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Twist sets for maps of the circle

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Abstract. Let f be a continuous map of degree one of the circle onto itself. We prove that for every number a from the rotation interval of f there exists an invariant closed set A consisting of points with rotation number a and such that f restricted to A preserves the order. This result is analogous to the one in the case of a twist map of an annulus.

0. *Introduction*

Let $f: S^1 \to S^1$ be a continuous map of degree one of the circle onto itself and let $F: \mathbb{R} \to \mathbb{R}$ be its lifting. We denote by $e: \mathbb{R} \to S^1$ the natural projection (here $S^1 = \mathbb{R}/\mathbb{Z}$).

Definition 1. We call a set $A \subset S^1$ a *twist set* (respectively an *almost twist set*) if F restricted to $e^{-1}(A)$ is increasing (respectively non-decreasing).

The reader should remember that 'increasing' means the same as 'preserving order'. Notice that the above definition does not depend on the choice of F.

The notion of a twist set is a natural generalization of the notion of a twist periodic orbit of Alseda and Llibre ([**1**], [**2**]). It happens sometimes that non-invertible maps in one dimension have properties similar to those of invertible maps in two dimensions. The twist sets studied here (or rather the mat sets see definition 3 below) are similar to the Mather sets for twist maps (see [**5**]).

Definition 2. We call a point $x \in S^1$ a *twist point* (respectively, an *almost twist point*) if its orbit $\{f^n(x)\}_{n=0}^{\infty}$ is a twist set (respectively, an almost twist set).

We denote the set of all almost twist points of f by AT (f). The standard proof of the existence of a rotation number of a homeomorphism of the circle applies also to f restricted to an almost twist set. Hence, if $A \subset S^1$ is an almost twist set, then for every $X \in e^{-1}(A)$ the limit

$$\rho(X) = \lim_n \frac{1}{n}(F^n(X) - X)$$

exists and is independent of the choice of X. We call it a *rotation number* of A. We also use the notation $\rho(A)$ and $\rho(x)$ (where $x = e(X)$). As always, the rotation number depends on F; if we take $F' = F + k$, $k \in \mathbb{Z}$, instead of F then k adds to the rotation number.

The rotation numbers have also been defined for other than almost twist points (see [**7**]), in particular for periodic points (cf. [**3**], [**6**]).

If f has periodic points, denote by L the closure of the set of rotation numbers of all periodic points. It is known ([**7**], [**3**]) that L is a closed interval (perhaps degenerated to one point). It is called the *rotation interval* of f (or more precisely, of F). If f has no periodic points then the situation is very similar to the case of a homeomorphism and every point has the same rotation number (cf. [**6**]). In this case L consists of this number.

Definition 3. We shall call a set $A \subset S^1$ a *mat set* ('mat' stands for 'minimal almost twist') if A is non-empty, closed, invariant (i.e. $f(A) \subset A$), minimal (i.e. for every $x \in A$, its orbit $\{f^n(x)\}_{n=0}^{\infty}$ is dense in A) and an almost twist set.

The main result of this paper is the following theorem:

THEOREM A. *Let $f: S^1 \to S^1$ be a continuous map of degree one. Then for every a from the rotation interval of f there exists a mat set A with $\rho(A) = a$.*

Theorem A can be easily deduced from the following theorem:

THEOREM B. *Let $f: S^1 \to S^1$ be a continuous map of degree one. Then for every rational a from the rotation interval of f there exists a periodic twist point $x \in S^1$ with $\rho(x) = a$.*

The paper is organized as follows. In § 1 we prove some simple properties of twist and almost twist points, deduce theorem A from theorem B and derive as a corollary a result of Ito [**4**]. In § 2 we describe possible mat sets. In § 3 we prove theorem B.

We denote by $\mathbb{Z}^+$ the set of all positive integers and by $\mathbb{N}$ the set of all natural numbers (i.e. non-negative integers).

1. *Twist and almost twist points*

We start by proving several lemmas.

LEMMA 1.1. *The set* AT (f) *is invariant and closed.*

Proof. It follows immediately from the definition that any subset of an almost twist set is an almost twist set. Consequently, the image of an almost twist point is an almost twist point. Hence, the set AT (f) is invariant.

Let $\lim_n x_n = x_0$, $x_n \in \mathrm{AT}(f)$ for $n \in \mathbb{Z}^+$. To prove that $x_0 \in \mathrm{AT}(f)$, we have to show that if $X < Y$ and $e(X) = f^i(x_0)$, $e(Y) = f^j(x_0)$ for some $i, j \in \mathbb{N}$, then $F(X) \le F(Y)$. But for such X and Y there exist X_n, Y_n, $n \in \mathbb{Z}^+$, such that $e(X_n) = f^i(x_n)$, $e(Y_n) = f^j(x_n)$ and $\lim_n X_n = X$, $\lim_n Y_n = Y$. Since $X < Y$, we have $X_n < Y_n$ for n sufficiently large. But since $x_n \in \mathrm{AT}(f)$, we have that $F(X_n) < F(Y_n)$. Since F is continuous, we obtain $F(X) \le F(Y)$. Consequently, $x_0 \in \mathrm{AT}(f)$ and hence AT (f) is closed. □

LEMMA 1.2. *Let $X \in e^{-1}(\mathrm{AT}(f))$, $p \in \mathbb{Z}$, $q \in \mathbb{Z}^+$. Then:*

(a) *if $F^q(X) - X \ge p$, then $\rho(X) \ge p/q$; if $F^q(X) - X \le p$, then $\rho(X) \le p/q$;*

(b) *if $\rho(X) > p/q$, then $F^q(X) - X > p$; if $\rho(X) < p/q$, then $F^q(X) - X < p$.*

Proof. If $F^q(X) - X \ge p$ (respectively $\le$) then by induction we obtain $F^{nq}(X) - X \ge np$ (respectively $\le$) for all $n \in \mathbb{Z}^+$, and consequently $\rho(X) \ge p/q$ (respectively $\le$). This proves (a). Then (b) follows immediately. □

LEMMA 1.3. *The function $\rho: \mathrm{AT}(f) \to \mathbb{R}$ is continuous.*

Proof. Let $x \in \mathrm{AT}(f)$ and $a < \rho(x)$. We shall show that if $y \in \mathrm{AT}(f)$ is sufficiently close to x, then $a < \rho(y)$. Take $X \in e^{-1}(x)$. Since $\rho(X) > a$, there exists $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$ such that $\rho(X) > p/q > a$. By lemma 1.2(b), $F^q(X) - X > p$. If $y \in \mathrm{AT}(f)$ is sufficiently close to x, then there exists $Y \in e^{-1}(y)$ such that $F^q(Y) - Y > p$. By lemma 1.2(a), we then have $\rho(Y) \geq p/q$, and hence $\rho(y) > a$.

