n=N_j+1}^{\infty} \frac{\alpha^n}{[a_jn]_q!}\right| \ll \frac{|\alpha|^{N_j}}{|q|^{a_jdN}} \ll \left(\left|\frac{\alpha^{d_j}}{q^{a_jd}}\right|\right)^N,$$

since $N_j = Nd_j$. As $a_1 < a_2 < \cdots < a_k$, $d_1 > d_2 > \cdots > d_k$ and $|\alpha| \ge 1$,

$$|X_N| \ll \left(\left|\frac{\alpha^{d_1}}{q^{a_1d}}\right|\right)^N. \tag{2.8}$$

By the same argument as in the proof of Theorem 1.1, we can estimate the conjugates of X_N by

$$|\sigma_l(X_N)| \ll N_1(\max\{1, |\sigma_l(\alpha)|\})^{N_1}. \tag{2.9}$$

As before, the implied constant depends only on q, the a_i and the $\lambda_{j,k}$. Multiplying the bounds (2.8) and (2.9) for the absolute values of all the conjugates of X_N and noting that $|\alpha| \ge 1$, we derive

$$\prod_{l=1}^{n_q} |\sigma_l(X_N)| \ll N_1^{n_q-1}\left(\frac{|\alpha| \prod_{l=2}^{n_q} \max\{1, |\sigma_l(\alpha)|\}}{|q|^{a_1^2}}\right)^{d_1N}.$$

By (1.2), the right-hand side tends to zero as $N \to \infty$. However, the left-hand side is a rational integer since it is a power of the norm of an algebraic integer. Therefore, there exists a natural number N_0 such that $X_N = 0$ for *all* $N > N_0$, which, in turn, implies that $X_N = X_{N+1} = 0$. Consequently,

$$\widetilde{\lambda_0} + \sum_{n=1}^{Nd_1} \frac{\lambda_1\alpha^n}{[a_1n]_q!} + \cdots + \sum_{n=1}^{Nd_k} \frac{\lambda_k\alpha^n}{[a_kn]_q!} = 0$$

and

$$\widetilde{\lambda_0} + \sum_{n=1}^{Nd_1+d_1} \frac{\lambda_1\alpha^n}{[a_1n]_q!} + \cdots + \sum_{n=1}^{Nd_k+d_k} \frac{\lambda_k\alpha^n}{[a_kn]_q!} = 0$$

for all $N > N_0$. Subtracting these two relations gives

$$\lambda_1 \sum_{n=Nd_1+1}^{Nd_1+d_1} \frac{\alpha^n}{[a_1n]_q!} + \cdots + \lambda_k \sum_{n=Nd_k+1}^{Nd_k+d_k} \frac{\alpha^n}{[a_kn]_q!} = 0 \tag{2.10}$$

for all $N > N_0$. Note that, for $1 \le j \le k$, we have $Nd + a_j \le a_jn \le Nd + d$ in the above sums. Therefore,

$$\frac{\alpha^{Nd_k+1}}{[Nd + a_1]_q!}$$

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divides each term in (2.10). By extracting this factor, we obtain

$$\begin{aligned}
&\lambda_1\Big(\alpha^{N(d_1-d_k)} + \frac{\alpha^{N(d_1-d_k)+1}}{(q^{Nd+2a_1}-1)\cdots(q^{Nd+a_1+1}-1)} + \cdots + \frac{\alpha^{N(d_1-d_k)+d_1-1}}{(q^{Nd+d}-1)\cdots(q^{Nd+a_1+1}-1)}\Big)\\
&\quad + \lambda_2\Big(\frac{\alpha^{N(d_2-d_k)}}{(q^{Nd+a_2}-1)\cdots(q^{Nd+a_1+1}-1)} + \cdots + \frac{\alpha^{N(d_2-d_k)+d_2-1}}{(q^{Nd+d}-1)\cdots(q^{Nd+a_1+1}-1)}\Big)\\
&\quad + \cdots\\
&\quad + \lambda_k\Big(\frac{1}{(q^{Nd+a_k}-1)\cdots(q^{Nd+a_1+1}-1)} + \cdots + \frac{\alpha^{d_k-1}}{(q^{Nd+d}-1)\cdots(q^{Nd+a_1+1}-1)}\Big) = 0.
\end{aligned} \tag{2.11}$$

Now, for $1 \leq j \leq k$ and $0 \leq l \leq d_j - 1$, the absolute value of the general term is

$$\left|\frac{\alpha^{N(d_j-d_k)+l}}{(q^{Nd+(l+1)a_j}-1)\cdots(q^{Nd+a_1+1}-1)}\right| \ll \left|\frac{\alpha^{d_1-d_k}}{q^{\delta d}}\right|^N$$

except for $j = 1$ and $l = 0$, with $\delta = \min\{a_1, a_2 - a_1\}$. Since $1 \leq \delta$, this implies that each term in (2.11) is $\ll |\alpha^{d_1-d_k}/q^d|^N$. By (1.2), this quotient is less than 1, as $1 \leq |\alpha| < |q|^{a_1}$. Hence, taking the limit as $N \to \infty$, all terms in (2.11) tend to zero except the first, that is, $\alpha^{N(d_1-d_k)}$. This implies that $\lambda_1 = 0$, which is a contradiction. This completes the proof.

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References

[1] M. Amou, T. Matala-Aho and K. Väänänen, 'On Siegel–Shidlovskii's theory for q-difference equations', *Acta Arith.* **127** (2007), 309–335.

[2] P. Corvaja and U. Zannier, *Applications of Diophantine Approximation to Integral Points and Transcendence*, Cambridge Tracts in Mathematics, 212 (Cambridge University Press, Cambridge, 2018).

[3] D. Duverney, 'Irrationalité de la somme des inverses de la suite de Fibonacci', *Elem. Math.* **52** (1997), 31–36.

[4] P. Erdős, 'On arithmetical properties of Lambert series', *J. Indian Math. Soc. (N.S.)* **12** (1948), 63–66.

[5] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd edn, Encyclopedia of Mathematics and Its Applications, 96 (Cambridge University Press, Cambridge, 2004).

[6] C. Krattenthaler, T. Rivoal and W. Zudilin, 'Series hypergeometriques basiques, q-analogues de valeurs de la fonction zeta et series d'Eisenstein', *J. Inst. Math. Jussieu* **5**(1) (2006), 53–79.

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[7] R. Murty, 'The Fibonacci zeta-function', in: *Automorphic Representations and L-functions*, Tata Institute of Fundamental Research Conference Proceedings, 18 (eds. D. Prasad, C. S. Rajan, A. Sankaranarayanan and J. Sengupta) (Hindustan Book Agency, New Delhi, India, 2013).

[8] C. Pisot, 'Kriterium für die algebraischen Zahlen', *Math. Z.* **48** (1942), 293–323.

[9] Y. Tachiya, 'Irrationality of certain Lambert series', *Tokyo J. Math.* **27**(1) (2004), 75–85.

[10] W. Zudilin, 'On the irrationality of generalized q-logarithm', *Res. Number Theory* **2** (2016), Article no. 15.

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