Analogously, if $b > \rho(x)$, then for all $y \in \mathrm{AT}(f)$ sufficiently close to x, we have $b > \rho(y)$. The continuity of ρ follows from these two properties. □

LEMMA 1.4. *Assume that f has no periodic points. Then every point of S^1 is a twist point.*

Proof. Suppose that $x \in S^1$ is not a twist point. Then there exist $X, Y \in \mathbb{R}$ such that $e(X) = f^i(x)$, $e(Y) = f^j(x)$ for some $i, j \in \mathbb{N}$, $X < Y$ and $F(X) \geq F(Y)$. Assume that $i \leq j$ (if $j \leq i$ then the proof is similar). Then $Y = F^k(X) + n$ for some $n \in \mathbb{Z}$ and $k = j - i \in \mathbb{N}$. We have

$$F(Y) = F(F^k(X) + n) = F^k(F(X)) + n.$$

The map $G = F^k + n$ is a lifting of f^k. We have $G(X) = Y > X$ and

$$G(F(X)) = F(Y) \leq F(X).$$

Therefore G has a fixed point. This contradicts the assumption that f has no periodic points. □

Remark. It is not true in general that if f has no periodic points then the whole circle is a twist set.

Now we assume that theorem B holds and prove theorem A.

Proof of theorem A. Let $a \in L$. If f has no periodic points, then $L = \{a\}$, and by lemma 1.4 there exists a twist point x with $\rho(x) = a$. Assume now that f has a periodic point. Then there exists a sequence of rational numbers $(a_n)_{n=1}^{\infty}$ such that $\lim_n a_n = a$ and $a_n \in L$ for all $n \in \mathbb{Z}^+$. By theorem B, for every $n \in \mathbb{Z}^+$ there exists a twist point x_n with $\rho(x_n) = a_n$. The sequence $(x_n)_{n=1}^{\infty}$ has a subsequence converging to some $x \in S^1$. Since $\mathrm{AT}(f)$ is closed, $x \in \mathrm{AT}(f)$. By lemma 1.3, we obtain $\rho(x) = a$.

Hence in all cases there exists an almost twist point x with $\rho(x) = a$. By continuity of F, the closure of the set $\{f^n(x)\}_{n=0}^{\infty}$ is an almost twist set. Its rotation number is a. By Zorn's lemma, it contains a mat set (minimality as defined here is the same as the minimality in the family of non-empty invariant closed subsets of a given set, ordered by inclusion). □

COROLLARY 1.5 (cf. [4]). *For every $a \in L$ there exists $X \in \mathbb{R}$ such that*

$$\lim_n (1/n)(F^n(X) - X) = a.$$

2. *Mat sets*

In this section we investigate more closely the mat sets. The results are similar to those obtained in the case of homeomorphisms.

PROPOSITION 2.1. *If A is a mat set and $\rho(A)$ is rational, then A is a periodic orbit.*

Proof. We shall show that A contains a periodic point. Then, by the minimality of A, it will follow that A is equal to the orbit of this point. Take $X \in e^{-1}(A)$. Let

$\rho(A) = p/q$, $p \in \mathbb{Z}$, $q \in \mathbb{Z}^+$. If $F^q(X) - p = X$, then $e(X)$ is periodic. Suppose that $F^q(X) - p > X$ (the proof in the case $F^q(X) - p < X$ is similar). Then, since A is invariant and almost twist, we have

$$F^q(X) - p \leq F^{2q}(X) - 2p \leq F^{3q}(X) - 3p \leq \cdots$$

and the points $F^{nq}(X) - np$, $n \in \mathbb{Z}^+$, belong to $e^{-1}(A)$. Since for every $n \in \mathbb{Z}^+$ we have $\rho(A) < (np+1)/nq$, then by lemma 1.2(b), we have

$$F^{nq}(X) - np < X + 1.$$

Therefore the limit

$$Y = \lim_n \, (F^{nq}(X) - np)$$

exists. Since A is closed, $Y \in e^{-1}(A)$. By continuity of F, we have $F^q(Y) - p = Y$, and hence $e(Y)$ is periodic. □

The following fact is obvious:

LEMMA 2.2. *If A is a periodic orbit, then the following properties are equivalent*:
(i) *A is a twist set*;
(ii) *A is an almost twist set*;
(iii) *A is a mat set.*

We shall call a periodic orbit which is a twist set, a *twist periodic orbit* (this definition coincides with the one in [**2**] and is slightly different from the one in [**1**]). Now we have to describe the dynamics on a twist periodic orbit (with respect to the ordering of points of the orbit).

PROPOSITION 2.3. *Let A be a twist periodic orbit with* $\rho(A) = p/q$, $p \in \mathbb{Z}$, $q \in \mathbb{Z}^+$, *p and q coprime. Let*

$$\cdots < X_{-2} < X_{-1} < X_0 < X_1 < X_2 < \cdots$$

all be elements of $e^{-1}(A)$. *Then for all* $i, j \in \mathbb{Z}$ *we have*

$$X_{i+qj} = X_i + j \quad \text{and} \quad F(X_i) = X_{i+p}.$$

Proof. Since $\rho(A) = p/q$ and p and q are coprime, the period of A is nq for some $n \in \mathbb{Z}^+$ and $F^{nq}(X_i) = X_i + np$ for all $i \in \mathbb{Z}$. Therefore for all $k \in \mathbb{Z}$ the number of elements of $e^{-1}(A) \cap [X_k, X_k + 1)$ is nq, and hence $X_{k+nq} = X_k + 1$. From this it follows easily that $X_{i+nqj} = X_i + j$ for all $i, j \in \mathbb{Z}$. Since A is a twist set and a periodic orbit, the map

$$F|_{e^{-1}(A)} : e^{-1}(A) \to e^{-1}(A)$$

is an order preserving bijection. Hence, there exists $m \in \mathbb{Z}$ such that $F(X_i) = X_{i+m}$ for all $i \in \mathbb{Z}$. Since

$$F^{nq}(X_i) = X_i + np = X_{i+n^2qp},$$

we obtain $m = np$.

Now to finish the proof it is enough to show that $n = 1$. Since A is a periodic orbit and $X_0, X_1 \in e^{-1}(A)$, for some $k \in \mathbb{N}$ and $l \in \mathbb{Z}$ we have $F^k(X_0) = X_1 + l$. Therefore $X_{knp} = X_{1+nql}$, and hence $1 = n(kp - ql)$. Consequently, $n = 1$. □

Corollary 2.4. *A twist periodic orbit as defined here is also a twist periodic orbit in the sense of* [**1**].

Now we start to investigate mat sets with irrational rotation number. Nevertheless, the following lemma applies to all mat sets (in fact, to all minimal closed invariant sets).

Lemma 2.5. *If A is a mat set then $f(A)=A$.*

Proof. Since A is invariant, $f(A)\subset A$. Since A is minimal, $f(A)$ is dense in A. But A is compact, and so is $f(A)$. Therefore $f(A)=A$. □

Proposition 2.6. *Let A be a mat set with $\rho(A)$ irrational. Then*:

(a) *either $A=S^1$ or A is homeomorphic to the Cantor set*;

(b) *if $x\in A$ is an endpoint of an interval disjoint from A, then there exists a unique $y\in A$ with $f(y)=x$; this y is also an endpoint of an interval disjoint from A*;

(c) *if $x\in A$, then either there exists a unique $y\in A$ with $f(y)=x$, or there are two such points*; *in this case they are the endpoints of some interval disjoint from A.*

Proof. To prove (a), assume that $A\neq S^1$. From the minimality of A and lack of periodic points in A it follows that A has no isolated points.

Suppose that A contains an interval. Take a maximal such interval K. Let x be an interior point of K. By the minimality of A, there exists $n\in\mathbb{Z}^+$ such that $f^n(x)\in K$. By the maximality of K, we have $f^n(K)\subset K$. Hence, there is a fixed point of f^n in K. This contradicts the assumption that $\rho(A)$ is irrational. Hence, A does not contain any interval.

Since A is a closed non-empty subset of S^1 without isolated points and nowhere dense, it is homeomorphic to the Cantor set. This proves (a).

Now we prove (b). Let $x\in A$ be an endpoint of an interval disjoint from A. By lemma 2.5, there exists $y\in A$ such that $f(y)=x$. Suppose also that for some $z\in A$, $z\neq y$, we have $f(z)=x$. By the minimality of A, there exist sequences $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ such that for every n we have $y_n=f^i(x)$, $z_n=f^j(x)$ for some $i, j\in\mathbb{Z}^+$, and $\lim_n y_n=y$, $\lim_n z_n=z$. Since A is invariant, y_n, $z_n\in A$ for all n. There exist X, Y, Z, Y_n, Z_n, $n\in\mathbb{Z}^+$, such that $e(X)=x$, $e(Y)=y$, $e(Z)=z$, $e(Y_n)=y_n$, $e(Z_n)=z_n$, $F(Y)=F(Z)=X$, $\lim_n Y_n=Y$ and $\lim_n Z_n=Z$. Since $y\neq z$, we have $Y\neq Z$. We may assume that $Y<Z$. If m, n are sufficiently large, then $Y_m<Z_n$, and consequently $F(Y_m)\le F(Z_n)$. Since $\lim_m F(Y_m)=F(Y)=X$ and $\lim_n F(Z_n)=F(Z)=X$, we have $F(Y_m)\le X\le F(Z_n)$ for m, n sufficiently large. But $e(Y_m)$ and $e(Z_n)$ are images of x under some iterates of f. Since $\rho(A)$ is irrational, we obtain $F(Y_m)<X<F(Z_n)$. This contradicts the assumption that x is an endpoint of an interval disjoint from A. Consequently, a point z with the described properties does not exist. This proves that y is the unique element of $A\cap f^{-1}(x)$.

If y is not an endpoint of an interval disjoint from A, then we can use very similar arguments to obtain a contradiction. This ends the proof of (b).

To prove (c), we assume that there are y and z as in the proof of (b) and we make the same construction. If there is $T\in A$ with $Y<T<Z$ and $F(T)=X$, then we obtain, as for Y and Z, a sequence $(T_n)_{n=1}^{\infty}$ of elements of $e^{-1}(A)$ such that

$\lim_n F(T_n) = X$, $F(T_n) \neq X$ for all n and $F(Y_m) < F(T_k) < F(Z_n)$ for m, k, n sufficiently large. This is impossible and hence the set $A \cap f^{-1}(x)$ cannot contain three different points. Moreover, if it contains two different points, they are the endpoints of an interval disjoint from A. This proves (c). □

COROLLARY 2.7. *If A is a mat set and $\rho(A)$ is irrational, then the system $(A, f|_A)$ arises from* (S^1, rotation by $\rho(A)$) *by 'blowing up' at most a countable number of two-sided orbits and/or negative semi-orbits (from every 'blown up' interval we leave only its endpoints).*

3. *Proof of theorem* B

We have to prove the existence of a twist periodic orbit with a given rotation number p/q from the rotation interval of f (where $p \in \mathbb{Z}$, $q \in \mathbb{Z}^+$, p and q coprime). It is known that if p/q belongs to the rotation interval and p and q are coprime, then there exists a periodic orbit of period q and rotation number p/q (see [7], [3]). We denote this orbit by A.

If $q = 1$, then A is a twist orbit. If not, then by taking a different lifting if necessary, we can reduce the problem to the case $0 < p < q$.

Hence, in the sequel we shall assume that $0 < p < q$. The set $e^{-1}(A)$ divides $\mathbb{R}$ into a countable number of intervals. We denote them, from left to right,

$$\ldots, I_{-2}, I_{-1}, I_0, I_1, I_2, \ldots.$$

Clearly, we have $I_{j+iq} = I_j + i$ for all $i, j \in \mathbb{Z}$. For $k \in \mathbb{Z}$ we define $\alpha(k) \subset \mathbb{Z}$ as follows:

$$\alpha(k) = \{i \in \mathbb{Z}\colon F(\inf I_i) \leq \inf I_k, \quad F(\sup I_i) \geq \sup I_k\}.$$

Clearly, $\alpha(j + iq) = \alpha(j) + iq$ for all $i, j \in \mathbb{Z}$.

We consider a family $\mathcal{P}$ of all maps $\varphi : \mathbb{Z} \to \mathbb{Z}$ such that

(i) $\varphi(k) \in \alpha(k)$ for all $k \in \mathbb{Z}$;

(ii) $\varphi(j + iq) = \varphi(j) + iq$ for all $i, j \in \mathbb{Z}$,

(iii) if $k \leq j$, then $\varphi(k) \leq \varphi(j)$, (i.e. φ is non-decreasing).

By (ii), for every $k \in \mathbb{Z}$ there exist $n, s \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$ such that $\varphi^{s+n}(k) = \varphi^n(k) + rq$ (factorizing φ mod q gives a map of a finite set into itself). Then for all $m \geq n$ we have $\varphi^{s+m}(k) = \varphi^m(k) + rq$, and we obtain

$$\lim_m \frac{1}{smq} \varphi^{sm}(j) = \frac{r}{s} \qquad \text{for } j = \varphi^n(k).$$

By (iii), we obtain from this

$$\lim_l \frac{1}{lq} \varphi^l(j) = \frac{r}{s} \qquad \text{for all } j \in \mathbb{Z}.$$

We shall call r/s the rotation number of φ and denote it by $\rho(\varphi)$. Notice, that since φ goes 'backwards' with respect to F, the rotation numbers of elements of $\mathcal{P}$ considered here will usually be negative.

Now we shall prove several lemmas.

LEMMA 3.1. *If φ, $\psi \in \mathcal{P}$ and $\varphi \leq \psi$, then $\rho(\varphi) \leq \rho(\psi)$.*

Proof. Let $k\in\mathbb{Z}$, $n\in\mathbb{Z}^+$. We have
$\varphi^n(k)=\varphi^{n-1}(\varphi(k))\le\varphi^{n-1}(\psi(k))=\varphi^{n-2}(\varphi(\psi(k)))\le\varphi^{n-2}(\psi^2(k))\le\cdots\le\psi^n(k)$.
Hence,

$$\rho(\varphi)=\lim_n\frac{1}{nq}\varphi^n(k)\le\lim_n\frac{1}{nq}\psi^n(k)=\rho(\psi).\qquad\square$$

We define maps $\underline{\varphi}$ and $\bar{\varphi}$ by

$$\underline{\varphi}(k)=\min\alpha(k),\quad\bar{\varphi}(k)=\max\alpha(k)\qquad\text{for } k\in\mathbb{Z}.$$

LEMMA 3.2. *The maps $\underline{\varphi}$ and $\bar{\varphi}$ belong to $\mathcal{P}$ and we have $\rho(\underline{\varphi})\le -p/q\le\rho(\bar{\varphi})$.*

Proof. The condition (i) of the definition of $\mathcal{P}$ is satisfied for $\underline{\varphi}$ and $\bar{\varphi}$ by their definition.

The condition (ii) is satisfied, since $\alpha(j+iq)=\alpha(j)+iq$. To show that the condition (iii) is satisfied, we have to use the definition of α. Since $\lim_{X\to-\infty}F(X)=-\infty$ and $\lim_{X\to+\infty}F(X)=+\infty$, $F(e^{-1}(A))=e^{-1}(A)$ and F is continuous, we have

$$\underline{\varphi}(l)=\min\{i\in\mathbb{Z}\colon F(\sup I_i)\ge\sup I_l\}$$

and

$$\bar{\varphi}(l)=\max\{i\in\mathbb{Z}\colon F(\inf I_i)\le\inf I_l\}.$$

From this (iii) follows.

Since A is a periodic orbit, for every $k\in\mathbb{Z}$ there exists a unique $i\in\mathbb{Z}$ such that $F(\sup I_i)=\sup I_k$. Denote this i by $\chi(k)$. Since the rotation number of A is equal to p/q, we have $\chi^q(k)=k-pq$. Clearly, $\underline{\varphi}\le\chi$. In the same way as in the proof of lemma 3.1, we can show that $\underline{\varphi}^n\le\chi^n$ for all $n\in\mathbb{Z}^+$. Hence, $\rho(\underline{\varphi})\le -p/q$. Analogously (taking inf instead of sup and $\ge$ instead of $\le$) we obtain $\rho(\bar{\varphi})\ge -p/q$. $\square$

LEMMA 3.3. *Let $k<j$, $l(k)\in\alpha(k)$, $l(j)\in\alpha(j)$, $l(k)\le l(j)$. Then there exist $l(k+1)\in\alpha(k+1),\dots,l(j-1)\in\alpha(j-1)$ such that*

$$l(k)\le l(k+1)\le\cdots\le l(j-1)\le l(j).$$

Proof. Set

$$l(i)=\min\{t\in\mathbb{Z}\colon t\ge l(k),\ F(\sup I_t)\ge\sup I_i\}\qquad\text{for } i=k+1,\dots,j-1.$$

We have $\inf I_t=\sup I_{t-1}$, and hence from the definition of $\alpha(i)$, in view of the inequality

$$F(\inf I_{l(k)})\le\inf I_k\le\inf I_i,$$

we obtain $l(i)\in\alpha(i)$. From the definition of $l(i)$ it follows that

$$l(k)\le l(k+1)\le\cdots\le l(j-1).$$

Since $F(\sup I_{l(j)})\ge\sup I_j>\sup I_{j-1}$, we obtain $l(j-1)\le l(j)$. $\square$

LEMMA 3.4. *Assume that $\varphi_1,\varphi_2\in\mathcal{P}$, $\varphi_1\ne\varphi_2$, $\varphi_1\le\varphi_2$ and that there is no ψ with $\psi\ne\varphi_1$, $\psi\ne\varphi_2$, $\varphi_1\le\psi\le\varphi_2$. Then there exists a unique $m\in\{0,\dots,q-1\}$ such that $\varphi_1(m)\ne\varphi_2(m)$.*

Proof. Since φ_1 and φ_2 satisfy the condition (ii) of the definition of $\mathcal{P}$ and $\varphi_1 \neq \varphi_2$, such an m exists. Set

$$j = \min\{i \in \mathbb{Z}: i > m, \varphi_1(i) \geq \varphi_2(m)\},$$
$$k = \max\{i \in \mathbb{Z}: i < j, \varphi_2(i) \leq \varphi_1(j)\}$$

(see figure 1). Clearly, $m \leq k < j$.

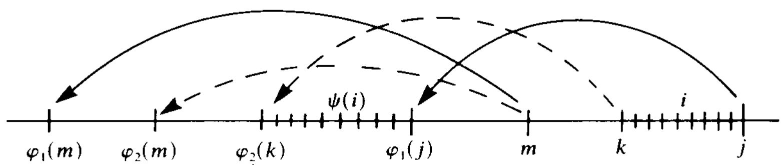

FIGURE 1

Suppose that $j > k+1$. If $j \leq k+q$, we define ψ as follows: for $i = k+1, \ldots, j-1$ we set $\psi(i) = l(i)$, where $l(i)$, $i = k+1, \ldots, j-1$, are the numbers from lemma 3.3, obtained for $l(k) = \varphi_2(k)$, $l(j) = \varphi_1(j)$; for $i = j, \ldots, k+q$ we set $\psi(i) = \varphi_1(i)$; if $t = i + nq$ for some $i \in \{k+1, \ldots, k+q\}$, $n \in \mathbb{Z}$, then $\psi(t) = \psi(i) + nq$. It is easy to see that $\psi \in \mathcal{P}$ (to check that $\psi(k) \leq \psi(k+1)$, notice that we have $\psi(k) = \varphi_1(k) \leq \varphi_2(k) = l(k) \leq l(k+1) = \psi(k+1)$). For $i = j, \ldots, k+q$ we have $\psi(i) = \varphi_1(i) \leq \varphi_2(i)$. For $i = k+1, \ldots, j-1$ by the definition of j we have $\psi(i) \geq \varphi_2(k) \geq \varphi_2(m) > \varphi_1(i)$ and by the definition of k we have $\psi(i) \leq \varphi_1(j) < \varphi_2(i)$. Hence, $\varphi_1 \leq \psi \leq \varphi_2$. Since $k+1 < j$, the set $\{k+1, \ldots, j-1\}$ is non-empty. For i from this set we have $\varphi_1(i) < \psi(i) < \varphi_2(i)$, and hence $\psi \notin \{\varphi_1, \varphi_2\}$. This contradicts our assumptions.

If $j > k+q$, we define ψ as follows: for $i = k+1, \ldots, k+q-1$ we set $\psi(i) = l(i)$ from lemma 3.3 (as before) and we set $\psi(k+q) = \varphi_2(k+q)$. We extend ψ to the whole of $\mathbb{Z}$ as before. To show that $\psi \in \mathcal{P}$ it is enough to check that $\psi(k+q-1) \leq \varphi_2(k+q)$ (the rest of the checking is trivial). But, as before, we have $\varphi_1(i) < \psi(i) < \varphi_2(i)$ for $i = k+1, \ldots, k+q-1$, so in particular $\psi(k+q-1) < \varphi_2(k+q-1) \leq \varphi_2(k+q)$. Again we obtain $\varphi_1 \leq \psi \leq \varphi_2$ and $\psi \notin \{\varphi_1, \varphi_2\}$. This contradicts our assumptions. Consequently the supposition that $j > k+1$ was false.

Thus, we have $j = k+1$. This means that $\varphi_2(k) \leq \varphi_1(k+1)$. Suppose that there exist $i, l \in \{k+1, \ldots, k+q\}$ such that $i < l$ and $\varphi_1(i) < \varphi_2(i)$, $\varphi_1(l) < \varphi_2(l)$. Then we set $\psi(t) = \varphi_1(t)$ for $t = k+1, \ldots, i$; $\psi(t) = \varphi_2(t)$ for $t = i+1, \ldots, k+q$, and we extend ψ to the whole of $\mathbb{Z}$ as before. It is easy to see that $\psi \in \mathcal{P}$. Clearly, we have $\varphi_1 \leq \psi \leq \varphi_2$. Since $\psi(i) = \varphi_1(i) \neq \varphi_2(i)$ and $\psi(l) = \varphi_2(l) \neq \varphi_1(l)$, we have $\psi \neq \varphi_1$ and $\psi \neq \varphi_2$. This contradicts our assumptions.

Hence, there exists at most one $i \in \{k+1, \ldots, k+q\}$ such that $\varphi_1(i) \neq \varphi_2(i)$. Since φ_1 and φ_2 satisfy condition (ii) of the definition of $\mathcal{P}$, this ends the proof. □

LEMMA 3.5. *There exist $\varphi_1, \varphi_2 \in \mathcal{P}$ such that $\varphi_1 \leq \varphi_2$, $\rho(\varphi_1) \leq -p/q \leq \rho(\varphi_2)$ and for at most one $m \in \{0, \ldots, q-1\}$ we have $\varphi_1(m) \neq \varphi_2(m)$.*

Proof. Since $\rho(\underline{\varphi}) \leq -p/q$ and the family $\mathcal{P}$ is finite, then there exists $\varphi_1 \in \mathcal{P}$ such that $\rho(\varphi_1) \leq -p/q$ and for every $\psi \in \mathcal{P}$ with $\psi \leq \varphi_1$ and $\psi \neq \varphi_1$, we have $\rho(\psi) > -p/q$.

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If $\rho(\varphi_1) = -p/q$, then we can take $\varphi_2 = \varphi_1$. If not, since $\bar{\varphi} \geq \varphi_1$, $\rho(\bar{\varphi}) \geq -p/q$ and $\mathscr{P}$ is finite, then there exists $\varphi_2 \in \mathscr{P}$ such that $\varphi_2 \geq \varphi_1$, $\varphi_2 \neq \varphi_1$ and for every $\psi \in \mathscr{P}$ with $\varphi_1 \leq \psi \leq \varphi_2$ we have either $\psi = \varphi_1$ or $\psi = \varphi_2$. By lemma 3.4, we then have $\varphi_1(m) \neq \varphi_2(m)$ for a unique $m \in \{0, \ldots, q-1\}$. □

For every pair (k, i) such that $i \in \alpha(k)$, we define a map $\Phi(k, i)$: $I_k \to I_i$ by the formula

$$\Phi(k, i)(X) = \inf\{Y \in I_i: F(Y) = X\}.$$

This map can be discontinuous. Clearly, we have $F|_{I_i} \circ \Phi(k, i) = \mathrm{id}_{I_k}$. If $X, Z \in I_k$ and $X < Z$, then in view of the (in)equalities $F(\inf I_i) \leq \inf I_k$ and $F(\Phi(k, i)(Z)) = Z$, there exists Y such that $\inf I_k \leq Y < \Phi(k, i)(Z)$ and $F(Y) = X$. Hence,

$$\Phi(k, i)(X) < \Phi(k, i)(Z).$$

This proves that $\Phi(k, i)$ is increasing.

If there exists $\varphi \in \mathscr{P}$ such that $\rho(\varphi) = -p/q$, then we consider the map

$$\Phi_0 = \Phi(\varphi^{q-1}(0), \varphi^q(0)) \circ \cdots \circ \Phi(\varphi(0), \varphi^2(0)) \circ \Phi(0, \varphi(0)).$$

This map is increasing and maps I_0 into $I_{\varphi^q(0)} = I_{-pq}$. Hence, the map $\Phi_0 + p$ is increasing and maps I_0 into itself. Therefore it has a fixed point X_0 (it is easy to see that $\sup\{X \in I_0: \Phi_0(X) + p \leq X\}$ is such point). Every element of $e^{-1}(\{f^n(e(X_0))\}_{n=0}^{\infty})$ belongs to some interval I_i with $i = \varphi^j(0) + kq, j \in \{0, \ldots, q-1\}$, $k \in \mathbb{Z}$. Two different elements cannot belong to the same interval, since then the denominator of $\rho(\varphi)$ would be smaller than q. Therefore it follows from the properties of φ that $e(X_0)$ is a periodic point of period q and rotation number p/q and its orbit is a twist set.

Thus, to complete the proof of theorem B, it is enough to consider the case of $\rho(\varphi_1) < -p/q < \rho(\varphi_2)$, where φ_1 and φ_2 are from lemma 3.5. Without any loss of generality, we may assume that $m = 0$. Then we have $\varphi_1(k) = \varphi_2(k)$ if q does not divide k and $\varphi_1(k) < \varphi_2(k)$ if q divides k. We make these assumptions for the rest of the proof.

We have $\rho(\varphi_i) = r_i/s_i$, where r_i and s_i are coprime, for $i = 1, 2$. By the above assumptions, $r_1/s_1 < -p/q < r_2/s_2$.

LEMMA 3.6. (a) *We have* $\varphi_i^{s_i}(0) = r_i q$ *for* $i = 1, 2$.

(b) *The numbers* $\varphi_i^j(0)$ *for* $j = 1, \ldots, s_i - 1$ $(i = 1, 2)$ *are not divisible by* q.

Proof. If (a) is not true, then we have $\varphi_i^{s_i}(k) = k + r_i q$ for some $i \in \{1, 2\}$ and $k \in \mathbb{Z}$, where none of the numbers $\varphi_i^j(k)$ $(j = 0, \ldots, s_i - 1)$ is divisible by q. But then φ_1 and φ_2 attain the same values at these numbers, and consequently $\rho(\varphi_1) = \rho(\varphi_2)$, which contradicts our assumptions.

If (b) is not true, then we have $\varphi_i^j(0) = tq$ for some $i \in \{i, 2\}$, $t \in \mathbb{Z}$ and $j \in \{1, \ldots, s_i - 1\}$. Then $\rho(\varphi_i) = t/j$. Since $0 < j < s_i$, this contradicts our assumption that r_i and s_i are coprime. □

LEMMA 3.7. *We have* $s_1 r_2 - s_2 r_1 = 1$.

Proof. The set $\{\varphi_2^n(iq): n \in \mathbb{N}, i \in \mathbb{Z}\}$ is of the form $\{\ldots, k_{-2}, k_{-1}, k_0, k_1, k_2, \ldots\}$, where $\cdots < k_{-2} < k_{-1} < k_0 < k_1 < k_2 < \cdots$ and $k_0 = 0$. By lemma 3.6 we have $k_{is_2} = iq$ for all

$i \in \mathbb{Z}$ and no other k_j is divisible by q. Since φ_2 is non-decreasing, it is increasing on the above set (otherwise the denominator of $\rho(\varphi_2)$ would be smaller than s_2), and consequently $\varphi_2(k_i) = k_{i+r_2}$ for all $i \in \mathbb{Z}$.

Let us take $i, j \in \mathbb{Z}$ such that $j \geq k_i$. If q does not divide j, then

$$\varphi_1(j) = \varphi_2(j) \geq \varphi_2(k_i) = k_{i+r_2}.$$

If q divides j, then we distinguish two cases. The first case is $s_2 > 1$. Then q does not divide k_{i-1}, and hence

$$\varphi_1(j) \geq \varphi_1(k_{i-1}) = \varphi_2(k_{i-1}) = k_{i+r_2+1}.$$

The second case is $s_2 = 1$. Then

$$j-1 \geq j-q \geq k_i - q = iq - q = k_{i-1}.$$

Since $q \geq 2$, we have $\varphi_1(j-1) = \varphi_2(j-1)$, and hence

$$\varphi_1(j) \geq \varphi_1(j-1) = \varphi_2(j-1) \geq \varphi_2(k_{i-1}) = k_{i+r_2-1}.$$

In both cases when q divides j, we obtain $\varphi_1(j) \geq k_{i+r_2-1}$.

By lemma 3.6, only one of the numbers $\varphi_1^i(0)$, $i = 0, \ldots, s_1 - 1$, is divisible by q. Therefore we obtain

$$\varphi_1^{s_1}(0) = \varphi_1^{s_1}(k_0) \geq k_{s_1 r_2 - 1}.$$

But $\varphi_1^{s_1}(0) = r_1 q = k_{r_1 s_2}$. Hence, $r_1 s_2 \geq s_1 r_2 - 1$. Since $r_1/s_1 < r_2/s_2$, we have $r_1 s_2 < s_1 r_2$, and therefore $r_1 s_2 = s_1 r_2 - 1$. □

Set $n_1 = s_2 p + r_2 q$, $n_2 = -s_1 p - r_1 q$.

LEMMA 3.8. *We have*:

(a) $n_1, n_2 > 0$, $n_1 + n_2 \leq q$, n_1 *and* n_2 *are coprime*;

(b) $n_1 r_1 + n_2 r_2 = -p, \quad n_1 s_1 + n_2 s_2 = q$.

Proof. From the inequaltiy $r_1/s_1 < -p/q < r_2/s_2$ it follows that n_1 and n_2 are positive. From the definition of n_1 and n_2 and from lemma 3.7 it follows by a direct computation that (b) holds. Since p and q are coprime, we obtain from (b) that n_1 and n_2 are coprime. Since $s_1, s_2 \geq 1$, we have $n_1 + n_2 \leq n_1 s_1 + n_2 s_2 = q$. □

Now we go back to the maps $\Phi(k, i)$.

LEMMA 3.9. *Let* $i \in \{1, 2\}$; $j, k \in \mathbb{Z}$; $X \in I_j$, $Y \in I_k$ *and* $X < Y$. *Then*

$$\Phi(j, \varphi_i(j))(X) < \Phi(k, \varphi_i(k))(Y).$$

Proof. If $\varphi_i(j) < \varphi_i(k)$ then the conclusion is obvious (we cannot have the equality of these points since then $X = Y$). If $\varphi_i(j) = \varphi_i(k)$, then the definitions of $\Phi(j, \varphi_i(j))$ and $\Phi(k, \varphi_i(k))$ are the same and the conclusion follows in the same way as the monotonicity of Φ (see the proof following the definition of $\Phi(k, i)$). If $\varphi_i(j) > \varphi_i(k)$, then $j > k$, which contradicts the assumption $X < Y$. Hence this case cannot occur. □

For $i \in \{1, 2\}$ write

$$\Phi_i = \Phi(\varphi_i^{s_i - 1}(0), \varphi_i^{s_i}(0)) \circ \cdots \circ \Phi(\varphi_i(0), \varphi_i^2(0)) \circ \Phi(0, \varphi_i(0)).$$

The map Φ_i maps I_0 into $I_{r_i q}$ and is increasing.

LEMMA 3.10. *Let X_1, $X_2 \in I_0$. Then $\Phi_2(X_2) - r_2 \leq \Phi_1(X_1) - r_1$.*

Proof. For $i = 1, 2$ we have

$$\Phi_i(X_i) - r_i = [\Phi(\varphi_i^{s_i-1}(-r_iq), \varphi_i^{s_i}(-r_iq)) \circ \cdots \circ \Phi(\varphi_i(-r_iq), \varphi_i^2(-r_iq)) \circ \Phi(-r_iq, \varphi_i(-r_iq))](X_i - r_i).$$

Assume that $s_1 < s_2$. Since $\rho(\varphi_2) > r_1/s_1$, we have $\varphi_2^{s_1}(-r_1q) > 0$, and hence $\varphi_2^{s_2-s_1}(-r_2q) < -r_1q$. Thus

$$F^{s_1}(\Phi_2(X_2) - r_2) \leq F^{s_1}(\Phi_1(X_1) - r_1).$$

By lemma 3.6(b), we have

$$\varphi_1^{s_1}(\varphi_2^{s_2-s_1}(-r_2q)) = \varphi_2^{s_1}(\varphi_2^{s_2-2_1}(-r_2q)) = (\varphi_2^{s_2}(-r_2q) = 0 = \varphi_1^{s_1}(-r_1q).$$

Using lemma 3.9 s_1 times, we obtain consecutively

$$F^{s_1-1}(\Phi_2(X_2) - r_2) \leq F^{s_1-1}(\Phi_1(X_1) - r_1), \ldots, \Phi_2(X_2) - r_2 \leq \Phi_1(X_1) - r_1.$$

Analogously, if $s_2 < s_1$, then $-r_2q < \varphi_1^{s_1-s_2}(-r_1q)$,

$$F^{s_2}(\Phi_2(X_2) - r_2) \leq F^{s_2}(\Phi_1(X_1) - r_1),$$

$$\varphi_2^{s_2}(-r_2q) = \varphi_2^{s_2}(\varphi_1^{s_1-s_2}(-r_1q)),$$

and we also obtain

$$\Phi_2(X_2) - r_2 \leq \Phi_1(X_1) - r_1.$$

Assume that $s_1 = s_2$. Then we have $-r_2 < -r_1$. Using the notation from the proof of lemma 3.7, we have $-r_2q = k_{-r_2s_2}$ and $-r_1q = k_{-r_1s_2}$. Thus, by the inequality obtained in the proof of lemma 3.7, we have $\varphi_1(-r_1q) \geq k_{-r_1s_2+r_2-1}$. Since $-r_1s_2 - 1 \geq -r_2s_2$, we obtain

$$\varphi_1(-r_1q) \geq k_{-r_2s_2+r_2} = \varphi_2(k_{-r_2s_2}) = \varphi_2(-r_2q).$$

If $\varphi_2(-r_2q) < \varphi_1(-r_1q)$, then

$$F^{s_1-1}(\Phi_2(X_2) - r_2) \leq F^{s_1-1}(\Phi_1(X_1) - r_1).$$

If $\varphi_2(-r_2q) = \varphi_1(-r_1q)$ then the definitions of $\Phi(-r_1q, \varphi_1(-r_1q))$ and $\Phi(-r_2q, \varphi_2(-r_2q))$ are the same and the inequality

$$F^{s_1-1}(\Phi_2(X_2) - r_2) \leq F^{s_1-1}(\Phi_1(X_1) - r_1)$$

follows from $X_2 - r_2 \leq X_1 - r_1$ (which is true since $X_2 - r_2 \in I_{-r_2q}$ and $X_1 - r_1 \in I_{-r_1q}$) in the same way as the monotonicity of Φ. In both cases we obtain

$$F^{s_1-1}(\Phi_2(X_2) - r_2) \leq F^{s_1-1}(\Phi_1(X_1) - r_1).$$

If $s_1 = 1$, then the proof is complete. If $s_1 > 1$, then we proceed as for the case $s_1 \neq s_2$ (use lemma 3.6(b) and then lemma 3.9 $s_1 - 1$ times) and we obtain

$$\Phi_2(X_2) - r_2 \leq \Phi_1(X_1) - r_1. \qquad \square$$

LEMMA 3.11. *For $i = 0, \ldots, n_1 - 1$ we have $i - s_1p = (n_2 + i) + r_1q$, and s_1 is the smallest positive k such that $i - kp \equiv l \pmod q$ for some $l \in \{0, \ldots, n_1 + n_2 - 1\}$.*

For $i = n_1, \ldots, n_1 + n_2 - 1$ we have $i - s_2p = (i - n_1) + r_2q$ and s_2 is the smallest positive k such that $i - kp \equiv l \pmod q$ for some $l \in \{0, \ldots, n_1 + n_2 - 1\}$.

Proof. The equalities $i - s_1p = (n_2 + i) + r_1q$ and $i - s_2p = (i - n_1) + r_2q$ follow directly from the definitions of n_1 and n_2. We can restate these equalities as follows. Let

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$\xi: \{0, \dots, q-1\} \to \{0, \dots, q-1\}$ be given by $\xi(t) = t - p \pmod q$, and let $\zeta: \{0, \dots, n_1+n_2-1\} \to \{0, \dots, n_1+n_2-1\}$ be given by $\zeta(t) = t + n_2 \pmod{n_1+n_2}$. Since p and q are coprime and n_1 and n_2 are coprime (see lemma 3.8(a)), both ξ and ζ are cyclic permutations. By lemma 3.8(a), we have $\{0, \dots, n_1+n_2-1\} \subset 0, \dots, q-1\}$. Write

$$\varepsilon(i) = \begin{cases} 1 & \text{if } i \in \{0, \dots, n_1-1\}, \\ 2 & \text{if } i \in \{n_1, \dots, n_1+n_2-1\}. \end{cases}$$

Then for all $i \in \{0, \dots, n_1+n_2-1\}$ we have $\xi^{s_{\varepsilon(i)}}(i) = \zeta(i)$. Hence, for a fixed $i \in \{0, \dots, n_1+n_2-1\}$, we have

$$\{0, \dots, n_1+n_2-1\} = \{i, \zeta(i), \dots, \zeta^{n_1+n_2-1}(i)\} = \{i, \xi^{b(1)}(i), \dots, \xi^{b(n_1+n_2-1)}(i)\},$$

where $b(t) = s_{\varepsilon(i)} + s_{\varepsilon(\zeta(i))} + \cdots + s_{\varepsilon(\zeta^{t-1}(i))}$. Since the elements $i, \zeta(i), \dots, \zeta^{t-1}(i)$ are mutually distinct, we have $b(t) < n_1 s_1 + n_2 s_2$. Hence, by lemma 3.8(b), we obtain $b(t) < q$ for $t = 1, \dots, n_1+n_2-1$. There are only n_1+n_2 numbers $k \in \{0, \dots, q-1\}$ such that $\xi^k(i) \in \{0, \dots, n_1+n_2-1\}$. Thus they are the numbers 0, $b(1), \dots, b(n_1+n_2-1)$. The smallest positive one among them is $b(1) = s_{\varepsilon(i)}$. This completes the proof. □

We define a sequence $(c_j)_{j=0}^q$ as follows:

(1°) $c_0 = 0$;

(2°) if $-jp \equiv i \pmod q$ for some $i \in \{0, \dots, n_1-1\}$, then $c_{j+1} = \varphi_1(c_j)$; if not, then $c_{j+1} = \varphi_2(c_j)$.

LEMMA 3.12. *We have*:

(a) *q divides c_j if and only if $-jp \equiv i \pmod q$ for some $i \in \{0, \dots, n_1+n_2-1\}$*;

(b) *$c_q = -pq$.*

Proof. Assume that for some $j \in \{0, \dots, q-1\}$, $i \in \{0, \dots, n_1+n_2-1\}$ and $k \in \{1, 2\}$ we have $-jp \equiv i \pmod q$, q divides c_j, and $c_{j+1} = \varphi_k(c_j)$. Then by lemma 3.6, q does not divide $c_{j+1}, \dots, c_{j+s_k-1}$ and q divides c_{j+s_k} (remember that $\varphi_1(l) = \varphi_2(l)$ if q does not divide l). By lemma 3.11, for $t = j+1, \dots, j+s_k-1$ there is no $l \in \{0, \dots, n_1+n_2-1\}$ such that $-tp \equiv l \pmod q$ and for $t = j+s_k$ such an l exists. Hence (a) follows by induction.

When j varies from 0 to $q-1$, then it happens n_1 times that $-jp \equiv i \pmod q$ for some $i \in \{0, \dots, n_1-1\}$ and then

$$c_{j+s_1} = \varphi_1^{s_1}(c_j) = c_j + r_1 q;$$

and it happens n_2 times that $-jp \equiv i \pmod q$ for some $i \in \{n_1, \dots, n_1+n_2-1\}$ and then

$$c_{j+s_2} = \varphi_2^{s_2}(c_j) = c_j + r_2 q.$$

Hence, $c_q = 0 + n_1 r_1 q + n_2 r_2 q = -pq$. □

We define a map Φ_0 by

$$\Phi_0 = \Phi(c_{q-1}, c_q) \circ \cdots \circ \Phi(c_1, c_2) \circ \Phi(c_0, c_1).$$

LEMMA 3.13. *The map Φ_0 is a composition of maps $\Phi_i - c_j/q$ for those $j \in \{0, \dots, q-1\}$ for which q divides c_j, where $i = 1$ if $-jp \equiv l \pmod q$ for some $l \in \{0, \dots, n_1-1\}$ and $i = 2$ if $-jp \equiv l \pmod q$ for some $l \in \{n_1, \dots, n_1+n_2-1\}$. The map Φ_0 maps I_0 into I_{-pq}.*

Proof. The above properties of Φ_0 follow immediately from the definitions of Φ_i $(i=0,1,2)$ and the sequence $(c_j)_{j=0}^{q}$ and from lemma 3.12 and its proof. □

LEMMA 3.14. (a) *We have* $-1\le r_1/s_1$ *and* $r_2/s_2\le 0$.

(b) *For all* $i\in\{1,2\}$, $j\in\mathbb{Z}$ *and* $k\in\mathbb{N}$ *we have*

$$\varphi_i(\varphi_i^k(0)+jq)\le\varphi_i^k(0)+jq.$$

Proof. Since $r_2/s_2>-p/q>-1$, $r_1/s_1<-p/q<0$ and $r_2s_1-r_1s_2=1$, we have $-s_1s_2<s_1r_2=r_1s_2+1$ and $0>r_1s_2=r_2s_1-1$. Hence $-s_1s_2\le r_1s_2$ and $0\ge r_2s_1$. Consequently $-s_1\le r_1$ and $0\ge r_2$. This proves (a).

For $i=2$, (b) follows from the arguments used in the beginning of the proof of lemma 3.7 and the fact that $r_2\le 0$. To prove (b) for $i=1$, we use similar arguments. □

The map $\Phi_0+p:I_0\to I_0$ is increasing, and hence it has a fixed point. We call this point X_0. The rotation number of X_0 is p/q, and since p and q are coprime, the period of $e(X_0)$ is q (it is clear that $e(X_0)$ is periodic).

LEMMA 3.15. *The orbit of* $e(X_0)$ *is a twist set.*

Proof. Write $B=e^{-1}(\{f^n(e(X_0))\}_{n=0}^{\infty})$. The set B consists of points of the form $F^i(X_0)+j$, $i\in\mathbb{N}$, $j\in\mathbb{Z}$. Let us assume that $X, Y\in B$ and $X<Y$. We have to show that $F(X)<F(Y)$. Let $F(X)\in I_j$, $F(Y)\in I_k$. We can choose j and k in such a way that they are of the form $\varphi_i^n(0)+mq$ (of course, we have a choice only if $F(X)$ or $F(Y)$ is an endpoint of the corresponding interval; but as yet we have not excluded this possibility).

If $j<k$ then clearly $F(X)\le F(Y)$. But since B is a lifting of a periodic orbit, we cannot have $F(X)=F(Y)$, and hence $F(X)<F(Y)$.

Assume that $j\ge k$. If there exists $i\in\{1,2\}$ such that $X\in I_{\varphi_i(j)}$ and $Y\in I_{\varphi_i(k)}$, then $F(X)<F(Y)$ (we cannot in this case have $F(X)>F(Y)$, since then, by lemma 3.9, $X>Y$). Assume that such an i does not exist. Then q divides both j and k. If $j>k$, then $X\in I_{\varphi_1(j)}$ and $Y\in I_{\varphi_2(k)}$. By lemma 3.14(b), $\varphi_2(k)\le k$, and we have $\varphi_1(j)\le k\le j-q$. Consequently, $\rho(\varphi_1)\le -1$, and by lemma 3.14(a), $r_1/s_1=-1$. Then $X\in I_k$, and thus $Y\in I_k$. Therefore $\rho(\varphi_2)=0$ and hence $r_2/s_2=0$. Thus we obtain in the case of $j>k$ that $X=\Phi_1(F(X)-j/q)+k/q$ and $Y=\Phi_2(F(Y)-k/q)+k/q$. By lemma 3.10, we obtain $Y\le X$, which contradicts our assumptions.

We are left with the case of $j=k$, q divides j. Without any loss of generality we may assume that $j=0$. Recall the situation: $F(X)$, $F(Y)\in I_0$, X, $Y\in B$, $X=\Phi(0,\varphi_1(0))(F(X))$, $Y=\Phi(0,\varphi_2(0))(F(Y))$, and we want to prove that $F(X)<F(Y)$.

Consider the set $B\cap I_0$. By Lemma 3.13 we have $B\cap I_0\subset B_1\cup B_2$, where $B_1=B\cap(\Phi_1(I_0)-r_1)$ and $B_2=B\cap(\Phi_2(I_0)-r_2)$. By lemma 3.10, all points of B_2 lie to the left of all points of B_1, except perhaps one common point. But for this common point Z we would have $F^{s_1}(Z)\in I_{-r_1q}$ and $F^{s_2}(Z)\in I_{-r_2q}$. Either $s_1>s_2$, but then $\varphi_1^{s_1-s_2}(-r_1q)\ne -r_2q$; or $s_1<s_2$, but then $\varphi_2^{s_2-s_1}(-r_2q)\ne -r_1q$; or $s_1=s_2$, but then $-r_1q\ne -r_2q$. In all cases the only possibility is that either $F^{s_1}(Z)$ is an endpoint of I_{-r_1q} or $F^{s_2}(Z)$ is an endpoint of I_{-r_2q}. But then the orbit of $e(X_0)$ is equal to A

(recall that A was the orbit chosen at the very beginning of the proof of theorem B). This contradicts the fact that Z is an interior point of I_0 (which follows from the definition of Z and lemma 3.10). Hence $B_1 \cap B_2 = \emptyset$.

By the arguments from the previous part of this proof, the maps $F^{s_i}|_{B_i}$ $(i = 1, 2)$ are increasing. Since $F^{s_1}(B_1) + r_1 \subset B \cap I_0$ and $F^{s_2}(B_2) + r_2 \subset B \cap I_0$, we can define by induction for every $T \in B \cap I_0$ a sequence $(\varepsilon_n(T))_{m=0}^{\infty}$ as follows:

$$\varepsilon_0(T) = \begin{cases} 1 & \text{if } T \in B_1, \\ 2 & \text{if } T \in B_2; \end{cases} \tag{1°}$$

$$\varepsilon_n(T) = \begin{cases} 1 & \text{if } F^t(T) + l \in B_1, \\ 2 & \text{if } F^t(T) + l \in B_2; \end{cases} \tag{2°}$$

where $t = s_{\varepsilon_0(T)} + \cdots + s_{\varepsilon_{n-1}(T)}$, $l = r_{\varepsilon_0(T)} + \cdots + r_{\varepsilon_{n-1}(T)}$. By lemmas 3.11, 3.12 and 3.13, if $T = F^m(X_0) + u$ for some $m \in \mathbb{N}$, $u \in \mathbb{Z}$ and $mp \equiv k \pmod q$ with $k \in \{0, \ldots, n_1 + n_2 - 1\}$, then

$$\varepsilon_0(T) = \begin{cases} 1 & \text{if } k \in \{n_2, \ldots, n_1 + n_2 - 1\} \\ 2 & \text{if } k \in \{0, \ldots, n_2 - 1\}, \end{cases}$$

$$\varepsilon_n(T) = \begin{cases} 1 & \text{if } k + pt \equiv v \quad \text{for some } v \in \{n_2, \ldots, n_1 + n_2 - 1\} \\ 2 & \text{if } k + pt \equiv v \quad \text{for some } v \in \{0, \ldots, n_2 - 1\}, \end{cases}$$

where t is defined as before (notice that here we have the maps which are inverses of the maps from the proof of lemma 3.11).

Since $F^{s_i}|_{B_i}$ $(i = 1, 2)$ are increasing, we have $F(X) < F(Y)$ if and only if $(\varepsilon_n(F(X)))_{n=0}^{\infty} \propto (\varepsilon_n(F(Y)))_{n=0}^{\infty}$, where $\propto$ is the lexicographical order induced by the order $2 \propto 1$. But it is easy to see (look at the map $t \mapsto t - n_2 (\mathrm{mod}\, n_1 + n_2)$) that we have $(\varepsilon_n(F(X)))_{n=0}^{\infty} \propto (\varepsilon_n(F(Y)))_{n=0}^{\infty}$ if and only if $k(F(X)) < k(F(Y))$, where $k(F(X))$ and $k(F(Y))$ are defined as k above for $T = F(X)$ and $T = F(Y)$ respectively. Since $X = \Phi(0, \varphi_1(0))(F(X))$ and $Y = \Phi(0, \varphi_2(0))(F(Y))$, we have $k(F(X)) \in \{0, \ldots, n_1 - 1\}$ and $k(F(Y)) \in \{n_1, \ldots, n_1 + n_2 - 1\}$. Therefore $k(F(X)) < k(F(Y))$, and consequently $F(X) < F(Y)$. □

Now the proof of theorem B is complete.

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